QUASI-INVERTIBILITY IN A STAIRCASE DIAGRAM

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QUASI-INVERTIBILITY IN A STAIRCASE DIAGRAM

by Walter Noll*

We deal with objects and morphisms in an abelian category, e.g., with modules and module-homomorphisms. Any morphism a: A-»B has a standard factorization

where $<X_1$ is injective (i.e., a monomorphism) and $(X_s$ surjective (i.e., an epimorphism).

<u>Definition</u>: A morphism a: A-»B is said to be <u>quasi-invertible</u> if it satisfies any one of the following equivalent conditions[1)]:

(i) There is a morphism a^1 : B->A such that

co'a • a

(ii) There is a morphism $\bar{a}$: B->A such that

$$o\bar{a}a - a \quad \text{and} \quad a\bar{r}i\bar{a} \ll \bar{a} \qquad (1)$$

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1) Cf. [l], p. 264, Prop. 5.1, where, in a different context, the term "allowable" is used for what we call quasi-invertible.

(iii) a_{im} has a left inverse, and a_s has a right inverse.

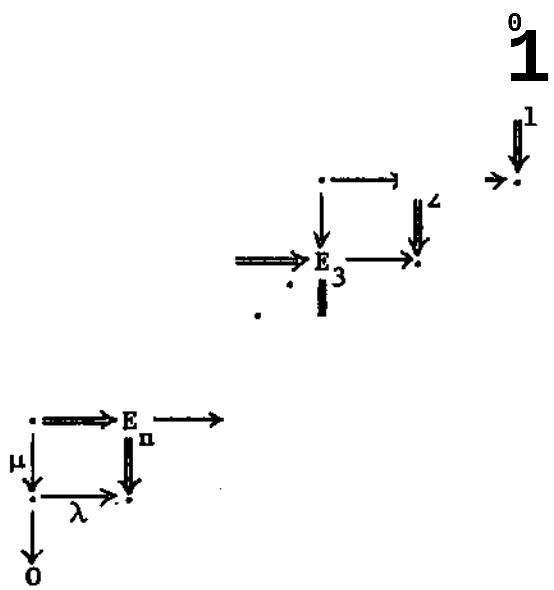
(iv) ker a has a left inverse, and coker a has a right inverse.

If $\bar{a}$ satisfies (1) we call it a quasi-inverse of a.

For monomorphisms, quasi-inverses coincide with left inverses, i.e., monomorphisms are quasi-invertible if and only if they are left-invertible (or "coretractions[11]). For epimorphisms, quasi-inverses coincide with right inverses, i.e., epimorphisms are quasi-invertible if and only if they are right-invertible (or "retractions[11]).

The purpose of this note is to state and prove the following result, which was needed in an investigation of annihilators of differential operators[2], but may have other applications.

Theorem: Consider the "staircase" diagram



(2)

seen that $\bar{6}$ is a right inverse of 6 and that $\bar{7}$ is a left inverse of 7, but this fact will not be needed.

Let $\bar{0}$ be a quasi-inverse of 67, so that

$$67\bar{0}67 - 67, \qquad (6)$$

and put

$$\bar{f} \; m \; \bar{a}(l_E-\bar{7}\bar{0}6)p.$$

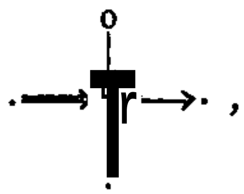
By (4) and (5) *ye* then obtain

$$\beta c_{\wedge}\beta\alpha = \beta\alpha\bar{\alpha}(1_E-\bar{7}0\bar{6})p>a = p(l_g-06)a_F-7*6)U_E-YY)a$$

$$= \beta(1_{\text{£i}}+6677-6670677)a.$$

It follows from (6) that the last two terms cancel and hence that (pa)^Oa) - pa. Therefore, pa is quasi-inyertible. Q.E.D.

<u>Proof</u> <u>of</u> <u>Theorem</u>: The upper end of the staircase diagram (2) can be used for the construction of a cross diagram



where the single arrow horizontal morphism is the cokernel of the double arrow horizontal morphism. It is clear that the hypotheses of the lemma are satisfied. The conclusion of the lemma and commutativity imply that the hypotheses of the lemma are satisfied for the cross diagram centered at E_2. Proceeding by induction, we see that the conclusion of the lemma holds for the cross diagram centered at E_n, i.e. that A|i is quasi-invertible. Since p. is surjective, we can use Prop. A to conclude that \ must be quasi-invertible. Q.E.D.

References:

[1] MacLane, S.: Homology. Springer, Berlin-Göttingen-Heidelberg, 1963.

[2] Dombrowski, H. D., & W. Noll: Left Annihilators of Differential Operators, to appear.

in which dots denote unnamed objects. Assume that the diagram is commutative« that all rows and columns are exact, and that the morphisms indicated by double arrows are quasi-invertible. Then *X* jLs also quasi-invertible.

The following facts will be needed:

Proposition A: If ap is. quasi-invertible and p surjective, then a is quasi-invertible.

Proposition B: If. ap is. quasi-invertible and a iniective. then p is. quasi-invertible.

If a and p are quasi-invertible, we cannot conclude that pa is also quasi-invertible. However, the following lemma allows us to draw this conclusion under an additional condition.

Lemma: Consider the "cross" diagram

$$\begin{tikzcd} & C \arrow[d, "\gamma"] & \\ A \arrow[r] & E \arrow[r] & D \\ & B \arrow[u, "\beta"] & \end{tikzcd} \qquad (3)$$

Assume that row and column are exact, and that *Ot,*p and 67 are quasi-invertible. Then pa is. also quasi-invertible.

Proof: Consider the standard decompositions 7 • 7.7 and 6 • 6,6 .
— — is is
Since 67 - (6' 6 7')7 is quasi-invertible, it follows by Prop. A. that
A, S 1 S
6:6 7. * 6.(6 7.) is also quasi-invertible. By Prop. B we can conclude
I S J. 1 SI.
that 6 7. is quasi-invertible. Noting that in 7 - Im 7 and
S 1 . ' i
Ker 6 = Ker $6_{S'}$ we see that there is no loss of generality if we assume that 6 is surjective and 7 injective. In view of the exactness of the row and the column of the diagram, we may actually assume that

7 • ker p, 6 - coker a.

Now let $\bar{a}$ be a quasi-inverse of a, so that

$$(1_E - a\bar{a})a = a - a\bar{a}a \ll 0.$$

It follows that $1_E - C\bar{O}$ annihilates a and hence must factor through coker a » 6. Thus, the exactness of the row of the diagram (3) is expressed by

$$6a - 0, \qquad 1_E - o5 - \bar{6}6, \tag{4}$$

where $\bar{6}$: D-»E. Similarly, one can prove that the exactness of the column of the diagram (3) is expressed by

$$P7 = 0, \qquad 1_E - \bar{p}p - 7\bar{Y}, \tag{5}$$

where $\bar{p}$ is a quasi-inverse of p and $\bar{7}$: E -»C. Incidentally, it is easily