ON THE EXISTENCE OF EXPECTATION TYPE MAPS FOR B*-ALGEBRAS

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by

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I. Introduction: Consider a finite measure space (Ω, Σ, m), let Σ_1 be a subfield of Σ and let m_1 be the restriction of m to Σ_1. One can define a linear map $\phi: L'(\Sigma,m) \to L'(\Sigma_1,m_1)$ by the equation $\int g\phi(f)dm = \int gfdm_1$, for all f in $L'(\Sigma,m)$ and for all bounded g in $L'(\Sigma_1,m_1)$. The map ϕ is called an expectation. The existence of ϕ follows from the classical Radon-Nikodym theorem and ϕ has the following properties: 1) ϕ is linear and positive, 2) $\phi(gf) = g\phi(f)$, 3) $\phi(\bar{f}) = \overline{\phi(f)}$, and 4) ϕ preserves the identity. The notion of expectation was extended to von Neumann Algebras by such authors as Dixmier [3] and Umegaki [9].

Let N and M be two von Neumann Algebras with $N \subset M$. An expectation of M on N is defined to be a positive, linear, *-map from M to N which preserves the identity and such that $\phi(ax) = a\phi(x)$ for all a in N and for all x in M. An expectation ϕ is called faithful if $\phi(x) = 0$ and x is positive implies $x = 0$. This notion of faithfullness can be extended to define what is meant by a complete set of expectations. Existence and properties of complete sets of expectations were studied by de Korvin [1], and there the expectations were obtained in terms of a family of states satisfying certain conditions. The expectations were obtained in a manner similar to the Radon-Nikodym theorem above, where the integral is replaced by a state and the functions by operators. The purpose of this paper is to extend the above results to B*-algebras and the elementary properties used can be found in Dunford and Schwartz [5] and Rickart [8]. In order to obtain the expectations, a map similar to the map $\natural$ used by Dixmier [2] and [4] is constructed. The main results of this paper are as follows:

Let N and M be B*-algebras with $N \subset M$. Suppose M is generated by its unitary elements (such is the case for Banach algebras with locally continuous involution [7]). If enough states exist on M which satisfy a continuity condition on the center of M, a boundedness condition on the positive elements of N, if the states diagonalize M, and further if the closure of the convex hull of the collection of utu*, as u ranges over the unitaries of N, for each t in N, is large enough, then there exist linear maps ϕ_α of M into certain subalgebras of N such that $\phi_\alpha(uvu^*) = u\phi_\alpha(v)u^*$ for all unitaries u of N and for all unitaries v of M. The ϕ_α's could be thought of as linear Radon-Nikodym type derivatives for states. Again if enough states exist on M which satisfy a continuity condition on the center of M, a boundedness condition on the positive elements of N, if the states diagonalize only N, and further if the closure of the convex hull of the collection of all utu*, as u ranges over the unitaries of N, for each t in N is large enough, then there exist maps Ψ_α, not necessarily linear, of M to certain subalgebras of N such that $\Psi_\alpha(xy) = x\Psi_\alpha(y)x^*$ for all x in N and y in M. Moreover, if the union of the carriers of the states is the identity, then the Ψ_α's form a complete set. Here the Ψ_α's could be thought of as Radon-Nikodym type derivatives for states, where the Ψ_α's need not be linear.

II. <u>Notation and Preliminaries</u>: Let M and N be B*-algebras such that $N \subset M$. A scalar valued function ρ defined on M will be called a state if ρ is linear, of norm one, positive in the sense that $\rho(x^*x) \geq 0$ for all x in M, and satisfies $\rho(x^*) = \overline{\rho(x)}$.

A state ρ on M is said to diagonalize N if $\rho(nm) = \rho(mn)$ for all m in M and n in N. A state ρ is said to be faithful if $\rho(x^*x) = 0$ implies $x = 0$. A collection $\{\rho_\alpha\}$ of states is said to be complete if $\rho_\alpha(x^*x) = 0$ for all α implies $x = 0$. By the commutant, N', of N in M we mean all elements of M which commute with all elements of N. By the center of N we will mean $Z_N = N \cap N'$.

Definition 2.1 We shall say that a state ρ defined on M is normal on N, if

$$\rho(x^*x \sup q_\beta) = \sup \{\rho(x^*xq_\beta)\}$$

for all increasing nets $\{q_\beta\}$ of projections in Z_N with $\sup q_\beta$ in Z_N, and for all x in N.

Definition 2.2 A state ρ is defined on M will be called continuously faithful on N, if there exists a projection $q \neq 0$ in Z_N such that ρ is faithful on $N_q = \{qaq : a \in N\}$ and further, if whenever $\{q_\beta\}$ is an increasing net of projections in Z_N, with ρ faithful of each N_{q_β}, and if $\sup q_\beta \in Z_N$, then ρ is faithful on $N_{\sup q_\beta}$.

Definition 2.3 By the carrier, relative to N, of a state ρ defined on M we will mean the maximal projection e in Z_N such that ρ is faithful on N_e.

It follows that if the carrier exists it is unique.

III. Preliminary Results

Theorem 3.1 Suppose $\{\rho_\alpha\}$ is a complete set of states on M and that each ρ_α is normal on N. Then each ρ_α is continuously faithful on N.

Proof: Let $e_\alpha = \sup q_{\alpha\beta}$ where $\{q_{\alpha\beta}\}$ is an increasing net of projections in Z_N, with ρ_α faithful on $N_{q_{\alpha\beta}}$ for each β and with

$\sup_\beta q_{\alpha\beta}$ in Z_N. Consider $e_\alpha x e_\alpha \in N_{e_\alpha}$ and suppose

$\rho_\alpha((e_\alpha x e_\alpha)^*(e_\alpha x e_\alpha)) = 0$. Since $e_\alpha \in Z_N$, $\rho_\alpha(x^*xe_\alpha) = 0$ and by normality $\sup_\beta \rho_\alpha(x^*xq_{\alpha\beta}) = 0$ and hence $\rho_\alpha(x^*xq_{\alpha\beta}) = 0$ for all β. Therefore since $q_{\alpha\beta} \in Z_N$ and since ρ_α is faithful on each $N_{q_{\alpha\beta}}$, we have $q_{\alpha\beta}x^*xq_{\alpha\beta} = 0$ for all β. Hence, $\rho_\gamma(q_{\alpha\beta}x^*xq_{\alpha\beta}) = 0$ for all γ and β, and again by normality $\rho_\gamma(e_\alpha x^*xe_\alpha) = 0$. Since this is true for all γ, by completeness, we conclude that $e_\alpha x e_\alpha = 0$.

Corollary 3.1.1 If ρ is a faithful normal state on N, then ρ is continuously faithful on N.

Theorem 3.2 If ρ is a state defined on M which diagonalizes N, is faithful on N_q, where q is any projection in Z_N, and which satisfies the boundedness condition that for some k,
$\rho(x^*xy^*y) \leq k\rho(x^*x)\,\rho(y^*y)$ for all $x,y \in N$, then (N_q,ρ) forms a Hilbert algebra under $(x,y) = \rho(xy^*)$.

Proof: The fact that (x,y) forms an inner product follows easily from the fact that ρ is a state. The property that $(y^*,x^*) = (x,y)$ follows from diagonalization and $(xy,w) = (y,x^*w)$ for the same reason. We now must show that left multiplication is continuous relative to this inner product. This follows from the boundedness condition, since $|xy|^2 = (xy,xy) = \rho(xyy^*x^*) = \rho(x^*xyy^*) \leq k\rho(x^*x)\,\rho(yy^*)$. Finally N_q^2 is norm dense in N_q since q is the identity in N_q.

For the rest of this paper, we will let $U(N)$ denote the collection of unitary elements of N and for each $x \in N$, we will denote the norm closure of the convex hull of all uxu^*, $u \in U(N)$, by $C_N(x)$.

Theorem 3.3 If $\{\rho_\alpha\}$ is a complete set of states of M which diagonalize N and if $C_N(x) \cap Z_N \neq \emptyset$, then the intersection consists of exactly one point.

Proof: Consider $u \in U(N)$ and $x \in N$. Since each ρ_α diagonalizes N,

$$\rho_\alpha(uxu^*) = \rho_\alpha(x)$$

and therefore by continuity ρ_α is constant on $C_N(x)$. Now suppose that $y \in C_N(x) \cap Z_N$ and $a \in Z_N$, then y is the limit in norm of elements of type

$$\Sigma\alpha_i u_i x u_i^* , \quad \alpha_i \geq 0, \quad \Sigma\alpha_i = 1, \quad u_i \in U(N).$$

By continuity ay is the limit in norm of

$$\Sigma\alpha_i a u_i x u_i^* = \Sigma\alpha_i u_i a x u_i^*$$

and so $ay \in C_N(ax) \cap Z_N$. Since ρ_α is constant on $C_N(ax)$, $\rho_\alpha(ay) = \rho_\alpha(ax)$ for all α and for all $a \in Z_N$. Let s be any other element of $C_N(x) \cap Z_N$, then

$$\rho_\alpha(ay) = \rho_\alpha(ax) = \rho_\alpha(as)$$

and hence $\rho_\alpha(a(y - s)) = 0$. Letting $a = (y - s)^*$ we conclude that $y = s$.

Note: We will denote the unique point in $C_N(x) \cap Z_N$, when it exists, by $x^{\natural}$.

Theorem 3.4 If $\{\rho_\alpha\}$ is a complete set of states on M which diagonalize N, if $C_N(x) \cap Z_N \neq \emptyset$ for each $x \in N$, and if each ρ_α is normal on Z_N, then each ρ_α is normal on N.

Proof: Since

$$x^{\natural} = \lim \Sigma\, \alpha_i u_i x u_i^*$$

where the limit is in the norm sense, then for $a \in Z_N$

$$ax^{\natural} = \lim \Sigma\, \alpha_i u_i a x u_i^*.$$

Hence

$$\rho_\beta(ax^{\natural}) = \lim \Sigma\, \alpha_i \rho_\beta(u_i a x u_i^*) = \lim \Sigma\, \alpha_i \rho_\beta(ax)$$

$$= \rho_\beta(ax),\ a \in Z_N,\ x \in N.$$

Now consider $x \in N$ and as increasing net $\{q_\beta\}$ of projections in Z_N, with $\sup q_\beta \in Z_N$. We have $\rho_\alpha(x^*x \sup q_\beta) = \rho_\alpha((x^*x)^{\natural} \sup q_\beta)$ $= \sup \rho_\alpha((x^*x)^{\natural} q_\beta) = \sup \rho_\alpha(x^*x q_\beta)$.

<u>Note</u>: We point out that the condition $C_N(x) \cap Z_N \neq \emptyset$ would be satisfied if any compact convex subset of $C_N(x)$ is left invariant by the collection of maps $y \to uyu^*$, $u \in U(N)$, from the Reisz-Kakutani fixed point theorem [7].

IV. <u>The Existence of Expectation like Maps</u>.

<u>Theorem 4.1</u> Suppose M is generated as a vector space by its unitaries, and that $\{\rho_\alpha\}$ is a complete set of states on M. Suppose that each ρ_α diagonalizes M, is normal on Z_N, and satisfies the boundedness condition, that there exists a k_α such that $\rho_\alpha(x^*xy^*y) \leq k_\alpha \rho_\alpha(x^*x)\, \rho_\alpha(y^*y)$ for all $x,y \in N$. Moreover suppose that for each $x \in N$, $C_N(x) \cap Z_N \neq \emptyset$. Then, if e_α is any projection in Z_N such that ρ_α is faithful on N_{e_α}, there exist expectation like maps $\phi_\alpha : M \to N_{e_\alpha}$ such that

1) $\rho_\alpha(u^*au) = \rho_\alpha(\phi(u)\cdot a)$, $u \in U(M)$, $a \in M$, and

2) ϕ_α's are linear and satisfy $\phi_\alpha(u^*vu) = u^*\phi_\alpha(v)u$, $u \in U(N)$, $v \in U(M)$.

<u>Proof</u>: From the previous section, for each α, ρ_α is normal on N, faithful on N_{e_α}, and $(N_{e_\alpha}, \rho_\alpha)$ forms a Hilbert algebra under $(x,y) = \rho_\alpha(xy^*)$. Consider $u \in U(M)$ and for $a,b \in N$, define

$$[a,b] = \rho_\alpha(u^*ab^*u).$$

It follows that $[a,b]$ is a bilinear hermitian form and $|[a,b]| =$

$|\rho_\alpha(u^*ab^*u)| = |\rho_\alpha(uu^*ab^*)| \leq \rho_\alpha(uu^*uu^*)\ \rho_\alpha(ab^*ba^*)$ by Schwartz's inequality. Furthermore

$$\rho_\alpha(ab^*ba) = \rho_\alpha(a^*ab^*b) \leq k_\alpha \rho_\alpha(a^*a)\ \rho_\alpha(b^*b)$$

which says that [a,b] is bounded with respect to the inner product. Hence, we may apply a Reisz representation theorem to obtain a bounded operator $\phi_\alpha(u)$ defined on the completion of N_{e_α} such that

$$[a,b] = (\phi_\alpha(u)(a),b).$$

Now for $d \in N_{e_\alpha}$ consider R_d defined on N_{e_α} by $R_d(x) = xd$. We have

$$(R_d(x),y) = (x,R_d{}^*(y)),\text{ and}$$

$$(R_d(x),y) = (xd,y) = (d,x^*y) = (y^*x,d^*) = (x,yd^*).$$

Therefore $R_d{}^*(y) = yd^*$. Also

$$(R_d(\phi_\alpha(u)(a),b) = (\phi_\alpha(u)(a),\ R_d{}^*(b)) = (\phi_\alpha(u)(a),bd^*)$$

$$= \rho_\alpha(u^*adb^*u) = (\phi_\alpha(u)(ad),b).$$

Hence, $R_d\phi_\alpha(u) = \phi_\alpha(u)R_d$ and by the commutation theorem $\phi_\alpha(u)$ must be a left multiplication i.e.

$$\phi_\alpha(u)(a) = \phi_\alpha(u)\cdot a$$

where we denote the element of N_{e_α} by $\phi_\alpha(u)$. Now

$$\rho_\alpha(u^*ab^*u) = (\phi_\alpha(u)\cdot a,b) = \rho_\alpha(\phi_\alpha(u)ab^*)$$

and setting b equal to the identity, we obtain

$$\rho_\alpha(u^*au) = \rho_\alpha(\phi_\alpha(u)a).$$

Since M is generated by its unitaries, we extend ϕ_α to M by linearity. Now consider $u \in U(N)$, $v \in U(M)$, and $a \in M$. We have

$$\rho_\alpha(\phi_\alpha(uv)a) = \rho_\alpha(u\phi_\alpha(v)u^*a)$$

since ρ_α diagonalizes M, hence

$$\phi_\alpha(uv) = u\phi_\alpha(u)u^*.$$

Similarily, one can obtain the identity

$$\phi_\alpha(vu) = \phi_\alpha(v)$$

and therefore we have 2).

Note: One could view the above ϕ_α's as a collection of linear Radon-Nikodym type derivatives for states.

Corollary 4.1 If Z_N has the property for each α, (whenever $\{q_{\alpha\beta}\}$ is an increasing net of projections in Z_N with ρ_α faithful on each $N_{q_{\alpha\beta}}$, then $\sup_\beta q_{\alpha\beta}$ is again a projection in Z_N), then the carrier of each ρ_α exists and the above theorem holds where e_α is the carrier of ρ_α. In the case that N and M are von Neumann algebras, Z_N has the above property.

Theorem 4.2 Suppose N and M are B*-algebras of operators with $N \subset M$ and let $\{\rho_\alpha\}$ be a complete set of states on M. Suppose that each ρ_α diagonalizes N, is normal on Z_N, and satisfies the boundedness condition of Theorem 4.1. Also suppose that for each $x \in N$, $C_N(x) \cap Z_N \neq \emptyset$. Then there exist expectation like maps $\Psi_\alpha : M \to N_{e_\alpha}$ where e_α is any projection in Z_N with ρ_α faithful on N_{e_α}, such that

1) $\rho_\alpha(axx^*) = \rho_\alpha(\Psi_\alpha(x) \cdot a)$, $x \in M$, $a \in N$

2) $\Psi_\alpha(xy) = x\Psi_\alpha(y)x^*$, $x \in N$, $y \in M$,

3) if Z_N satisfies the condition of Corollary 4.1, then the above is true where e_α is the carrier of ρ_α for each α, and

4) in the case where the e_α's are the carriers of the ρ_α's, if M has an identity e and $\sup e_\alpha = e$, then $\{\Psi_\alpha\}$ is a complete set on N.

Proof: The results 1), 2), and 3) follow as in the proof of Theorem 4.1, where here for each $x \in M$ we define

$$[a,b] = \rho_\alpha(ab^*xx^*), \quad a,b \in N.$$

For 4), consider $x \in N$ and suppose $\Psi_\alpha(x^*x) = 0$ for all α. Then

$$\rho_\alpha(ax^*xx^*x) = 0, \quad a \in N_{e_\alpha}$$

and in particular if $a = e_\alpha^* e_\alpha$

$$\rho_\alpha((e_\alpha x^*x)(e_\alpha x^*x)^*) = 0.$$

By faithfulness of ρ_α on N_{e_α}, since $x^*xe_\alpha \in N_{e_\alpha}$,

$$x^*xe_\alpha = 0 \text{ for all } \alpha.$$

Now by normality and completeness

$$0 = \sup_\alpha \rho_\beta(x^*xe_\alpha) = \rho_\beta(x^*x) \quad \text{for all } \beta,$$

and hence $x = 0$.

<u>Remark</u>: If N is a von Neumann algebra then the carrier e_α of each ρ_α exists. Furthermore with the above hypothesis $\sup e_\alpha = e$.

<u>Proof</u>: Suppose q is a projection in Z_N and that q is orthogonal to all the e_α. Then each ρ_α is not faithful on $N_{q'}$, where q' is any non-zero subprojection of q in Z_N. By Zorn's lemma there would exist a set of orthogonal projections $\{q_\beta\}$ such that $\rho_\alpha(q_\beta) = 0$ and $\sup q_\beta = q$. Now $\rho_\alpha(q) = \rho_\alpha(\sup q_\beta) = \sup \rho_\alpha(q_\beta) = 0$ for all α and by completeness $q = 0$. Hence $\sup e_\alpha = e$.

BIBLIOGRAPHY

1. A. de Korvin, Complete sets of expectations on von Neumann algebras, submitted Journal of Functional Analysis, 1968.

2. J. Dixmier, Applications ♮ dans les anneaux d'operateurs, Comp. Math. 10 (1952), 1-55.

3. ________, Formes lineaires sur un anneau d'operateurs, Bull. Soc. Math. France 81 (1953), 9-39.

4. ________, Les algebres d'operateurs dans l'espace Hilbertien, Paris (1957).

5. N. Dunford and J. Schwartz, Linear Operators, Vol. I and II, Intersicence, New York (1958).

6. M. Loeve, Probability Theory, D. Van Nostrand Co., Inc., New Jersey (1963).

7. S. Kakutani, Two fixed point theorems concerning bicompact convex sets, Proc. Imp. Acad., Tokyo 14, (1938) 242-245.

8. C. E. Rickart, Banach Algebras, D. Van Nostrand Co., Inc., New Jersey (1960).

9. H. Umegaki, Conditional expectations in an operator algebra, Tohoku Math. Journal, 6. (1954) 177-181.