ON SUBOBJECTS AND IMAGES IN CATEGORIES

Oswald Wyler
and
Hans Ehrbar

Report 68-34

November/ 1968

ON SUBOBJECTS AND IMAGES IN CATEGORIES

by Hans Ehrbar and Oswald Wyler*

1. Introduction

Subobjects and images in categories have been defined in many ways. MacLane [9]] proposed an axiomatic theory of "bicategories" which was simplified by Isbell [5J. Grothendieck [41 defined subobjects (sous-trues) as equivalence classes of monomorphisms and suggested a definition of images which many authors have adopted (see e.g. CIO; I.10J). This works well in algebra, but not in general topology. Isbell £&), Jurchescu and Lascu £7j , Sonner fllj and others have suggested categorical remedies for this situation.

No "absolute" definition of subobjects and images in a category has been proposed which is adequate for all situations. Moreover, in some situations, e.g. in general topology and in the theory of partial algebras, several reasonable definitions of subobjects are possible. Thus a "relative" theory of subobjects and images is needed. In the present paper, we define .J-subobjects and ^-images in a category £ for an arbitrary class J, of morphisms of £ . Our definitions are equivalent to those of Grothendieck [4~\ if J. is the class of

• Research supported in part by N.S.F. Grant GP-8773.

all monomorphisms of $\underline{C}$. We introduce a new concept, strong $\underline{J}$-images, and we study $\underline{J}$-images and strong $\underline{J}$-images, and the resulting image functors, in sections 3 - 5. Isbell's theory [5] is generalized in section 6. In sections 7 - 8, we define and study direct and inverse images as strictly dual concepts, and we discuss briefly the resulting pseudofunctors and fibrations. Examples and comments are given in section 9. Further developments and applications have been obtained by each of us and will be published separately.

One feature of our theory is that we do not require images to be monomorphic. There are several reasons for this. For instance, coimages in operational categories need not be epimorphic, and to require monomorphic images would almost destroy applications to reflective subcategories ([2; cap. 2] and [8]).

We shall refer by m.n to the n^{th} item of section m, and the symbol ▮ will denote the end or absence of a proof. The notations introduced in section 2 will be used throughout the paper.

2. Preliminaries

Throughout this paper, let $\underline{C}$ be a category and $\underline{J}$ a class of morphisms of $\underline{C}$. From 3.5 on, we shall require $\underline{J}$ to be left transportable (2.6).

2.1. We write composition in $\underline{C}$ from left to right, so that $f\,g$ means: first f , then g . We identify $\underline{C}$ with its class of morphisms, and we denote by $|\underline{C}|$ the class of all objects of $\underline{C}$. We often identify an object $A \in |\underline{C}|$ with the identity morphism $\mathrm{id}\, A \in \underline{C}$. We denote domain and codomain of a morphism $f \in \underline{C}$ by $f\,D_o$ and $f\,D_1$, so that $(f\,D_o)\,f = f\,(f\,D_1) = f$ in $\underline{C}$. We write functors and natural transformations as right operators, with compositions from left to right. Id $\underline{C}$ denotes the identity functor on $\underline{C}$ and id T the identity natural transformation, with $A\,(\mathrm{id}\,T) = \mathrm{id}\,(A\,T)$ for an object A. $\underline{C}^{op}$ denotes the dual category of $\underline{C}$.

2.2. We denote by $\underline{C}^2$ the category with morphisms of $\underline{C}$ as objects and commutative squares in $\underline{C}$ as morphisms. A morphism of $\underline{C}^2$, from $f \in \underline{C}$ to $g \in \underline{C}$, is a quadruple (f,u,v,g) of morphisms of $\underline{C}$ such that $f\,v = u\,g$ in $\underline{C}$. Composition in $\underline{C}^2$ is given by

$$(f,u,v,g)(g,u',v',h) = (f,u\,u',v\,v',h) .$$

We write $(u,v) : f \longrightarrow g$, and sometimes just (u,v), for (f,u,v,g).

We define a domain functor D_o and a codomain functor D_1 from $\underline{C}^2$ to $\underline{C}$ by putting

$$(f,u,v,g)\,D_o = u \quad , \quad (f,u,v,g)\,D_1 = v$$

for a morphism (f,u,v,g) of $\underline{C}^2$. This agrees with the notation of 2.1 for an object f of $\underline{C}^2$, i.e. a morphism f of $\underline{C}$.

We denote by $\underline{C}^2[\underline{J}]$ the full subcategory of $\underline{C}^2$ with morphisms in $\underline{J}$ as its objects, and by $I_{\underline{J}} : \underline{C}^2[\underline{J}] \longrightarrow \underline{C}^2$ the inclusion functor. A morphism of $\underline{C}^2[\underline{J}]$ is a commutative square in $\underline{C}$ with two opposite sides from $\underline{J}$. The dual category of $\underline{C}^2[\underline{J}]$ is obtained by a variant of the usual reversal of arrows: only the arrows from $\underline{C}$ are reversed, the arrows from $\underline{J}$ are not reversed.

2.3. For an object A of $\underline{C}$, we denote by $\underline{C}/A$ the subcategory of $\underline{C}^2$

consisting of all morphisms $s \in C^2$ with $s D_1^2 = \mathrm{id}\, A$, and by $H_A : C/A \to C^2$ the inclusion functor. Objects of $\bar{C}/A$ are all morphisms $a \in C$ with $a D^1 = A$. We write $u : a \to b$ if (a,u,A,b) is a morphism of $\bar{C}/A$, i.e. $a = u b$ and $a D^1 = b D^1 = A$ in $\bar{C}$. The product of $u : a \to b$ and $v : b \to c$ in $\bar{C}/A$ is $uv : a \to c$.

We denote by $\bar{J}/A$ the full subcategory $\bar{C}/A \cap \bar{C}^2/\bar{J}$ of $\bar{C}/A$, and by $I_A : J/A \to C/A$ and $H_{A,J} : J/A \to C^2/J$ the inclusion functors. Thus $I_A H_A = H_{A,J} I_J$. Objects of J/A are all morphisms $j \in J$ with $j D_1 = A$.

2.4. $f\delta = f$, for f in $|C^2| = C_f$ defines a natural transformation $\delta : D_0 \to D_1$ with the following universal property. If $\mu : F_0 \to F_1$ is a natural transfomation of functors $F_i : K \to C$, then there is exactly one functor $M : K \to C^2$ such that $F_i = M D_i$ $(i = 0, 1)$ and $\mu = M\delta$.

If the functors F_i have limits $L_i = \lim F_i$, with projections $k\lambda_i : L_i \to k F_i$ for $k \in |K|$, then there is a unique morphism $m = \lim \mu :$ $L_0 \to L_1$ in C such that $m (k\lambda_1) = (k\lambda_0)(k\mu)$ in C for every $k \in |K|$. One sees easily that m is a limit of the functor $M : K \to C^2$, with projections $(k\lambda_0, k\lambda_1) : m \to k\mu$ for $k \in |K|$.

2.5. If K is a category and $A \in |C|$, then $\varphi E_A = \mathrm{id}\, A$, for all $\varphi \in K_f$ defines a constant functor $E_A : K \to C$. If $F : K \to C$ is a functor and $\mu : F \to E_A$ a natural transformation, then $F = M H_A D_0$, $\mu = M H_A \delta$ for exactly one functor $M : K \to C/A$. If F has a limit $L = \lim F$, with projections $k\lambda : L \to k F$ for $k \in |K|$, then μ has a limit $m = \lim \mu :$ $L \to A$, with $m = (k\lambda)(k\mu)$ in C for all $k \in |K|$. m is also a limit of M in $\underline{C}/A$, with projections $k\lambda : m \longrightarrow k\mu$ for $k \in |\underline{K}|$.

2.6. Let $\underline{J}^{\ell}$ be the class of all products $u\,j$ in $\underline{C}$ with $j \in \underline{J}$ and u isomorphic. We say that $\underline{J}$ is left transportable if $\underline{J}^{\ell} = \underline{J}$. The class $\underline{J}^{\ell}$ always is left transportable. Dually, we call $\underline{J}$ right transportable in $\underline{C}$ if $\underline{J}$ is left transportable in $\underline{C}^{op}$.

2.7. We recall that a functor $T : \underline{A} \longrightarrow \underline{B}$ has a left adjoint functor $S : \underline{B} \longrightarrow \underline{A}$ if and only if there is a natural transformation $\beta : \mathrm{Id}\,\underline{B} \longrightarrow S\,T$ such that every morphism $B\beta$, $B \in |\underline{B}|$, is universal for T, i.e. if $A \in |\underline{A}|$ and $g : B \longrightarrow A\,T$ in $\underline{B}$, then $g = (B\beta)(f\,T)$ in B for exactly one morphism $f : B\,S \longrightarrow A$ of $\underline{A}$. We call β a front adjunction for T.

If $\underline{A}$ is a subcategory of $\underline{B}$ and T the inclusion functor, then a universal morphism for T is called a reflection for $\underline{A}$ in $\underline{B}$.

3. Images and strong images

Definition 3.1. Let $f \in \underline{C}$ and $(p,j) \in \underline{C} \times \underline{C}$. We say that (p,j) is a strong $\underline{J}$-image of f in $\underline{C}$ if (a) $f = p\,j$ in $\underline{C}$ and $j \in \underline{J}$, and (b) whenever $f\,v = u\,j'$ in $\underline{C}$ with $j' \in \underline{J}$, then $u = p\,x$, $j\,v = x\,j'$ in $\underline{C}$ for exactly one morphism $x \in \underline{C}$. We call (p,j) a $\underline{J}$-image of f in $\underline{C}$ if (a) is satisfied, and (b) is satisfied for the special case $v = f\,D_1$. We say that $\underline{C}$ has $\underline{J}$-images, or that $\underline{C}$ has strong $\underline{J}$-images, if every morphism of $\underline{C}$ has a $\underline{J}$-image or a strong $\underline{J}$-image respectively in $\underline{C}$.

A morphism $f \in \underline{C}$ is called $\underline{J}$-extremal or strongly $\underline{J}$-extremal if f has a J,-image or strong J.-image (p,j) with j isomorphic in £ • If A€L \C\ , then we may define a J-subobject of A to be an object of j/A f or of a suitable skeleton of j/A •

Dually, let £ be a class of morphisms of £ . We say that (p,j) is a P-coimage.> or a strong P-coimage. of f in £ if (j,p) is a Primage, or a strong Primage, of f in $£^{op}$. P-coextremal and strongly P-coextremal morphisms, and P-quotient objects, are defined accordingly.

We usually omit the prefix J[in proofs and informal discussions.

Proposition 3.2. A morphism j of C is in J, if and only if (j T>09 j) is a strong J-image of j in C fi

Proposition 3.3. It (PJ) i§-£ J-image of f in £ , then $(p^f{}_f j^f)$ is a J-image of f in C if and only if j^1 € J. and p^f « p u 9 y - $u^{\wedge 1}$ j for an isomorphism u of. C

The same result is valid for strong images. Thus if f £ C. has one strong image, then every image of f is strong.

Remark 3.4* Every .J-image in £ is a Jf-image for the class jf of 2*6, and if f £ £ has a J>image, then f has a jJ-image by 3.3. This is also true for strong images. If A £ | £ | , then j/A and J/A have isomorphic skeletons. Thus we may replace J by J without changing subobjects, images and strong images essentially, and we assume from now on that J, is left transportable. It follows by 3.3 that a morphism f 6£ is J>extremal or strongly J.-extremal if and only if $(f, f D_1)$ is a -J-iraage or strong ^J-image of f •

Lemma 3.5. Let (p,j) be a J-image of u £ £ . If. uD_o s p x , j » x u for some x £ £ , then p is iaomorphic in £ and u $\in$ J .

Proof, If $u D_o$ » p- x , x u « j , then p » p (x p) , j = (x p) j f and hence x p « p D_1 by the unicity in 3.1. Thus p is isomorphic and u £ J[1

Proposition 3*6, A morphism u of C is J-extremal and in ^J if and only if u is isomorphic in £ and has a J-image.

Proof* If u is extremal and in J , then (u, u D_1) and (u D_o , u) are images of u . By 3.3, u is an isomorphism.

Conversely, if u is an isomorphism and (p,j) an image of u , then uD_o = » p x , j = x u for x = j u^{-1} f so that p is isomorphic and u £ J, by 3.5» But then j is an isomorphism too, and u is extremal %

Proposition 3«7» If J, is a subcategory of £ and if (p,j) is a J-image in £ , then p is^ J^extremal.

Proof, p D_1 a j D_o is in J since J is a subcategory. If p » p^1 j^f f with y £ jj , then j^1 j 6j[, and thus p^1 s p x , j * x j^f j for a unique x ££ • Then p » p x j^f , j s x j^1 j , and thus x y a p D_1 by unicity in 3.1. If also p ' ^ p x^1 , p D_1 * x^1 j^f , then x^f j^1 j = j f and x^f «x follows. Thus (p, p.p_1.) is an image of p §

Proposition 3*8. If £ has J-images» and if u v £ J[and v G jJ. imply u BJ, whenever u v is defined in jJ f then every couple (p,j) in £X £ with p D_1 » j D, , j £ J, f and p J-»extremal. is a J-imagein C •

Proof. If (p',j') is an image of $p\,j$, then $p = p'\,x$, $j' = x\,j$ for a unique $x \in \underline{C}$. Now $x \in \underline{J}$ by our hypothesis, and p is extremal. Thus $p' = p\,y$, $p\,D_1 = y\,x$ for a unique $y \in \underline{C}$. But then $p' = p'\,x\,y$, $x\,y\,j' = x\,y\,x\,j = x\,j = j'$, and $x\,y = p'\,D_1$ by the unicity in 3.1. Thus x is an isomorphism, and (p,j) is an image by 3.3 ∎

Proposition 3.9. *Let $\underline{C}$ have strong $\underline{J}$-images, and let $u\,v\,w$ be defined in $\underline{C}$. If $u\,v \in \underline{J}$ and $v\,w \in \underline{J}$, then $u \in \underline{J}$.*

Proof. Let (p,j) be an image of u. Then $u\,D_0 = p\,x$, $u\,v = x\,j\,v$ for some $x \in \underline{C}$, since $u\,v \in \underline{J}$, and $p\,x\,u = u = p\,j$, $x\,u\,v\,w = j\,v\,w$ follow. But then $x\,u = j$, by the unicity in 3.1, for $j' = v\,w$. Thus $u\,D_0 = p\,x$, $j = x\,u$, and $u \in \underline{J}$ by 3.5 ∎

Corollary 3.10. *If $\underline{C}$ has strong $\underline{J}$-images, then $\underline{J}$ is right transportable.*

Proof. If $j\,u$ is defined in $\underline{C}$ with $j \in \underline{J}$ and u isomorphic, then $j = (j\,u)\,u^{-1}$ and $u^{-1}\,(u\,D_0)$ are in $\underline{J}$, with 3.6. Thus $j\,u \in \underline{J}$ by 3.9 ∎

4. Local and global image functors

Proposition 4.1. *Let $f = p\,j$ and $f\,D_1 = A$ in $\underline{C}$, with $j \in \underline{J}$.*

4.1.1. *(p,j) is a $\underline{J}$-image of f in $\underline{C}$ if and only if $p : f \longrightarrow j$ is a reflection for $\underline{J}/A$ in $\underline{C}/A$.*

4.1.2. *(p,j) is a strong $\underline{J}$-image of f in $\underline{C}$ if and only if $(p, \mathrm{id}\,A) : f \longrightarrow j$ is a reflection for $\underline{C}^2[\underline{J}]$ in $\underline{C}^2$.*

Proof. Both statements follow immediately from the definitions ∎

Definition 4.2. A local $\underline{J}$-image functor for $\underline{C}$, at an object A of $\underline{C}$, is a pair (ψ_A, im_A) consisting of a functor $\mathrm{im}_A : \underline{C}/A \longrightarrow \underline{J}/A$, left adjoint to the inclusion functor I_A, and a front adjunction $\psi_A : \mathrm{Id}\ \underline{C}/A \longrightarrow \mathrm{im}_A I_A$. A global $\underline{J}$-image functor $(\psi_{\underline{J}}, \mathrm{im}_{\underline{J}})$ for $\underline{C}$ consists of a functor $\mathrm{im}_{\underline{J}} : \underline{C}^2 \longrightarrow \underline{C}^2[\underline{J}]$, left adjoint to the inclusion functor $I_{\underline{J}}$, and a front adjunction $\psi_{\underline{J}} : \mathrm{Id}\ \underline{C}^2 \longrightarrow \mathrm{im}_{\underline{J}} I_{\underline{J}}$ such that $f\,\psi_{\underline{J}} D_1 = \mathrm{id}\,(f\,D_1)$ for every $f \in \underline{C}$.

Let $f \in \underline{C}$ and $f\,D_1 = A$. By 4.1, $(f\,\psi_A, f\,\mathrm{im}_A)$ is an image of f if (ψ_A, im_A) is a local image functor at A, and $(p_f, f\,\mathrm{im}_{\underline{J}})$ is a strong image of f for $p_f = f\,\psi_{\underline{J}} D_o$ if $(\psi_{\underline{J}}, \mathrm{im}_{\underline{J}})$ is a global image functor.

If $(\psi_{\underline{J}}, \mathrm{im}_{\underline{J}})$ is a global image functor, then $\mathrm{im}_{\underline{J}} I_{\underline{J}} D_1 = D_1$ and $\psi_{\underline{J}} D_1 = \mathrm{id}\, D_1$. This is easily verified. If $A \in |\underline{C}|$, it follows that $\mathrm{im}_{\underline{J}}$ maps $\underline{C}/A$ into $\underline{J}/A$, and thus $H_A\,\mathrm{im}_{\underline{J}} = \mathrm{im}_A H_{A,\underline{J}}$, $H_A \psi_{\underline{J}} = \psi_A H_A$ for a local image functor (ψ_A, im_A) at A.

Theorem 4.3. $\underline{C}$ has $\underline{J}$-images if and only if there is a local $\underline{J}$-image functor (ψ_A, im_A) for $\underline{C}$ at every object A of $\underline{C}$.

Proof. This follows immediately from 4.1.1 and the definitions ∎

Theorem 4.4. The following three statements are logically equivalent.

4.4.1. $\underline{C}$ has strong $\underline{J}$-images.

4.4.2. $\underline{C}$ admits a global $\underline{J}$-image functor.

4.4.3. $\underline{J}$ is right transportable and contains all isomorphisms of $\underline{C}$, and $\underline{C}^2[\underline{J}]$ is a reflective subcategory of $\underline{C}^2$.

Proof. 4.4.2 =^ 4.4.1 by 4.1.2 and the definitions, and 4.4.1 ==^ 4.4.3 follows immediately from 3.10, 3.6, and 4.1.2. Assume now that 4.4.3 is valid.

If f£ £ , let $(p_f,u) : f \to i'_{\{}$, with y_f £ J , be a reflection for $C^2[J]$ in $\mathcal{E}^p$. Since (f, f D1 J : f —* f D1 in $\mathcal{C}^p$ and f D_1 £ J[, we have (f, f D_1J * $(p_f,u)(j_f,v)$ in $\mathcal{C}^2$ for some (j_f,v) : JJ. —> f ^ . Now

$$(f,\ f\ D_1)(u,u)\ .\ (p_f,u)(j£,\ u D_1)\ :\ f \to u\ D_x$$

in C7 f and thus $(j_f,v)(u_f u)$ a $(j^*_f,\ u\ D_1)$: j_f —> u D_1 , by the universal property of (p^tu) . It follows that u is isomorphic in C/ with inverse v_f and that j_r * $j'_f v$ in C . As J[is right transportable, J^fii • But then

$$f\,\gamma_J = (p_f,\ f\ D_1) = (p_f,u)(p_f\ D_1,\ v)\ :\ f \longrightarrow j_f$$

defines a reflection f % for 0^p_7^'in £-^{2}. Using the reflections f^L in the usual way to construct a left adjoint functor im of I , with f im = j for f € £ t and a front adjunction U_{γ_J} : Id C^{2} —^ im I , we obtain an image functor (\J/T9 imj . Thus 4.4*3 ==£> 4.4.2 0

Theorem 4.5. *Let C. have strong J-images, and let* F_i : K *~£> C. *be functors with limits* L. (i s 0, l) • *If* ylc : F '—> F_n *is a natural transformation such that* kyu£ .J *for all* k<£ JK \ f *then* lim ^IL : L_0 —> L_1 *is in* j; .

Proof. If i\, has these properties, then the functor M of 2.4 maps IC into STUL] and has a limit lim /\ in £T •^{2} By 4.4 and [lO; V.5.lJ$_f$. M has a limit m in C^2 with m £ J . But then lim u = u m v^{-1} in C for isomorphisms u and v of £ f and lim yu is in J, by 3.10 ∎

Theorem 4.6. *Let $\underline{C}$ have $\underline{J}$-images, let $F : \underline{K} \longrightarrow \underline{C}$ be a functor with a limit L, and let $\mu : F \longrightarrow E_A$ be a natural transformation, where E_A is a constant functor (2.5). If $k\mu \in \underline{J}$ for all $k \in |\underline{K}|$, then $\underleftarrow{\lim}\, \mu : L \longrightarrow A$ is in $\underline{J}$.*

Proof. Similar to that of 4.5, using 2.5 and 4.3 ∎

We note the most important special case of 4.6.

Corollary 4.7. *Let $\underline{C}$ have $\underline{J}$-images, and let A be an object of $\underline{C}$. If a family $(i_k)_{k \in K}$ of objects of $\underline{J}/A$ has an intersection (fibred product) $p = \bigcap j_k$ in $\underline{C}$, then $p \in \underline{J}$* ∎

5. Miscellaneous results

Results proved in this section for $\underline{J}$-images are also valid, with only minor changes in proofs, for strong $\underline{J}$-images. We denote by $\underline{P}$ and $\underline{P}^{st}$ the classes of all $\underline{J}$-extremal and of all strongly $\underline{J}$-extremal morphisms of $\underline{C}$.

Proposition 5.1. *Let $p\,q$ be defined in $\underline{C}$. If $\underline{J}$ consists of monomorphisms of $\underline{C}$ and $(p\,q,\, j)$ is a $\underline{J}$-image, then (q, j) is a $\underline{J}$-image.*

Proof. If $q\,j = q'\,j'$ with $j' \in \underline{J}$, then $p\,q' = p\,q\,x$, $j = x\,j'$ for some $x \in \underline{C}$. Then $q\,x\,j = q'\,j'$, and $q\,x = q'$ since j' is monomorphic. x is unique for the same reason ∎

Proposition 5.2. *Let $p\,q$ be defined in $\underline{C}$. If $(p\,q,\, j)$ is a $\underline{J}$-image* and p an epimorphism of £ f then (qf j) is a J-dmage«

Proof, If q j * q^f j^1 with j ' £ J f then q' » q x , j « x j (in £ iff p q « p q x 9 j «x j^1 in 0. The latter is true for exactly one x 6 £ @

Corollary 5.3* If, (Pid) is a J-image. and if up v~^1 is defined in £ for isomorphisms uf y of C , then (u p •^-1 , v *j)* is a J-image«

Proof. (u~^1 u p, j) is an image, and thus (u p, j) is an image by 5.2, Now (u p v~^1 f v j) is an image by 3.3 fl

Proposition 5.4. Let p q be defined in C , with p ∈ P^st . Then p q ∈ P ⟺ q ∈ P and p q ∈ P^st ⟺ q ∈ P^st .

Proof, This result (which we do not use in the present paper) is a special case of 7.4; let *jf* 1, j2 be identity morphisms ©

Proposition^^, if P j is defined in £, with p€£^St «Ji i£l>

then \v9j) is a strong J-image and a strong £ -coimage.

Proof. This follows immediately from the definitions |

Proposition 5^6* If every morphism f of C has a factorization f »e j in £ with e epimorphic in £ and j £ J. f then jP consists of epimorphisms of £ f and all equalizers in £ are in J[•

Proof, If p <£P, and p » e j 9 j ^ j; f then p x « e f x j »pD.1 for some x ££ f and then e j x » e • If e is epimorphic, then j x *m* e D1 f and thus j is isomorphic and p epimorphic in £ .

If m is an equalizer of morphisms f_k in £, and if m » e j with e epimorphic and j €. J, $ then all products j f are equal, and hence j » x m for some x £ £ . But then e x m » m , and thus e x = m Do . As e is epimorphic, it follows that e is isomorphic. Thus m €: £ %

Proposition 5«7* If £ has equalizers and all equalizers of £ are in J[f then jP consists of epimorphisms of £ •

Proof. Let p £ jP f and let p f » p g in £ • If j £ <J is an equalizer of f and g in C , then p «u j for some u6: £ . But then u « p x , p D_1 m-x j for some x£ £ . It follows that f * x j f = x j g s g f

Proposition 5*8. If £ consists of monomorphisms of £ f then all coequalizers in £ are in j?[st] .

Proofs Let q be a coequalizer of morphisms f_k in £ , and let q v s« u j in £ with j £ J^ • Since j is monomorphic, all products f_k u are equal, and thus u » q x for a unique x £ £ . Now q v a q x j , and (q P^) v sx j follows. Thus (q, qD_1) is a strong image of q@

6« Self-dual theories

Let P, be a right transportable class of morphisms of £ •

Proposition 6.1. The following two statements are logically equivalent.

6.1.1. *$\underline{C}$ has $\underline{J}$-images, and every $\underline{J}$-image in $\underline{C}$ is a $\underline{P}$-coimage.*

6.1.2. *$\underline{C}$ has $\underline{P}$-coimages, and every $\underline{P}$-coimage in $\underline{C}$ is a $\underline{J}$-image* ∎

We say that $\underline{C}$ *has* $(\underline{P},\underline{J})$-*decompositions* if $\underline{C}$ satisfies 6.1.1 and 6.1.2.

Theorem 6.2. *If $\underline{C}$ has $(\underline{P},\underline{J})$-decompositions, then $\underline{P}$ is the class of all $\underline{J}$-extremal morphisms of $\underline{C}$, $\underline{J}$ is the class of all $\underline{P}$-coextremal morphisms of $\underline{C}$, $\underline{J}$ and $\underline{P}$ are subcategories of $\underline{C}$, and $\underline{J} \cap \underline{P}$ is the class of all isomorphisms of $\underline{C}$.*

Proof. $p \in \underline{P} \Longleftrightarrow (p,\ p\,D_1)$ is a $\underline{P}$-coimage $\Longleftrightarrow (p,\ p\,D_1)$ is a $\underline{J}$-image $\Longleftrightarrow p$ is $\underline{J}$-extremal. Dually, $\underline{J}$ is the class of all $\underline{P}$-coextremal morphisms. Now $\underline{J} \cap \underline{P}$ is the class of all isomorphisms by 3.6.

Let now (p,j) be a $\underline{J}$-image of $u\,v$ in $\underline{C}$ with u and v in $\underline{J}$. Then $u = p\,x$, $j = x\,v$ for some $x \in \underline{C}$. Since u is $\underline{P}$-coextremal, we have $u\,D_o = p\,y$, $x = y\,u$ for some $y \in \underline{C}$. But then $u\,D_o = p\,y$, $y\,u\,v = j$, and $u\,v \in \underline{J}$ by 3.5. Dually, $\underline{P}$ is a subcategory of $\underline{C}$ ∎

Theorem 6.3. *The following five statements are logically equivalent.*

6.3.1. *Every $f \in \underline{C}$ has a factorization $f = p\,j$ in $\underline{C}$ with $p \in \underline{P}$ and $j \in \underline{J}$. If $u\,j = p\,v$ in $\underline{C}$ with $j \in \underline{J}$ and $p \in \underline{P}$, then $u = p\,x$, $v = x\,j$ in $\underline{C}$ for exactly one $x \in \underline{C}$.*

6.3.2. *$\underline{C}$ has strong $\underline{J}$-images, $\underline{J}$ is closed under composition in $\underline{C}$, and $\underline{P}$ is the class of all $\underline{J}$-extremal morphisms of $\underline{C}$.*

6.3.3. *$\underline{C}$ has $(\underline{P},\underline{J})$-decompositions, and if $p\,j = p\,D_o = j\,D_1$ in $\underline{C}$ with $p \in \underline{P}$ and $j \in \underline{J}$, then p and j are isomorphisms of $\underline{C}$.*

6.3.4. *$\underline{C}$ has J-images, $\underline{P}$ is closed under composition in $\underline{C}$, and if* $p\,j$ *is defined in* $\underline{C}$, *then* (p,j) *is a* $\underline{J}$*-image in* $\underline{C}$ *if and only if* $p \in \underline{P}$ *and* $j \in \underline{J}$.

6.3.5. *$\underline{P}$ and $\underline{J}$ are closed under composition in $\underline{C}$, every* $f \in \underline{C}$ *has a factorization* $f = p\,j$ *in* $\underline{C}$ *with* $p \in \underline{P}$ *and* $j \in \underline{J}$, *and if* $p\,j = p'\,j'$ *in* $\underline{C}$ *with* p, p' *in* $\underline{P}$ *and* j, j' *in* $\underline{J}$, *then* $p' = p\,x$, $j = x\,j'$ *in* $\underline{C}$ *for exactly one morphism* $x \in \underline{C}$.

We say that $\underline{C}$ *has strong* (P,J)*-decompositions* if these five statements are valid.

Proof. If 6.3.1 is valid, and if $f = p\,j$ in $\underline{C}$ with $(p,j) \in \underline{P} \times \underline{J}$. then (p,j) clearly is a strong $\underline{J}$-image and a strong $\underline{P}$-coimage of f. With 6.2, this shows that 6.3.1 $\Longrightarrow$ 6.3.2.

If 6.3.2 is valid, then (p,j) is a $\underline{J}$-image iff $p\,j$ is defined in $\underline{C}$, $p \in \underline{P}$, and $j \in \underline{J}$, by 3.7 and 5.5, and then (p,j) is a $\underline{P}$-coimage by 5.5. This proves, with 3.6, that 6.3.2 $\Longrightarrow$ 6.3.3.

We show next that $u\,v \in \underline{J}$ and $v \in \underline{J}$ imply $u \in \underline{J}$ if 6.3.3 is valid. Let (p,j) be a $\underline{J}$-image of u, with $p \in \underline{P}$. Since $(u\,D_0, u\,v)$ is a $\underline{P}$-coimage, we have $u\,D_0 = p\,x$, $j\,v = x\,u\,v$ for some $x \in \underline{C}$. Let (p',j') be a $\underline{J}$-image of x, with $p' \in \underline{P}$. Then $p\,p'\,j' = u\,D_0$, and $p\,p' \in \underline{P}$ by 6.2. Thus j' is isomorphic, and $x \in \underline{P}$. Now $j\,v \in \underline{J}$ by 6.2, and thus $(p\,D_1, j\,v)$ is a $\underline{P}$-coimage of $j\,v$. But then $p\,D_1 = x\,y$, $u\,v = y\,j\,v$ for some $y \in \underline{C}$. Thus p is isomorphic with $x = p^{-1}$, and $u \in \underline{J}$.

Now 6.3.3 $\Longrightarrow$ 6.3.4 by the preceding paragraph, 3.8 and 6.2.

Let now 6.3.4 be valid, 6,3.5 is valid if J, is closed under composition. Thus let u v be defined in £ with uf v in J, , and let (p,j) be a J^-image of u v . Then u a p x , j a x v for some x £ £ . Let (p1,j1) be a J^-image of x . Then p p' f e P , and (p p*f y) is a JMLmage of u ^£ • Thus p p* y a u D , y u » j1 for an isomorphism y of £ • But then p ' y u v a p1 j1 v « J , and u v £ J by 3.5. Thus 6.3.4 «=#* 6.3.5.

Finally, let 6.3.5 be valid. If u j « p v with p 6.JP , j ^ JJ , then let u * p ' jr , v « p" j1f f with p1 , pfl in jP and y , jM in £ . Then p1 « p pM B , 3" * z y i for a \mique z ^ C, f and u » p x f v * x j for x « p" z j1 . If also u a p xJL , v. » x1 j f let Xj « p1 i1 with P1 ^ P , ij 6 i . Then p' a p Pl zV , ^ a z» j' , and PJL a p» z" f j" » zW ^ j for morphisms z1 f z" of £ . Now p1 a p plf zlf zf , j" a zM zf y j f and thus z" z1 a z . But then xx a p» zfl 2f j1 ax f and 6.3.5 => 6.3.11

Corollary 6.4. If £ has (P>J)HJecompositions and jJ consists of monomorphisms of £ f then £ has strong (P«j)-decompositions.

Proof. The hypothesis implies 6.3*3 §

Remark 6.5. If £ has strong (P,^-decompositions, then £ has strong P^coima^es AS well as strong jP-images. £ is an Isbell bicategory ^5} if P consists of epimorphisms and J. of monomorphisms of £ • It seems that very little in Isbell's theory depends on these additional assumptions.

7* Direct and inverse images

Definition J/a, Let (j,f)£ J[X<L , with j D * f D . We say that (f_1, j_1) is a direct J-image or a strong direct J-image of (j,f) in £ if (f-itj^) ia a Jr image or a strong J-image respectively of j f in C.

Thus $(f_1, t j_1)$ is a strong direct image of (j,f) iff (a) $(f_{1t} f)$: j —> J_x in firfji] t and (b) whenever (u, f v) : j —> j^f in £r[jj t then u * f_1 x in C. and (x,v) : j_1 —> j^f in Cr[ji1 ^or exactly one x£C. • For a direct image, we require (a), and (b) only for v » f D_1 .

Definition 7.2. Let (f,j)€; C X J , with j D. « f D. . We say that (jnt^J is a strong inverse J-image of (f,j) in C. if (a) $(f_{1f} f)$: ^—>j in C7"[j] , and (b) whenever (u, v f) : j^1 •—>j in C7^[J , then u « x f_1 in C and (x,v) : j^1 —$>$j_x$ in C^EJI for exactly one x££ . We say that (j_1, f_1) is an inverse J-image of $(f_f\, j)$ in C if (a) is satisfied, and (b) is satisfied for v s f D_o .

Inverse images and strong inverse images are dual in CT/J,T to direct images and strong direct images. Except for 7.7 and the self-dual 8.3, every result of sections 7 and 8 has a dual in this sense which we do not state.

If ^J contains all isomorphisms of £ , then a strong inverse J^image is a pullback in C ; put j^1 » v D_o in 7.2, (b). We note also that inverse (^-images in £ are the same as pullbacks in £ . Direct images are images, and every image is a direct image if J[contains all isomorphisms of £ .

Proposition 7.3. If. $(f_1, ^)$ is a direct J-image of (j,f) in C, then

(f\j^f) is a direct J-image of (Jtf) in £ if and only if f• • f1 u f
j^1 BU^-1 j1 in £ for an isomorphism u o£ £ £9

This result and its dual are also valid for strong direct and inverse images.

Theorem 7.4» Let (fi>ji) be a strong direct J-image of (j,f) in £ f and let (vlfv) : ^~-^J2 in Sr[lJ • Then U^j^) is a direct J-image or a strong direct J-image of (d1>v) in £ if and only if (f^ v1f j^) is a direct J-image or a strong direct J-image respectively of (j, f v) in £ .

The dual of this generalizes a well-known result for pullbacks*

Proof, (u, f v) : j ->y in Cr£jj iff u . ^ y in Cf (yfv) : ^ -3> y ^ £"[kjt for a unique y £ £ f since (f-tj.)i is a strong direct image. But then u « f1 v1 x , j2 «x j^1 in £ for exaetly on^ x € £ iff y m v1 x f j2 « x j^f in £ for exactly one x ££ . Thus (f1 • j1 J2> is a direct image iff (v1f j2) is a direct image.

The proof for strong images is exactly analogous*0

Theorgn 7.5« Let (j,f) ££X £ t with J D j . f D o. If C has J-images, then the following two statements are logically equivalent.

7*5.1. (^tjy) is a strong direct J-image of (jff) .

7.5*2. f1 j1. * j f in £ f j1 ^ J, and whenever f v is defined in £ Mi Cvj.fJg) a direct J-image of (dlfv) f then (^ vlf J2> is a direct J-image of (j, fv) in C .

Proof. 7^5«1 ««5^ 7.5.2 by 7.4. For the converse, let jf v « u j^1 in £f •)^f € i t and 1st (v1# j2) be a direct image of (1Lfv) . Then u » f1 x f

j1 v m x jf in £ iff x * v1 y for some y with: u * f1 • 1 • y t 2 *y jf •
If 7.5.1 is valid, then there is exactly one such y £ £ , and thus (f-1tj.1) is
a strong direct image f]

Proposition,7.6* If £ has strong inverse J-imagea, then every direct J-image in £ is a strong direct J-image»

Proof. Let (f^^) be a direct image of (j,f) . If j f v « u j1 in £ , y Si f let (d%vf) *©: a strong inverse image of (v,jf) . Then J f » e j11 , u• «• 2 v1 for a unique z£ £ , and also s • f1 y f j.1 « y j" for a unique y £ £ . Then u » f1 y v1 , i1 v « y v1 j1 . Conversely, if u » f.1 x , j1 y «x jf f then x «y1 vf f j1 * y1 jw for some yf 6 . £ . But then f1 yf vf m z v1 / fx yf j11 * 2 j" , and thus f1 y1 » z , y1 j" » j1 # It follows that y1 * y , x «* f1 y , and (f1^tj1) is a strong direct image of (jff) ©

The following result has no dual for direct images.

TheoremJ7.7. Let (j1 . ^j be a pullback of (f>i) in g,, with j ^J . If £ has strong J-images> then $1£J[, and (jj^f^) is a strong inverse J-image of (f,j) in C ,

Proof> We must only show that j1 ^1 . Thus let (p•,;}') be a strong image of j1 . Then f1 * p1 x , j1 f * x j for some x ££ . Since (j1tf1) is a pullback, y m y ^ , x ^ y ^ for a unique y ££ • Now pf y j1 « j1 , p1 y fx « fx , and thus p1 y » ^ DQ , y ^ * jf . But then i1^£ by 3.5

8. Direct and inverse image functors

Definition 8.1. Let $f : A \longrightarrow B$ in $\underline{C}$, and let $f_* : \underline{J}/A \longrightarrow \underline{J}/B$ be a functor and $\Psi_f : H_{A,\underline{J}} \longrightarrow f_* H_{B,\underline{J}}$ a natural transformation such that $j \Psi_f D_1 = f$ for every $j \in |\underline{J}/A|$. We put $\Psi_f I_{\underline{J}} D_o = \psi_f$, and we say that the pair (Ψ_f, f_*) is a direct $\underline{J}$-image functor at f if $(j \psi_f, j f_*)$ is a direct $\underline{J}$-image of (j,f) for every $j \in |\underline{J}/A|$.

Dually, let $f^* : \underline{J}/B \longrightarrow \underline{J}/A$ be a functor and $\Phi_f : f^* H_{A,\underline{J}} \longrightarrow H_{B,\underline{J}}$ a natural transformation such that $j \Phi_f D_1 = f$ for every $j \in |\underline{J}/B|$. We put $\Phi_f I_{\underline{J}} D_o = \varphi_f$, and we say that (f^*, Φ_f) is an inverse $\underline{J}$-image functor at f if $(j f^*, j \varphi_f)$ is an inverse $\underline{J}$-image of (f,j) for every $j \in |\underline{J}/B|$.

Proposition 8.2. If $\underline{C}$ has $\underline{J}$-images, then there is a direct $\underline{J}$-image functor (Ψ_f, f_*) at every $f \in \underline{C}$.

Proof. Let $f : A \longrightarrow B$. For $j \in |\underline{J}/A|$, let $(j \psi_f, j f_*)$ be a direct image of (j,f), and let $j \Psi_f = (j \psi_f, f) : j \longrightarrow j f_*$ in $\underline{C}^2[\underline{J}]$. For $u : j \longrightarrow j'$ in $\underline{J}/A$, we have $u (j' \psi_f) = (j \psi_f) x$, $j f_* = x (j' f_*)$ in $\underline{C}$ for exactly one $x \in \underline{C}$. We put $x = u f_* : j f_* \longrightarrow j' f_*$ in $\underline{J}/B$. One verifies easily that this defines a functor f_* and a natural transformation Ψ_f with the required properties. ▮

Theorem 8.3. Let $f \in \underline{C}$. If (Ψ_f, f_*) is a direct $\underline{J}$-image functor and (f^*, Φ_f) an inverse $\underline{J}$-image functor at f, then f_* is left adjoint to f^*.

Proof. Let $f : A \longrightarrow B$. For objects a of $\underline{J}/A$ and b of $\underline{J}/B$, we consider the equations

$$(1) \qquad a = u\,(b\,f^*)\ , \qquad u\,(b\,\varphi_f) = (a\,\psi_f)\,v\ , \qquad a\,f_* = v\,b$$

in $\underline{C}$. If $u : a \longrightarrow b\,f^*$ in $\underline{J}/A$, then $(u\,(b\,\varphi_f),\ f) : a \longrightarrow b$ in $C^2[\underline{J}]$. Since $(a\,\psi_f,\ a\,f_*)$ is a direct image of (a,f) , (1) is satisfied for exactly one morphism $v : a\,f_* \longrightarrow b$ of $\underline{J}/B$. Dually , if $v : a\,f_* \longrightarrow b$ in $\underline{J}/B$, then (1) is satisfied for exactly one morphism $u : a \longrightarrow b\,f^*$ of $\underline{J}/A$. Thus putting $u = v\,\eta_{a,b}$ if (1) is satisfied defines a bijection

$$\eta_{a,b} \ :\ \underline{J}/B\,(a\,f_*,\ b) \longrightarrow \underline{J}/A\,(a,\ b\,f^*)\ .$$

One verifies easily that $\eta_{a,b}$ is natural in a and in b ∎

<u>Theorem 8.4</u>. <u>If $\underline{C}$ has J-images and a direct J-image functor</u> $(\Psi_f,\ f_*)$ <u>is given at every</u> $f \in \underline{C}$, <u>then the equations</u>

$$\Psi_{\mathrm{id}\,A}\,(c_A\,H_{A,\underline{J}}) = \mathrm{id}\,H_{A,\underline{J}}\ , \qquad \Psi_{fg}\,(c_{f,g}\,H_{C,\underline{J}}) = \Psi_f\,(f_*\Psi_g)\ ,$$

<u>for</u> $A \in |\underline{C}|$ <u>and</u> $f\,g$ <u>defined in</u> $\underline{C}$, <u>with</u> $g\,D_1 = C$, <u>determine natural transformations</u> $c_A : (\mathrm{id}\,A)_* \longrightarrow \mathrm{Id}\,\underline{J}/A$ <u>and</u> $c_{f,g} : (f\,g)_* \longrightarrow f_*\,g_*$. <u>These natural transformations satisfy the coherence relations</u>

$$c_{fg,h}\,(c_{f,g}\,h_*) = c_{f,gh}\,(f_*\,c_{g,h})\ ,$$

$$c_{A,f}\,(c_A\,f_*) = \mathrm{id}\,f_* = c_{f,B}\,(f_*\,c_B)\ ,$$

<u>for</u> $f : A \longrightarrow B$ <u>and</u> $f\,g\,h$ <u>defined in</u> $\underline{C}$.

<u>Proof</u>. For $j \in |\underline{J}/A|$, the first two equations mean that

$$(j\,\psi_A)(j\,c_A) = j\,D_o\ , \qquad (j\,\psi_{fg})(j\,c_{f,g}) = (j\,\psi_f)(j\,f_*\psi_g)$$

in C, for morphisms j c_A : i A_* —> j of JJ/A and i $c_{f,g}$: j (f g)$_*$ —> j $f_\#$ $g_\#$ tit JJC . Since (j y^, J $A_\#$) and (j j ^ g . J (f g)$_\#$) are direct images of (i,k) and of (j, f gj , and

$$((j\,\psi_f)(j\,f_*\,\psi_g)$$

in £r(jlT 9 these conditions determine j c_A and j $c_{f,g}$ uniquely.

The remainder of the proof is straightforward diagram chasing 0

We note that Jc_A is an isomorphism for '$(;\£]k\ , by 3.2 and 3#3#

Thus c_A : $A_\#$ —> Id j/A is a natural equivalence for every A^/C^l •

Theorem 8.5. £ has strong J-images if and only if £ has J-imaffea and $c_{f,g}$: $(f\,g)_*$ —> f. g_A is a natural equivalence whenever f g is defined in C *

Proof, We test 7*5.2. If jf v is defined in £ with j € J , and if ($f_{1f}i_x$) is a direct image of (j,f) and ^^j g) one of (j_{1f}v) then

f l a &Vf *x > jf*mx h ; v l * ^ l ¥ $y » J i v » *yJ 2 ;

x ^ 1 ^ * (d f » / V ^ x V*^ f (x V*)(d l V*^ * j f. V«

in £ for isomorphisms x and y of £ . It follows that

f 1 V l ~(j YtJ z ' J(fv)* *zj 2

in £ for % m (j c_f ^)(x $v_\#$) y . Now (f^ $v_{1\#}$ J_2J is a direct image of (j_f f v) iff z is an isomorphism of £ , and this is the case iff j $c_{f,v}$ is an isomorphism ▮

Remark 8.6. If £ has strong J^-images and £ consists of monomorphisms of £ then we obtain a global direct image functor f f—^f_* as follows.

We assume that $\underline{J}$ has a subclass $\underline{J}_o$ such that every $j \in \underline{J}$ has exactly one factorization $j = u j_o$ in $\underline{C}$ with $j_o \in \underline{J}_o$ and u isomorphic in $\underline{C}$. Then every morphism of $\underline{C}$ has exactly one $\underline{J}_o$-image, and it follows that there is a unique direct J_o-image functor $f_A : J/A \to J/B$ for $f : A \to B$ in $\underline{C}$. One verifies easily that $f \mapsto f_\#$ defines a functor on $\underline{C}$.

If $\underline{C}$ has strong inverse $\underline{J}_o$-images and $\underline{J}_o$ consists of monomorphisms of $\underline{C}$ then we obtain a global inverse image functor $f \mapsto f^*$ on $\underline{C}_o$ in the same way.

Remarks 8.7. Comparing 7.1 and 8.1 with the definitions of [3; § 1], one sees easily that giving a direct image functor $(V_i, f_\#)$ at every $f \in \underline{C}$ is the same as giving an opcleavage for the functor $I_J D_1 : \underline{C} \to \underline{C}_t$ except that the natural transformations c_A are equivalences only if $\underline{C}$ has strong images. Dually, inverse image functors $(f^*, (f_i))$ at every $f \in \underline{C}$ define a cleavage for the functor $I_J D_1$ with the corresponding reservation.

In the terminology of [1], especially [1; 5.6] and [1; 8], the data of 8.4 define a pseudofunctor $F_\# : D\underline{C}^{op} \to \underline{Cat}^3$ and a transformation $\phi_\# : F_o \Rightarrow F_\#$ of pseudofunctors, where $\underline{Cat}$ is the bicategory of categories (denoted by $\underline{Tac}$ in [1]), and F_o is a constant strict pseudofunctor, with $f F_o = \mathrm{Id}\, \underline{C}$ for $f \in \underline{C}$. The pair $(F_\#, \phi_\#)$ may be called a direct $\underline{J}$-image pseudofunctor.

Dually, inverse image functors $(f^*, (\varphi)_f)$, $f \in \underline{C}$, determine natural transformations $c_A : \mathrm{Id}\, J/A \to (\mathrm{id}\, A)^*$ and $c_{f,g} : g^* f^* \to (fg)^*$ with the expected coherence relations. These data define a pseudofunctor $F^* : D\underline{C} \to \underline{Cat}^t$ and a transformation $\phi^* : F^o \Rightarrow F^*$ of pseudofunctors, where F^o is constant. The pair (F^*, ϕ^*) may be called an inverse $\underline{J}$-image pseudofunctor.

9« Examples and complements

9.1. If £* is the class of all isomorphisms of £ , then £ has strong (£,£*J-decompositions and strong (£*,£)-decompositions. Inverse £•-images are trivial; inverse £-images are pullbacks in £ .

9«2. We denote by K the class of all monomorphisms and by E. the class of all epimorphisms of £ • M,-subobjects and Jl-images in our sense are essentially the same as subobjects and images in the sense of £4j and [lOj • Inverse images in the usual sense (see LlOt I.llj) are strong inverse Jt-images in our sense. The categories of sets and of groups, and all abelian categories, have strong (jS,M)-decompositions and inverse Mf-images. The category of rings has strong (j?,14)-decompositions for a proper subcategory £ of IS .

9-3. If every morphism f ££ has a factorization f » e m in £ with e ^ JE and m £ M , then the M^-extremal and E>-coextremal morphisms in our sense are the extremal epimorphisms and monomorphisms of C6j and [llj, and (e,m) is an Jl-image in our sense of f * e m in £ if and only if e is a coimage of f in the sense of [ll] .

The following result follows immediately from 7.6 and 5.4.

Proposition 9.4• If £ has strong inverse M-imagea. and if every morphism f o£. £ has a factorization f » e m in £ with e £ E. and m ^ >[, then the extremal epimorphisms of £ form a subcategory of £ @

9.5* The strict monomorphisms and epimorphisms of (73 are the strongly

$\underline{E}$-coextremal monomorphisms and the strongly $\underline{M}$-extremal epimorphisms in our sense. The nice properties obtained in [7] for strict monomorphisms and epimorphisms, and their proofs, remain valid for strongly $\underline{P}$-coextremal and strongly $\underline{J}$-extremal morphisms in general, without any restrictions, except for equalizers and coequalizers (see 5.8). If $f = e\ m$ in $\underline{C}$ and e is a coimage of f in the sense of [7], then (e,m) is a strong $\underline{M}$-image of f in our sense. $\underline{C}$ has coimages in the sense of [7] if and only if $\underline{C}$ has strong $(\underline{P},\underline{M})$-decompositions in our sense for a class $\underline{P}$ of epimorphisms of $\underline{C}$.

9.6. The category of topological spaces furnishes some interesting examples of strong $(\underline{P},\underline{M})$-decompositions. We call $f : A \longrightarrow B$ in this category coarse if A has the coarsest topology such that f is continuous for the given topology of B. Dually, we call $f : A \longrightarrow B$ fine if B has the finest topology such that f is continuous for the given topology of A.

9.6.1. $\underline{J}_1$ consists of all coarse injective maps, and $\underline{P}_1$ consists of all surjective maps. $\underline{J}_1$-subobjects correspond to subspaces.

$\underline{J}_2$ consists of all injective maps, and $\underline{P}_2$ consists of all fine surjective maps. $\underline{P}_2$-quotient objects correspond to quotient spaces.

9.6.2. $\underline{J}_3$ consists of all coarse maps and $\underline{P}_4$ of all fine maps. $\underline{J}_4 = \underline{P}_3$ is the class of all bijective maps.

9.6.3. $\underline{J}_5$ consists of all closed injective maps, and $\underline{P}_5$ consists of all maps $f : A \longrightarrow B$ with $f(A)$ dense in B. $\underline{J}_5$-subobjects correspond to closed subspaces.

For two more unusual examples, see [2; 3.19].

9.7. Let $\underline{C}$ be a pointed category, and let $\underline{K}$ be the class of all kernels of morphisms of $\underline{C}$. $\underline{K}$ is left and right transportable.

Theorem. If $\underline{C}$ has kernels and cokernels, then $\underline{C}$ has strong $\underline{K}$-images, and $f\ \mathrm{im}_{\underline{K}} = \ker \operatorname{coker} f$, for $f \in \underline{C}$, defines a global $\underline{K}$-image functor.

Of course, ker coker is determined only up to natural equivalence.

Proof. Let $c = \operatorname{coker} f$ and $j = \ker c$. Then $f\ c = 0$, and hence $f = p\ j$ for a unique morphism $p \in \underline{C}$. By 4.1.2, it is sufficient to show that (p,j) is a strong $\underline{K}$-image of f . Thus let $f\ v = u\ j'$ with $j' = \ker g'$ in $\underline{C}$. Then $f\ v\ g' = u\ j'\ g' = 0$, and hence $v\ g' = c\ y$ for some $y \in \underline{C}$. Now $j\ v\ g' = j\ c\ y = 0$, and thus $j\ v = x\ j'$ for some $x \in \underline{C}$. Since x is monomorphic and $p\ x\ j' = p\ j\ v = f\ v = u\ j'$, x is unique and $p\ x = u$ ∎

Our results 4.5, 4.7 and 7.7 are well-known for this case.

9.8. If $\underline{C}$ is the category of groups and $\underline{K}$ the class of all kernels of morphisms of $\underline{C}$, then $\underline{C}$ has strong $\underline{K}$-images and inverse $\underline{K}$-images, by 9.7 and 7.7, but $\underline{K}$ is not a subcategory of $\underline{C}$. Thus $\underline{C}$ does not have $(\underline{P},\underline{K})$-decompositions for any class $\underline{P}$ of morphisms of $\underline{C}$.

A preordered set C induces a category $\underline{C}$ consisting of all pairs (x,y) in $C \times C$ with $x \leq y$, with composition $(x,y)(y,z) = (x,z)$, and with $|\underline{C}| = C$. If $C = \{0, 1, 2, 3\}$ with the usual order, and if $\underline{J}$ consists of all morphisms of $\underline{C}$ except $(1,2)$ and $(2,3)$, then $\underline{J}$ is a subcategory of $\underline{C}$, and the conclusion of 3.9 is valid for $\underline{J}$. One sees easily that $\underline{C}$ has $\underline{J}$-images and inverse $\underline{J}$-images, but not strong $\underline{J}$-images.

The hypotheses of our results are mostly justified by examples like these, with one important exception. We do not have at present an example of a category $\underline{C}$ with classes $\underline{J}$ and $\underline{P}$ of morphisms such that $\underline{C}$ has $(\underline{P},\underline{J})$-decompositions, but not strong $(\underline{P},\underline{J})$-decompositions.

References

1. Bénabou, J. Introduction to bicategories. Reports of the Midwestern Category Seminar, Springer Lecture Notes 47, 1967.
2. Ehrbar, H. Zerlegende Morphismenklassen, reflektierende Unterkategorien und ein Modell der Kategorie der topologischen Räume. Diplomarbeit 1967, Universität München. Unpublished.
3. Gray, J. W. Fibred and cofibred categories. Proceedings of the Conference on Categorical Algebra -- La Jolla 1965. Springer-Verlag New York, 1966.
4. Grothendieck, A. Sur quelques points d'algèbre homologique. Tôhoku Math. J. 9, 119 - 221 (1957).
5. Isbell, J.R. Some remarks concerning categories and subspaces. Canad J. Math. 9, 563 - 577 (1957).
6. -- Subobjects, adequacy, completeness, and categories of algebras. Rozprawy Mat. 36 (1964).
7. Jurchescu, M. and A. Lascu. Morfisme stricte, categorii cantoriene, functori de completare. Studii Cerc. Mat. 18, 219 - 234 (1966)
8. Kennison, J. F. Full reflective subcategories and generalized covering spaces. Illinois J. Math. 12, 353 - 365 (1968).

9* MacLane, JS. Duality for groups. Bull. A.M.S. ^0., 485 - 516 (1950).

10. Mitchell^^. Theory of Categories. Academic Press, New York, 1965.

11. Sonner^ J. Canonical categories. Proceedings of the Conference on Categorical Algebra — La Jolla 1965. Spiinger-Verl&g New York, 1966.

12. Wyler, jO. Operational categories. Proceedings of the Conference on Categorical Algebra — La Jolla 1965. Springer-Verlag New York, 1966.

UNIVERSITAET MUENCHEN

CARNEGIE-MELLON UNIVERSITY