MORPHISM CATEGORIES AND UNIVERSAL OBJECTS

Oswald Wyler

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We obtain a representation theorem for morphism categories with enough universal or couniversal objects which generalizes the well-known triple description of an adjoint functor situation by adjunction, front adjunction, back adjunction.

General formal properties of solutions of universal problems can easily be formulated in our theoiy, but we shall not do this here. We wish to point out that action of two categories on a class is closely related to a profunctor in the sense of Bénabou £l]? but W© shall not pursue this theme.

2. Action of categories on classes. We adopt in this note the usual convention of writing all compositions from right to left, but nearly everything else from left to right. We write f£ JK and A^|]C| if f is a morphism and A an object of a category K. id A and Id K, or just A and K, denote an identity morphism and an identity functor, and IC^{op} denotes the dual (or opposite) category of K.

Definition 2.1. We say that a category jC acts on a class 2~ from the left if a composition f•or with values in JT" is defined for some pairs (f,cr) in KXJL and satisfies the following conditions.

2.1.1. If f *cr is defined for f : A — > B in K and <sr<ejLt then (id A) *<T is defined, and (id A) * or s or.

2.1.2. If g f and f * cr are defined, for f, g in | and <=r€H then (g·f)»cT and g • (f• cr) are defined, and equal.

Dually, we say that K Acts on JT" from the right if a composition er-• f with values in 21, is defined for some pairs (<5~f) in £xK and satisfies the dual laws of 2.1.1 and 2.1.2.

Definition 2.2. We say that a pair (K_o, K_1) of categories acts on a class £ if K_o acts on 21 from the ri^it, K_1 acts on 21 from the left, and the two actions are compatible, i.e. the following condition is satisfied.

2.2.1. If CT* f_o and $f_1 \cdot$ cr are defined for f_i<± K_i and<3-<S2T, then $f_1 \cdot$ « (cr-* f_o) and (f_1 * or) • f_o are defined, and equal.

Remarks 2.3. Prom now on, we usually omit the dots in compositions $f_1 \cdot$ cr- and <T* f_o . We note the following consequence of 2.1.2 without proof,

2.3.1. iet. K, act on X from the left, and let f : A -> B in K snd cn^X, II (id k)cr is defined, then tar is defined ;[£ f<?~ is defined, then (id B)(fCT") is defined.

If (K_o, K_1) acts on 21 » then we say that cr : A_o --^> A_1 in 2T , or that $\sigma \in \Sigma(A_o, A_1)$, for cr^Xand A ^ I K.1 , if cr (id A_Q) and (id k^cr are defined. If f : A_i -->B_i in K_i and csr-^r∆T, then we have:

2.3.2. erf_o and $f_1$cr are both defined if and only if cr : B_o –$> A_1 in 5*t and then f_1^f_o •• A_o – > B_1 in ^ .

We note that 2.1.2 and 2.2.1 are weaker than the associative law for a category. But these laws are all that we need, and there are useful examples for which the stronger laws are not valid. All examples will be found in section 4.

3^ Morphism 9£tego£ies» If two categories K ,]C , two functors U : JL_1 –*]$£_1$ f fi^cL an action of (K_o, K|) on a class X are given, then we construct a category M, « jjJ_o »2I»^{U}iI_1[°f commutative squares as follows.

Objects of $\underline{M}$ are triples (A_o,σ,A_1) with $A_i \in |\underline{K}_i|$ and $\sigma : U_o A_o \longrightarrow U_1 A_1$ in Σ. Morphisms of $\underline{M}$ are "commutative squares"

$$\begin{array}{ccc} & \xrightarrow{f_o} & \\ \sigma \big\downarrow & & \big\downarrow \tau \\ & \xrightarrow[f_1]{} & \end{array}$$

with σ, τ in Σ, $f_i \in \underline{K}_i$, and with $(U_1 f_1)\sigma$ and $\tau(U_o f_o)$ defined and equal. We write such a square as (σ,f_o,f_1,τ), or inaccurately as $(f_o,f_1) : \sigma \longrightarrow \tau$. Identity morphisms and composition in $\underline{M}$ are given by the formulas

$$\mathrm{id}\,(A_o,\sigma,A_1) = (\sigma,\ \mathrm{id}\,A_o,\ \mathrm{id}\,A_1,\sigma) \ ,$$

$$(\tau,g_o,g_1,\rho)(\sigma,f_o,f_1,\tau) = (\sigma,\ g_o f_o,\ g_1 f_1,\rho) \ ,$$

provided of course that $g_i f_i$ is defined in $\underline{K}_i$. One verifies easily that $\underline{M}$ is closed under composition, and a category.

<u>Definition 3.1.</u> We call the category $\underline{M} = [U_o,\Sigma,U_1]$ just constructed a <u>morphism category</u>.

<u>Remark 3.2.</u> We put $\underline{M} = [\underline{K}_o,\Sigma,\underline{K}_1]$ if in particular $U_i = \mathrm{Id}\,K_i$ and acts on $(\underline{K}_o,\underline{K}_1)$. The general case can be retrieved from this special case as follows. Given $U_i : \underline{K}_i \longrightarrow \underline{K}'_i$ and an action of $(\underline{K}'_o,\underline{K}'_1)$ on Σ, we put

$$\sigma \cdot f_o = \sigma(U_o f_o) \ , \qquad f_1 \cdot \sigma = (U_1 f_1)\sigma \ ,$$

for $\sigma \in \Sigma$ and $f_i \in \underline{K}_i$, whenever the righthand sides are defined. This defines an action of $(\underline{K}_o,\underline{K}_1)$ on Σ, and the resulting category $[\underline{K}_o,\Sigma,\underline{K}_1]$ is the same as $[U_o,\Sigma,U_1]$.

4. Examples. Proofs are omitted in this section.

4.1. If $\underline{K}$ is a category, then $(\underline{K},\underline{K})$ acts on $\underline{K}$ by composition in $\underline{K}$. The resulting category $[\underline{K},\underline{K},\underline{K}]$ is usually denoted by $\underline{K}^2$ or Mor $\underline{K}$. More generally, a functor $F : \underline{C} \longrightarrow \underline{K}$ produces categories $[F,\underline{K},\underline{K}]$ and $[\underline{K},\underline{K},F]$.

4.2. Every pair $(\underline{K}_o,\underline{K}_1)$ of categories acts trivially on a singleton $\{0\}$, by $0\, f_o = f_1\, 0 = 0$ for all $f_i \in \underline{K}_i$. The resulting category $[\underline{K}_o,\{0\},\underline{K}_1]$ is isomorphic to the product category $\underline{K}_o \times \underline{K}_1$.

4.3. The category $\underline{1}$ consisting of one identity morphism 1 acts trivially on every class Σ, by $1\sigma = \sigma$ or $\sigma 1 = \sigma$ for every $\sigma \in \Sigma$. Action of $\underline{1}$ on Σ from the left is compatible with every action of a category $\underline{K}$ on Σ from the right (and vice versa), and the morphism category $[\underline{1},\Sigma,\underline{K}]$ is isomorphic to the category constructed in [2; ch. 2].

4.4. If $\underline{K}$ acts on Σ from the left, then $\underline{K}^{op}$ acts on Σ from the right (and vice-versa), by putting $\sigma * f = f\sigma$ whenever $f\sigma$ is defined. Similarly, an action of $(\underline{K}_o,\underline{K}_1)$ on Σ induces an action of $(\underline{K}_1{}^{op},\underline{K}_o{}^{op})$ on Σ. Functors $U_i : \underline{K}_i \longrightarrow \underline{K}_i'$ induce functors $U_i{}^{op}$ of the dual categories, and the resulting category $[U_1{}^{op},\Sigma,U_o{}^{op}]$ is isomorphic to the category $[U_o,\Sigma,U_1]^{op}$.

4.5. If $\underline{K}$ is a category, then putting $f \bullet A = B$ and $B \bullet f = A$ for $f : A \longrightarrow B$ in $\underline{K}$ defines actions of $\underline{K}$ on $|\underline{K}|$ from the left and from the right. These two actions are not compatible, i.e. 2.2.1 is not satisfied, if $\underline{K}$ has a morphism $f : A \longrightarrow B$ with $A \neq B$.

4.6. Two non-trivial examples. If R is a commutative ring with identity, $\underline{K}$ the category of R-modules, and Σ the class of bilinear mappings $\sigma : A \times B \longrightarrow C$ of R-modules, then $(\underline{K} \times \underline{K}, \underline{K})$ acts on Σ. Universal objects of the category $[\underline{K} \times \underline{K}, \Sigma, \underline{K}]$ are tensor products of pairs of R-modules.

If $\underline{C}$ and $\underline{K}$ are categories, then $E_A \varphi = \mathrm{id}\, A$ for $A \in |\underline{K}|$ and $\varphi \in \underline{C}$, and $E_f \alpha = f$ for $f : A \longrightarrow B$ in $\underline{K}$ and $\alpha \in |\underline{C}|$, defines constant functors $E_A : \underline{C} \longrightarrow \underline{K}$ and natural transformations $E_f : E_A \longrightarrow E_B$, and hence a functor $E : \underline{K} \longrightarrow \underline{K}^{\underline{C}}$. If $\underline{C}$ is small, then universal objects of the morphism category $[\underline{K}^{\underline{C}}, \underline{K}^{\underline{C}}, E]$ are colimits of functors $F : \underline{C} \longrightarrow \underline{K}$, and couniversal objects of the morphism category $[E, \underline{K}^{\underline{C}}, \underline{K}^{\underline{C}}]$ are limits of functors $F : \underline{C} \longrightarrow \underline{K}$.

4.7. Let $F : \underline{K}_0^{op} \times \underline{K}_1 \longrightarrow \underline{Ens}$ be a functor to the category of sets, and let Σ be a class containing disjoint copies of all sets $F(A_0, A_1)$, $A_i \in |\underline{K}_i|$. For $f_i : A_i \longrightarrow B_i$ and $\sigma \in \Sigma$ in $F(B_0, A_1)$, we put

$$\sigma f_0 = F(f_0, A_1)(\sigma) , \quad f_1 \sigma = F(B_0, f_1)(\sigma) .$$

This defines an action of $(\underline{K}_0, \underline{K}_1)$ on Σ.

Conversely, let us call an action of $(\underline{K}_0, \underline{K}_1)$ on a class Σ <u>legitimate</u> if the classes $\Sigma(A_0, A_1)$, $A_i \in |\underline{K}_i|$, are mutually disjoint sets. A legitimate action induces a functor $F : \underline{K}_0^{op} \times \underline{K}_1 \longrightarrow \underline{Ens}$ by $F(f_0, f_1)(\sigma) = f_1 \sigma f_0$, for $f_i : A_i \longrightarrow B_i$ in $\underline{K}_i$ and σ in $\Sigma(B_0, A_1) = F(B_0, A_1)$. The action of $(\underline{K}_0, \underline{K}_1)$ on Σ induced from F may be a restriction of the given action, but it leads to the same morphism category $[\underline{K}_0, \Sigma, \underline{K}_1]$.

5. Natural transformations. We consider a natural transformation λ :

F —> G in a functor category K^Q as a mapping from £ to JG , by putting \f « (G f)(AAj = (\B)(P f) for f : A—>B in £ . Then (/JL\)(g f) » (y^a g)(Af) if /U-^ is defined in R^C and g f in £ .

If K, acts on a class X" ^rom the left, then K^C acts from the left on the class 21"" of all mappings from £ to 21 as follows. For <p£m2Z^m~ and A : P — > G in K^C , we define A^> only if (P f)(GDA) is defined, and equal to <pt , for every morphism f : A —> B of £ . In this case, we put

$$(\lambda\varphi)(f) = (\lambda B)(\varphi f) = (\lambda f)(\varphi A)$$

for f : A — > B in £ . 2.1.1 and 2.1.2 are easily verified for this action.

An action from the right is lifted in similar fashion, and if (K_0, K_1) acts on X, then (K_0^C, K_1^C) acts on ^"^C. Thus a morphism category JU t 2T,U ^J leads to a morphism category U_0"~, X""t U_1""j, with natural transformations as objects, by the following definition.

??.Ci2:ii't?S^5_1. Let (K_0, K_1) act on X $ let P_i : £ —>• K_i be functors, and let d*<ir JL~~ • We call '<p a natural transformation from P_0 to P_1 , and write <f> : P_0 —> P_1 , if 'a>P and P'q> are defined in X"~ , i.e. if

$$(\varphi B)(F_0\ f) \ll\ {}^{\wedge}\!>f \gg (P_1\ f)(\varphi A)$$

in 21 for every morphism f : A •—> B in £ •

For the case that (K,K) acts on K by composition in K , this is equivalent to the usual definition of a natural transformation.

Let now functors U_i : K_i —> K! and a morphism category M = [U t Y ,u/l be given. One sees easily that we define functors D_i : J4 —^ K_i and a natural transformation 9 : $U_o D_o$ —> tL D_n by putting

\ q » f_t » 3q * (U_x f^cr * TT(U_Q f_Q) ,

for every moiphism q a (^r,f $_6$f $_1$~C) of M. •

If <£) : C, —>M, is a functor and B *Cp* the composition mapping, then $\partial\phi$: $U_o D_o \phi \to U_1$ $V_1 (b$ in I^C • Conversely, we have

<u>Theorem 5*2«</u> <u>Let</u> Jl » &o'^^{f}^^J <u>as above«</u> <u>For functors</u> F_1 : *C*—>K_1 <u>and a natural transformation</u> *cp* : $U_o F_o$ —> $U_1 F_1$ '<u>in</u> *£j*-^{C}, <u>there is exactly</u> <u>one functor</u> <*p* : £ —^ M, <u>such that</u> F_o = D_o (J>, F_1 «D_1 (b, <u>and</u> ^ = $\partial\phi$.

This characterizes M by a universal property.

<u>Proof</u>. The requirements for Cp can be satisfied only if we put

$$\phi f = (\varphi A,\ F_Q f,\ F_1 f,\ \text{^>}B)$$

for f : A —> B in C. . Conversely, one sees easily that this defines a functor CD 2 C,—>M[which meets the requirements.

<u>RemarkJ>>2»</u> Let #£= £U_Q^p-, X^p» D_1^p-J . Theorem 5.2 defines a bisection between the objects (F_o, C^tF^ of *T^Z and* the objects of £^{C} • We use this to identify a functor Cjf) : C. —>M, with the corresponding object (D_o 0, c>0, $D_1 \phi$) of ^9?T. The bisection of 5.2 can be extended to a canonical isomorphism of the categories *TffZand* M^C *f* but we omit this step.

<u>6. Universal objects and functors</u>. We consider again two functors U_1 : K_1 --> K_1 and a morphism category *M* = MJ$_o$,2Z»^{U}i^J •

Definition 6.1. An object (A_o, CT, A_1) of 14 is called a universal object of M at A_o if for every object $(B_o, T., B_1)$ of M and every morphism f_o : A —>. B of K there is exactly one morphism f,, : A, —> B, in KL such that $(cr_f f_0, f_1, TT) £ M$, i.e. $(lL_1 f_1) <r « TT (U_0 f_0)$ in X . We say that M has enough universal objects if M, has a universal object at every $A_o .£ |K_o|$.

Dually, we call $(A_o, o"_t A_l)$ a couniversal object of M if $(A^, cr, A_o)$ is a universal object of $^{op}U_1, 2^{op}U_0, j^$, and we say that J4 has enough couniversal objects if jU. ,J",U. 1 has enough universal objects*

Proposition 6»2. If $(A_o, cr_f A_1)$ is a universal object of M, , then an object $(A_o, 5"_f Ā_1)$ of M is universal at A_o if and only if 5r=t $(u_1 u)$cr- in ^T for an isomorphism u : A_1 —£> $Ā_1$ of JC_1 .

We omit the standard proof of this result.

Theorem 6.3. M, has enough universal objects if and only if there -is a functor r*: K_o —^ M such that D /""*» Id K_o and that f~*k is a universal object of M at A'_o for every $A_Q £ (jC_o$.

We call a functor P with these properties a universal functor for K •

Proof. The "if" part is trivial. Conversely, assume that a universal object (A_o, yA_o, PA_o) of M is given at every $A_o 6 |K_o|$. For f_o : A_o —^ B_o in K_o , we put yf_o » $(v/B_o)(U_o f_o)$ in JT, and the equation

$$V B_o)(U_o f_o) - /f_o - ^ U_1 P f_o)(y A_o)$$

in *21* determines $Pf_o : PA_o$ ——5> PB_o in K. uniquely. One verifies easily that this defines a functor P : K_o <——> IC_1 and a natural transformation *y* :

U —x,u. p . and hence by 5.2 a functor $P = (K, V, P) : K \to M$ with the required properties.

Our next result, which will be needed in section 8, shows that the functor P of the proof of 6.3 is not of some special kind.

Proposition 6.4. Let $P : K_0 \to K_1$ be a functor. The morphism category $M_P = [P, K_1, K_1]$ admits a universal functor $F_P = (\mathrm{id}\ K_0, \mathrm{id}\ P, P) : K_0 \to M_P$. Dually, the morphism category $M^P = [K_1, K_1, P]$ admits a couniversal functor $P^P = (P, \mathrm{id}\ P, \mathrm{Id}\ K_0) : K_0 \to M^P$.

Proof. $F_P A_0 = (A_0, \mathrm{id}\ P A_0, P A_0)$ in M_P for $A_0 \in |K_0|$, and if $f_0 : A_0 \to B_0$ in K_0 and $f_1 : P B_0 \to B_1$ in K_1, then $(f_0, f_1) : \mathrm{id}\ P A_0 \to$ in M_P iff $f_1 = T(P f_0)$ in K_1. Thus F_P is a universal functor.

7. Properties of universal functors. We use the same notations as before.

Theorem 7.1. A functor $P = (\mathrm{id}\ K, y, P) : K \to M$ is universal if and only if (T, yT, PT) is a universal object of $(U_0^C, 21\sim\%\ U_1^C)$ for every category C and every functor $T : C \to K_0$.

Proof. For the "if" part, we need only the case $C = 1$, identifying functors $T : 1 \to K$ with objects of K and fu-, 5Z~, U_n7 with M.

For the "only if" part, let V be a universal functor, let $T : C \to K$ be given, let 7K~ fu~, 21~, U.-l, and let $A_0 : T \to F_0$ in K_0^C and $A_1 : U_0 F_0 \to V_1 F_1$ in Z~, with $F_1 : C \to$ ^ . Then $(A_0, A_1) : yT \to \varphi$ in }Jt for $A_1 : PT \to P^1$ in K_1^C iff

$$(\varphi B)(U_o \lambda_o f) = (\varphi \cdot U_o \lambda_o) f = (U_1 \lambda_1 \cdot \gamma T) f = (U_1 \lambda_1 f)(\gamma T A)$$

in Σ for every morphism $f : A \longrightarrow B$ of $\underline{C}$. Since $\Gamma T A$ is a universal object of $\underline{M}$, this determines $\lambda_1 f : P T A \longrightarrow F_1 B$ in $\underline{K}_1$ uniquely, and a moderate amount of diagram chasing shows that the morphisms $\lambda_1 f$ of $\underline{K}_1$ define a natural transformation $\lambda_1 : P T \longrightarrow F_1$, with $(\lambda_o, \lambda_1) : \gamma T \longrightarrow \varphi$ in $\mathfrak{M}$ for this and no other $\lambda_1 : P T \longrightarrow F_1$ in $\underline{K}_1^{\underline{C}}$.

<u>Corollary 7.2.</u> <u>$\underline{M}$ has enough universal objects if and only if every morphism category $[U_o^{\underline{C}}, \Sigma^{\underline{C}}, U_1^{\underline{C}}]$ has enough universal objects</u>.

We consider now a second morphism category $\underline{N} = [V_o, \bar{\Sigma}, V_1]$, obtained from functors $V_i : \underline{L}_i \longrightarrow \underline{L}_i'$ and an action of $(\underline{L}_o', \underline{L}_1')$ on $\bar{\Sigma}$, with functors $\bar{D}_i : \underline{N} \longrightarrow \underline{L}_i$ and a natural transformation $\bar{\partial} : V_o \bar{D}_o \longrightarrow V_1 \bar{D}_1$. We call a functor $\Theta : \underline{M} \longrightarrow \underline{N}$ a <u>morphism functor</u> if $\bar{D}_i \Theta = T_i D_i$ for functors $T_i : \underline{K}_i \longrightarrow \underline{L}_i$. If this is the case, then we put $\Theta = [T_o, \vartheta, T_1]$ for $\vartheta = \bar{\partial} \Theta : \underline{M} \longrightarrow \bar{\Sigma}$. This mapping ϑ behaves like a natural transformation in two variables. If we write $\vartheta\sigma$ instead of $\vartheta(A_o, \sigma, A_1)$ for an object (A_o, σ, A_1) of $\underline{M}$, then

$$\Theta(\sigma, f_o, f_1, \tau) = (\vartheta\sigma, T_o f_o, T_1 f_1, \vartheta\tau)$$

in $\underline{N}$ for a morphism (σ, f_o, f_1, τ) of $\underline{M}$ and $\Theta = [T_o, \vartheta, T_1]$.

<u>Theorem 7.3.</u> <u>Let $\Gamma = (\mathrm{Id}\, \underline{K}_o, \gamma, P) : \underline{K}_o \longrightarrow \underline{M}$ be a universal functor, and let $\Phi = (F_o, \varphi, F_1 P) : \underline{K}_o \longrightarrow \underline{N}$ for functors $F_i : \underline{K}_i \longrightarrow \underline{L}_i$ and a natural transformation $\varphi : V_o F_o \longrightarrow V_1 F_1 P$ in $\bar{\Sigma}^{\underline{K}_o}$. Then $\Phi = \Theta \Gamma$ for a unique morphism functor $\Theta = [F_o, \vartheta, F_1] : \underline{M} \longrightarrow \underline{N}$</u>.

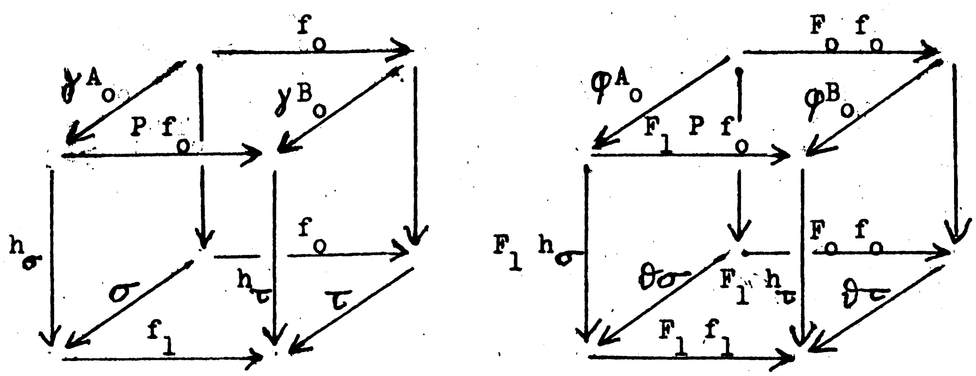
Proof. Θ is determined if we know $\vartheta\sigma$ for every object (A_0, σ, A_1) of $\underline{M}$, with $\vartheta\gamma A_0 = \varphi A_0$. Since ΓA_0 is a universal object of $\underline{M}$, we have $(\mathrm{id}\, A_0,\, h_\sigma) : \gamma A_0 \longrightarrow \sigma$ in $\underline{M}$ for a unique morphism $h_\sigma : P\, A_0 \longrightarrow A_1$ of $\underline{K}_1$, and then we must have

$$\Theta(\gamma A_0, A_0, h_\sigma, \sigma) = (\varphi A_0,\, F_0\, A_0,\, F_1\, h_\sigma,\, \vartheta\sigma)$$

in $\underline{N}$. Thus we must put $\vartheta\sigma = (V_1\, F_1\, h_\sigma)(\varphi A_0)$ in Σ.

We have constructed Θ, but does Θ map $\underline{M}$ into $\underline{N}$? In order to show that it does, let $(f_0, f_1) : \sigma \longrightarrow \tau$ in $\underline{M}$, with $f_i : A_i \longrightarrow B_i$ in $\underline{K}_i$. We consider the two cubes (7.4) in which the unmarked arrows are identity morphisms.

(7.4)



phisms. At left, all faces are commutative squares, except possibly the front face. But γA_0 acts like an epimorphism, and thus the front face at left commutes in $\underline{K}_1$, by simple diagram chasing. Now the bottom face at right commutes since all other faces do. Thus Θ maps $\underline{M}$ into $\underline{N}$, and now one sees immediately that Θ is a functor.

8. Applications. We obtain a representation theorem for morphism categories with enough universal morphisms, and we use this theorem and its dual for some

comments on adjoint functors. We use all previously established notations, including those of 6.4. We call a morphism functor $[T_0, \vartheta, T_1]$ strict if the functors T_i are identity functors. The composition of two morphism functors clearly is a morphism functor, and the composition of two strict morphism functors is strict. Id $\underline{M} = [\underline{K}_0, \partial, \underline{K}_1]$ is a strict morphism functor.

Theorem 8.1. A functor $\Gamma = (\text{Id } \underline{K}_0, \gamma, P) : \underline{K}_0 \longrightarrow \underline{M}$ is a universal functor for $\underline{M}$ if and only if $\Gamma = \Theta \Gamma_P$ for a strict morphism functor $\Theta : \underline{M}_P \longrightarrow \underline{M}$ which is an isomorphism of the two categories.

Proof. If a strict morphism functor $\Theta : \underline{M}_P \longrightarrow \underline{M}$ is an isomorphism of the two categories, then Θ clearly preserves universal objects, and thus $\Theta \Gamma_P$ is a universal functor for $\underline{M}$. Conversely, if $\Gamma = (\text{Id } \underline{K}_0, \gamma, P) : \underline{K}_0 \longrightarrow \underline{M}$ is a universal functor, then $\Gamma = \Theta \Gamma_P$ and $\Gamma_P = H \Gamma$ for unique strict morphism functors $\Theta : \underline{M}_P \longrightarrow \underline{M}$ and $H : \underline{M} \longrightarrow \underline{M}_P$, by 7.2 and 6.4, and then Θ and H are inverse isomorphisms by a standard argument. (see 6.2).

Consider now functors $T : \underline{K}_1 \longrightarrow \underline{K}_0$ and $S : \underline{K}_0 \longrightarrow \underline{K}_1$. These define morphism categories $\underline{M}_S = [S, \underline{K}_1, \underline{K}_1]$ and $\underline{M}^T = [\underline{K}_0, \underline{K}_0, T]$, as well as a universal functor $\Gamma_S : \underline{K}_0 \longrightarrow \underline{M}_S$ and a couniversal functor $\Gamma^T : \underline{K}_1 \longrightarrow \underline{M}^T$.

Proposition 8.2. A functor $T : \underline{K}_1 \longrightarrow \underline{K}_0$ has a left adjoint functor $S : \underline{K}_0 \longrightarrow \underline{K}_1$ if and only if $\underline{M}^T = [\underline{K}_0, \underline{K}_0, T]$ has enough universal morphisms.

Proof. By 6.3, $\underline{M}^T$ has enough universal morphisms iff there is a universal functor $\Phi = (\text{Id } \underline{K}_0, \varphi, S) : \underline{K}_0 \longrightarrow \underline{M}^T$, i.e. there is a functor $S : \underline{K}_0 \longrightarrow \underline{K}_1$ and a natural transformation $\varphi : \text{Id } \underline{K}_0 \longrightarrow T S$ such that $(A_0, \varphi A_0, S A_0)$ is a universal object of $\underline{M}^T$ for every $A_o \in |\underline{K}_o|$. But this means precisely that φ is a front adjunction for T .

Remark 8.3. We consider the adjoint situation of 8.2 further. We have seen that a front adjunction $\varphi : \mathrm{Id}\ \underline{K}_o \longrightarrow T\ S$ for T corresponds to a universal functor $\Phi = (\mathrm{Id}\ \underline{K}_o, \varphi, S)$ for $\underline{M}^T$. By 8.1, this functor satisfies $\Phi = \Theta \Gamma_S$ for a strict morphism functor $\Theta : \underline{M}_S \longrightarrow \underline{M}^T$. One sees easily that this functor Θ corresponds to an adjunction in the usual sense. By the dual of 8.1, $\Theta^{-1} \Gamma^T$ is a couniversal functor $\Psi = (T, \psi, \mathrm{Id}\ \underline{K}_1)$ for $\underline{M}_S$, and this means that $\psi : S\ T \longrightarrow \mathrm{Id}\ \underline{K}_1$ is a back adjunction for S . Thus our theory furnishes a very simple connection between the three aspects of an adjoint situation.

References

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