SOME FUNCTION-THEORETIC ASPECTS OF DISCONJUGANCY OF LINEAR-DIFFERENTIAL SYSTEMS

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1. Introduction.

In this paper we consider linear differential systems of the form

$$\underset{\sim}{y}'(z) = P(z)\underset{\sim}{y}(z) \tag{1.1}$$

where $\underset{\sim}{y}(z)$ is the column vector $(y_1(z),\ldots,y_n(z))$ and $P(z)$ is the $n\times n$ matrix $[p_{ik}(z)]_1^n$, where the n^2 analytic functions $p_{ik}(z)$ are regular in the domain D. Following Schwarz [8], we shall say that (1.1) is disconjugate in D if for every choice of n (not necessarily distinct) points $z_1, z_2, \ldots, z_n$ in D, the only solution of (1.1), which satisfies $y_i(z_i) = 0$ $i = 1,2,\ldots,n$ is the trivial one $\underset{\sim}{y}(z) \equiv 0$.

Various aspects and applications of systems disconjugancy were considered by Nehari [6], Schwarz [8], London and Schwarz [3], and Kim [1]. Considering disfocality of second-order differential equations Nehari pointed out that following principle [6, Theorem 1.1] which we state here as a necessary and sufficient condition for disconjugancy of the differential system

$$y_1' = p(z)y_2, \quad y_2' = q(z)y_1, \tag{1.2}$$

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where $p(z)$ and $q(z)$ are regular functions in the doamin D.

Let

$$f(z) = \frac{u_1(z)}{v_1(z)}, \qquad g(z) = \frac{u_2(z)}{v_2(z)} \tag{1.3}$$

where $\underset{\sim}{u} = (u_1, u_2)$ and $\underset{\sim}{v} = (v_1, v_2)$ are linearly independent solutions of (1.2). The system (1.2) is disconjugate in D if and only if $f(z)$ and $g(z)$ are 'relatively schlicht' in D, i.e. if

$$f(z_1) \neq g(z_2) \tag{1.4}$$

for every choice of $z_1, z_2 \in D$.

If $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are replaced by a different set of two linearly independent solutions of (1.2), then, according to (1.3), f and g are replaced by Tf and Tg, where T is given by

$$Tf = \frac{Af + B}{Cf + D}, \quad AD - BC \neq 0 . \tag{1.5}$$

It is therefore necessary, that any relation between the coefficients $p(z)$ and $q(z)$ of (1.2) and the functions $f(z)$ and $g(z)$ will remain invariant under the mapping $f \to Tf$, $g \to Tg$. Two combinations of f and g with this invariance property are

$$\Phi[f,g] = \frac{f'g'}{(f-g)^2}, \tag{1.6}$$

and

$$\Psi[f,g] = \frac{f''}{f'} - \frac{g''}{g'} - \frac{2(f'+g')}{f-g} . \tag{1.7}$$

The relations between the coefficients $p(z)$ and $q(z)$ of (1.2) and the functions $\Phi[f,g]$ and $\Psi[f,g]$ are given by

$$-p(z)q(z) = \Phi[f,g] \tag{1.8}$$

and

$$\frac{p'(z)}{p(z)} - \frac{q'(z)}{q(z)} = \Psi[f,g] . \tag{1.9}$$

Now, for functions $f(z)$ and $g(z)$ which are 'relatively schlicht' in $|z| < 1$ it is known [5, p.281, 6, Theorem 7.1] that

$$|\Phi[f,g]| = \frac{|f'(z)g'(z)|}{|f(z)-g(z)|^2} \le \frac{1}{(1-|z|^2)^2} , \quad |z| < 1 . \tag{1.10}$$

Utilizing this result one obtains the following necessary conditon. _If_ (1.3) _is disconjugate in_ $|z| < 1$ _then_

$$|p(z)q(z)| \le \frac{1}{(1-|z|^2)^2} , \quad |z| < 1. \tag{1.11}$$

Our principal aim in this paper is to generalize these results of Nehari to differential systems with $n \ge 3$. The ideas are also related to a recent paper by the author [2], where some function-theoretic aspects of disconjugancy of n-th order linear differential equations were considered.

2. <u>Mappings onto domains with empty intersection</u>.

Let $\chi_k(z) = (y_{1k}(z), y_{2k}(z), \ldots, y_{nk}(z))$ $k = 1,2,\ldots,n$, be n linearly independent solutions of (1.1), then the matrix $Y(z) = [y_{ik}(z)]_1^n$ is a fundamental solution of the matrix differential equation

$$Y'(z) = P(z)Y(z) \tag{2.1}$$

corresponding to (1.1), i.e. the determinant $\det[y_{ik}(z)]_1^n \neq 0$ for all $z \in D$. Without loss of generality we may assume that $y_{in}(z) \not\equiv 0$ $i = 1,2,\ldots,n$, and define the functions

$$f_{ik}(z) = \frac{y_{ik}(z)}{y_{in}(z)}, \quad i,k = 1,2,\ldots,n \tag{2.2}$$

which are meromorphic in D. Furthermore

$$\det[y_{ik}(z)]_1^n = \prod_{i=1}^{n} y_{in}(z) \det[f_{ik}(z)]_1^n. \tag{2.3}$$

Hence, $\det[f_{ik}(z)]_1^n \neq 0$ for all $z \in D$.

Let

$$H_i(z;a_1,\ldots,a_{n-1}) = \sum_{k=1}^{n-1} a_k f_{ik}(z), \quad i = 1,2,\ldots,n$$

and denote by $D_i(a_1,\ldots,a_{n-1})$ the image of D given by $H_i(z;a_1,\ldots,a_{n-1})$. We state now:

<u>Theorem 1</u>.

(1.1) <u>is disconjugate in</u> D <u>if and only if for every choice of complex constant</u> $a_1,\ldots,a_{n-1}$, <u>such that</u> $0 < \sum_{k=1}^{n-1} |a_k| < \infty$,

$$(2.5) \qquad \bigcap_{i=1}^{n} D_i(a_1,\ldots,a_{n-1}) = \emptyset$$

holds.

As pointed out by Schwarz [8, Theorem 3], disconjugancy of (1.1) in D is equivalent to the fact that for any fundamental solution $[y_{ik}(z)]_1^n$ of (2.1), we have $\det[y_{ik}(z_i)]_1^n \neq 0$ for every choice of n (not necessarily distinct) points $z_1, z_2, \ldots, z_n \in D$. According to (2.3) it follows now that disconjugancy of (1.1) in D is equivalent to

$$\prod_{i=1}^{n} y_{in}(z_i) \det[f_{ik}(z_i)]_1^n \neq 0$$

for every choice of $z_1,\ldots,z_n \in D$. Thus if $y_{in}(z) \neq 0$ $(i=1,2,\ldots,n)$ for all $z \in D$, the functions $f_{ik}(z)$ defined in (2.2) are regular in D and Theorem 1 follows from [8, Theorem 3]. But if we do not assume that $y_{in}(z) \neq 0$ the result does not follow immediately, and it is exactly the zeros of $y_{in}(z)$ that cause the difficulty in the proof of Theorem 1. To handle this we shall require the following two lemmas.

Lemma 1.

Given a set of n points $z_1, z_2, \ldots, z_n$ of D there always exists a solution $\underset{\sim}{y}(z)$ of (1.1) such that $y_i(z_i) \neq 0$, $i = 1,2,\ldots,n$.

Proof.

By the existence theorem there exists a solution $\underset{\sim}{u}(z)$ such that $u_1(z_1) = 1$. Suppose $u_2(z_2) = 0$, then by the same argument there exists a solution $\underset{\sim}{v}(z)$ such that $v_2(z_2) = 1$. If $v_1(z_1) = 0$ then $\underset{\sim}{y}(z) = \underset{\sim}{u}(z) + t\underset{\sim}{v}(z)$, $t \neq 0$, is a solution of (1.1) which satisfies $y_1(z_1) \neq 0$, $y_2(z_2) \neq 0$. Assume now that $\underset{\sim}{u}(z)$ and $\underset{\sim}{v}(z)$ are solutions of (1.1) which satisfy $u_i(z_i) = \alpha_i \neq 0$ $i = 1,2,\ldots,j < n$, $u_{j+1}(z_{j+1}) = 0$, $v_1(z_1) = 0$, $v_i(z_i) = \beta_i \neq 0$ $i = 2,\ldots,j+1$. If $t \neq -\alpha_i \beta_i^{-1}$, $i = 2,\ldots,j+1$, then $\underset{\sim}{y}(z) = \underset{\sim}{u}(z) + t\underset{\sim}{v}(z)$ will be a solution of (1.1) which satisfies $y_i(z_i) \neq 0$ $i = 1,2,\ldots,j+1$.

<u>Lemma 2</u>.

<u>If</u> (1.1) <u>is not disconjugate in</u> D, <u>and if</u> $y_{in}(z) \neq 0$ $i = 1,2,\ldots,n$, <u>then there exists a non-trivial solution</u> $\underset{\sim}{y}^*(z)$ <u>of</u> (1.1), <u>such that</u> $y_i^*(z_i^*) = 0$ <u>for</u> $z_i^* \epsilon D$, <u>and</u> $y_{in}(z_i^*) \neq 0$, $i = 1,2,\ldots,n$.

<u>Proof</u>.

Since (1.1) is not disconjugate in D, there exists a non-trivial solution $\underset{\sim}{y}(z)$, such that $y_i(z_i) = 0$ for $z_i \epsilon D$ $i = 1,2,\ldots,n$. If $y_{jn}(z_j) = 0$ for some $1 \leq j \leq n$, then apply a perturbation $\underset{\sim}{y}_\epsilon(z) = \underset{\sim}{y}(z) + \epsilon\underset{\sim}{u}(z)$, where $\underset{\sim}{u}(z)$ is a solution of (1.1) which satisfies $u_i(z_i) \neq 0$ $i = 1,2,\ldots,n$, and ϵ is a complex parameter. By making a proper choice of ϵ, say $\epsilon = \epsilon^*$, we obtain $\underset{\sim}{y}^*(z) = \underset{\sim}{y}_{\epsilon^*}(z)$, and $y_i^*(z_i^*) = 0$, where $z_i^* \epsilon D$, $i = 1,2,\ldots,n$. Furthermore, ϵ^* is chosen in such a way to guarantee that $y_{in}(z_i^*) \neq 0$.

We are ready now to prove Theorem 1.

<u>Proof of Theorem 1</u>.

(i) Suppose $b \in \bigcap_{i=1}^{n} D_i(a_1,\ldots,a_{n-1})$ for some choice of $a_1,\ldots,a_{n-1}$, such that $0 < \sum_{k=1}^{n-1} |a_k| < \infty$, then there exist n points $z_1,z_2,\ldots,z_n \epsilon D$ such that

$$H_i(z_i;a_1,\ldots,a_{n-1}) = \sum_{k=1}^{n-1} a_k f_{ik}(z_i) = b \quad i = 1,2,\ldots,n.$$

If $b = \infty$ then $y_{in}(z_i) = 0$ and (1.1) is not disconjugate. If $b \neq \infty$ then

$$y_i(z_i) = \sum_{k=1}^{n-1} a_k y_{ik}(z_i) - b y_{in}(z_i) = 0.$$

Indeed, if $y_{in}(z_i) \neq 0$ then evidently $y_i(z_i) = 0$, and if $y_{in}(z_i) = 0$, then it follows from $b \neq \infty$ that $\sum_{k=1}^{n-1} a_k y_{ik}(z_i) = 0$ and we have again $y_i(z_i) = 0$. Hence, disconjugancy of (1.1) in D implies (2.5).

(ii) Assume (1.1) is not disconjugate in D, i.e. there exists a non-trivial solution $\underset{\sim}{y}^*(z) = \sum_{k=1}^{n} a_k \underset{\sim}{y}_k(z)$ of (1.1) such that $y_i^*(z_i^*) = 0$ for $z_i^* \in D$ $i = 1,2,\ldots,n$. By Lemma 2 we may assume that $y_{in}(z_i^*) \neq 0$. Hence

$$\frac{y_i^*(z_i^*)}{y_{in}(z_i^*)} = \sum_{k=1}^{n-1} a_k f_{ik}(z_i^*) + a_n = 0, \quad i = 1,\ldots,n,$$

and $-a_n \in \bigcap_{i=1}^{n} D_i(a_1,\ldots,a_{n-1})$. This completes the proof of Theorem 1.

3. Relations between the coefficients $p_{ik}(z)$ and the functions $f_{ik}(z)$.

Replacement of $\underset{\sim}{y}_k(z)$ $(k = 1,2,\ldots,n)$ by another set of fundamental solutions $\underset{\sim}{w}_k(z)$ $(k = 1,2,\ldots,n)$ results in a transformation

$$(3.1) \quad f_{ik}(z) \to F_{ik}(z) = \frac{w_{ik}(z)}{w_{in}(z)} = \frac{\sum_{j=1}^{n} \alpha_{jk} f_{ij}(z)}{\sum_{j=1}^{n} \alpha_{jn} f_{ij}(z)}, \quad i,k=1,2,\ldots,n \quad \det[\alpha_{st}]_1^n \neq 0$$

applied to the matrix $[f_{ik}(z)]_1^n$. Hence, any relation between the entries of the matrices $[p_{ik}(z)]_1^n$ and $[f_{ik}(z)]_1^n$ must remain invariant under mappings of the type (3.1).

Without loss of generality we may assume that

$$(3.2) \qquad p_{ii}(z) \equiv 0 \quad i = 1,2,\ldots,n \,,$$

since this can be achieved by means of a transformation [8, p.489]

$$(3.3) \quad u_i(z) = \tau_i(z) y_i(z), \quad \tau_i(z) = c_i \exp \int_{z_0}^{z} p_{ii}(\zeta)\,d\zeta, \quad i=1,2,\ldots,n,$$

which leaves $f_{ik}(z)$ unchanged. Assuming (3.2) it is still possible to apply (3.3) with $\tau_i(z) = c_i \neq 0$ where c_i are arbitrary constants. This results in

$$(3.4) \qquad \underset{\sim}{u}'(z) = R(z)\underset{\sim}{u}(z), \quad R(z) = [r_{ik}(z)]_1^n$$

where

$$(3.5) \qquad r_{ik}(z) = p_{ik}(z)\frac{c_k}{c_i} \qquad i,k = 1,2,\ldots,n.$$

Therefore, the coefficients $p_{ik}(z)$ can be determined by the functions $f_{ik}(z)$ up to a relation of the type (3.5). It is easily verified by (3.5) that

$$(3.6)\qquad \sigma_{ij}(z) = p_{ij}(z)p_{ji}(z), \quad i \neq j, \quad i,j, = 1,2,\ldots,n$$

and

$$(3.7)\qquad \eta_{ij}(z) = \frac{p'_{ij}(z)}{p_{ij}(z)}, \qquad i \neq j, \quad i,j = 1,2,\ldots,n$$

are independent of the constants c_i. Next we prove that $\sigma_{ij}(z)$ and $\eta_{ij}(z)$ can be expressed in terms of the functions $f_{ik}(z)$, and therefore remain invariant under the group of transformations of the type (3.1). According to (2.2) we have $y_{ik}(z) = f_{ik}(z)y_{in}(z)$. Differentiating and using (1.1) we obtain

$$(3.8)\qquad \sum_{j=1}^{n} p_{ij}\frac{y_{jn}}{y_{in}}[f_{jk}- f_{ik}] = f'_{ik} \qquad k = 1,2,\ldots,n-1.$$

Thus for every fixed $1 \leq i \leq n$ we have (n-1) linear equations for the (n-1) unknown $p_{ij}\frac{y_{jn}}{y_{in}}$ $j \neq i$, $j = 1,2,\ldots,n$. The $(n-1)\times(n-1)$ matrix $m_{jk}(i,z) = f_{jk}(z) - f_{ik}(z)$ $j = 1,2,\ldots,i-1, i+1, \ldots,n$, $k = 1,2,\ldots,n-1$, satisfies $\det[m_{jk}(i,z)] = (-1)^{n+i}\det[f_{jk}(z)]_1^n \neq 0$ for all $z\in D$. Solving (3.8) we get

$$(3.9)\qquad p_{ij}\frac{y_{jn}}{y_{in}} = \frac{\det[h_{sk}(i,j,z)]_1^n}{\det[f_{sk}(z)]_1^n} \qquad i \neq j, \quad i,j = 1,2,\ldots,n$$

where

$$\left.\begin{aligned} h_{sk}(i,j,z) &= f_{sk}(z) \qquad s \neq j, \\ h_{jk}(i,j,z) &= f'_{ik}(z) \end{aligned}\right\} \begin{aligned} &s,k = 1,2,\ldots,n \\ &j \neq i,\ i,j = 1,2,\ldots,n. \end{aligned}$$

Setting now

$$(3.10)\qquad B_{ii}(z) = 0, \quad B_{ij}(z) = \frac{\det[h_{sk}(i,j,z)]}{\det[f_{sk}(z)]}, \ i \neq j, \quad i,j = 1,2,\ldots,n$$

it follows from (3.9) that

$$\sigma_{ij}(z) = p_{ij}(z)p_{ji}(z) = B_{ij}(z)B_{ji}(z)$$

$$(3.11)\qquad = \frac{\det[h_{sk}(i,j,z)]\det[h_{sk}(j,i,z))]}{(\det[f_{sk}(z)])^2}, \quad i\neq j, \quad i,j=1,2,\ldots,n,$$

and

$$(3.12)\qquad \eta_{ij}(z) = \frac{p'_{ij}(z)}{p_{ij}(z)} = \frac{B'_{ij}(z)}{B_{ij}(z)} + \sum_{k=1}^{n} [B_{ik}(z) - B_{jk}(z)], \ i\neq j, \ i,j=1,\ldots,n.$$

By Theorem 1, any condition for the functions $f_{ik}(z)$ $k = 1,2,\ldots,n$ to satisfy (2.5), which may be expressed in terms of $\sigma_{ij}(z)$ and $\eta_{ij}(z)$, is equivalent to conditions for disconjugancy of (1.1). For $n=2$, a known result in the theory of functions, namely inequality (1.10), was applied to yield the necessary condition for disconjugancy (1.11). Yet, for $n > 2$, we do not know of any necessary condition for the functions $f_{ik}(z)$ to satisfy (2.5). Conversely, in Section 7 a condition of this type will be deduced from necessary conditions for disconjugancy obtained in Theorem 5.

4. <u>A family of 'relatively schlicht' functions</u>.

Another way to generalize Nehari's principle [6, Theorem 1.1] is by generating a family of 'relatively schlicht' functions. Let

$$(4.1) \qquad g_j(z) = \frac{u_j(z)}{v_j(z)}, \quad g_k(z) = \frac{u_k(z)}{v_k(z)}, \quad j \neq k, \quad j,k = 1,2,\ldots,n$$

where $\underset{\sim}{u} = (u_1,\ldots,u_n)$ and $\underset{\sim}{v} = (v_1,\ldots,v_n)$ are linearly independent solutions of (1.1), which satisfy

$$(4.2) \qquad u_i(z_i) = v_i(z_i) = 0, \; i \neq j,k \quad i = 1,2,\ldots,n, \quad z_i \in D.$$

Denote by S_t the set of common zeros of $u_t(z)$ and $v_t(z)$ $t = 1,2,\ldots,n$. We assume that

$$(4.3) \qquad S_t \subset D, \; S_t \neq D \quad t = j,k.$$

In case $S_t = D$, $1 \leq t \leq n$, we do not define $g_t(z)$.

Evidently there always exists at least two linearly independent solutions of (1.1) which satisfy (4.2). (This is an immediate consequence of the existence of a fundamental set of n linearly independent solutions.) Moreover, if $z_i = a \in D$, $i \neq j,k$, $i = 1,2,\ldots,n$, then there exist exactly two linearly independent solutions such that $u_i(a) = v_i(a) = 0$, $i \neq j,k$ $i = 1,2,\ldots,n$. But in the general case, where some of the z_i may be distinct, it does not follow from the existence theorem that any three solutions of (1.1) which satisfy $y_i(z_i) = 0$ $i \neq j,k$ $i = 1,2,\ldots,n$, are linearly dependent. In Lemma 3, we discuss this situation.

Theorem 2.

Let $g_j(z)$ and $g_k(z)$ be defined by (4.1), where $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are any two linearly independent solutions of (1.1) which satisfy (4.2) and (4.3). In order that the system (1.1) be disconjugate in D, it is necessary and sufficient that for every choice of n points (not necessarily distinct) $z_1, z_2, \ldots, z_n$ of D, and every pair of functions $g_j(z)$ and $g_k(z)$

$$g_j(z_j) \neq g_k(z_k), \quad j \neq k, \quad j,k = 1,2,\ldots,n \tag{4.4}$$

will hold i.e. disconjugancy of (1.1) is equivalent to the 'relatively schlichtness' of all pairs of functions $g_j(z)$ and $g_k(z)$, $j \neq k$.

For the proof of Theorem 2 we require some preliminary prepositions which we state as a lemma.

Lemma 3.

Suppose there exist three linearly independent solutions $y(z)$, $v(z)$ and $w(z)$, which satisfy $y_i(z_i) = v_i(z_i) = w_i(z_i) = 0$ $i = 1,2,\ldots,n-2$, $z_i \epsilon D$ then

(i) (1.1) *is not disconjugate in* D.

(ii) *There exists a pair of functions $g_j(z)$ and $g_k(z)$ $j \neq k$ which are not 'relatively schlicht' in D. i.e. $g_j(\zeta_j) = g_k(\zeta_k)$ for some $\zeta_j, \zeta_k \epsilon D$.*

Proof.

(i) Let z_{n-1}, $z_n \epsilon D$. There always exists a non-trivial solution $\underset{\sim}{u}(z) = \alpha_1 \underset{\sim}{y}(z) + \alpha_2 \underset{\sim}{v}(z) + \alpha_3 \underset{\sim}{w}(z)$ which satisfies $u_{n-1}(z_{n-1}) = u_n(z_n) = 0$. Hence (1.1) is not disconjugate in D, since $u_i(z_i) = 0$ $i = 1,2,\ldots,n$.

(ii) We first make the following remark. Since $\underset{\sim}{y}(z)$ and $\underset{\sim}{v}(z)$ are linearly independent solutions, then at least one component of each solution, say $y_s(z)$, and $v_m(z)$, $1 \leq s$, $m \leq n$, $s \neq m$ are not identically zero. Hence, we may assume that at least two components of $\underset{\sim}{v}(z)$ are not identically zero. Suppose now that

$$v_{n-1}(z) \not\equiv 0, \quad v_n(z) \not\equiv 0, \quad z \in D \tag{4.5}$$

and let $z_{n-1}, z_n \in D$ be such that $v_{n-1}(z_{n-1}) \neq 0$ and $v_n(z_n) \neq 0$, then the functions $g_{n-1}(z)$ and $g_n(z)$, where $g_t(z) = \frac{u_t(z)}{v_t(z)}$ $t = n-1, n$ are not 'relatively schlicht' in D since $g_{n-1}(z_{n-1}) = g_n(z_n) = 0$.

In case (4.5) is false and $y_{n-1}(z) \equiv v_{n-1}(z) \equiv w_{n-1}(z) \equiv 0$ we assume that $v_1(z) \not\equiv 0$, $v_n(z) \not\equiv 0$. Let $\zeta_1, \zeta_n \in D$ be such that $v_1(\zeta_1) \neq 0$, $v_n(\zeta_n) \neq 0$. Proceeding as before there exists a non-trivial solution $\underset{\sim}{u}(z) = \alpha_1 \underset{\sim}{y}(z) + \alpha_2 \underset{\sim}{v}(z) + \alpha_3 \underset{\sim}{w}(z)$ such that $u_1(\zeta_1) = 0$, $u_i(z_i) = 0$, $i = 2,\ldots,n-2$, $u_{n-1}(z) \equiv 0$, $u_n(\zeta_n) = 0$, and $g_1(\zeta_1) = g_n(\zeta_n) = 0$. If $y_t(z) \equiv v_t(z) \equiv w_t(z) \equiv 0$ for $t = n-1, n$ we may assume that $v_1(z) \not\equiv 0$, $v_2(z) \not\equiv 0$ and proceed as before.

<u>Proof of Theorem 2</u>.

(i) Necessary. Suppose $g_j(z_j) = g_k(z_k) = \beta\alpha^{-1}$, then $\underset{\sim}{y}(z) = \alpha \underset{\sim}{u}(z) - \beta \underset{\sim}{v}(z)$ satisfies $y_i(z_i) = 0$ $i = 1,2,\ldots,n$.

(ii) Sufficient. Suppose there exists a solution $\underset{\sim}{u}(z)$ such that $u_i(z_i) = 0$ $i = 1,2,\ldots,n$, $z_i \in D$. Let $\underset{\sim}{v}(z)$ be a solution of (1.1), which is linearly independent on $\underset{\sim}{u}(z)$ and satisfies $v_i(z_i) = 0 \quad i = 1,2,\ldots,n-2$. Now if

$$(4.6) \qquad v_{n-1}(z_{n-1}) \neq 0, \quad v_n(z_n) \neq 0$$

then $g_{n-1}(z_{n-1}) = g_n(z_n) = 0$, So suppose (4.6) is false and $v_{n-1}(z_{n-1}) = 0$. Assume $S_n \neq D$, where S_n denotes the set of common zeros of $u_n(z)$ and $v_n(z)$ and let $\zeta_n \notin S_n$. There exists a non-trivial solution $\underset{\sim}{y}(z) = \alpha_1 \underset{\sim}{u}(z) + \alpha_2 \underset{\sim}{v}(z)$ such that $y_n(\zeta_n) = 0$ and $y_i(z_i) = 0 \quad i = 1,2,\ldots,n-1$. Moreover there exists another solution $\underset{\sim}{w}(z)$, which is linearly independent of $\underset{\sim}{y}(z)$ and satisfies $w_i(z_i) = 0 \; i=3,4,\ldots,n-1$, $w_n(\zeta_n) = 0$. Now $w_t(z_t) \neq 0 \quad t = 1,2$. Because, if $w_2(z_2) = 0$ then $u_i(z_i) = v_i(z_i) = w_i(z_i) = 0 \quad i = 2,\ldots,n-1$ and by Lemma 3, it follows from the 'relatively schlichtness' in D of every pair of functions $g_j(z)$ and $g_k(z)$ that $\underset{\sim}{w}(z) = \beta_1 \underset{\sim}{u}(z) + \beta_2 \underset{\sim}{v}(z)$. But since $\underset{\sim}{w}(z)$ and $\underset{\sim}{y}(z)$ are linearly independent it follows now from $w_n(\zeta_n) = y_n(\zeta_n) = 0$ that $u_n(\zeta_n) = v_n(\zeta_n) = 0$, which contradicts our assumption that $\zeta_n \notin S_n$. So $w_2(z_2) \neq 0$ and similarily $w_1(z_1) \neq 0$. Considering now the functions $g_t(z) = \dfrac{y_t(z)}{w_t(z)} \quad t = 1,2$, it follows that $g_1(z_1) = g_2(z_2) = 0$. If $S_n = D$, we may assume that $S_1 \neq D$ and proceed as before.

5. Quantities invariant under the mapping $f \to Tf$, $g \to Tg$.

Our next goal is to establish relations between the coefficients $p_{ik}(z)$ of the system (1.1) and the functions $g_j(z)$ and $g_k(z)$ defined by (4.1). As has become by now a standard procedure, we have to find out first what kind of transformations may be applied to g_j and g_k without affecting their relations with the coefficients p_{ik}. If $\underset{\sim}{u}(z)$ and $\underset{\sim}{v}(z)$ are replaced by the linearly independent solutions $A\underset{\sim}{u}(z) + B\underset{\sim}{v}(z)$ and $C\underset{\sim}{u}(z) + D\underset{\sim}{v}(z)$ respectively, then according to (4.1), g_j and g_k are replaced by Tg_j and Tg_k, where T is the linear transformation (1.5). Therefore any relation between the coefficients p_{ik} and the functions g_j and g_k should be expressed by quantities which remain invariant under the transformation $g_t \to Tg_t \quad t = j,k$.

This brings up the following question. Given two meromorphic functions, $f(z)$ and $g(z)$, in a domain D, what combinations of $f(z)$ and $g(z)$ and their derivatives remain invariant under the transformation $f \to Tf$, $g \to Tg$. Two combinations of this type were given by Nehari, namely $\Phi[f,g]$ and $\Psi[f,g]$ which are defined by (1.6) and (1.7). By differentiating $\Psi[f,g]$ and $\Phi[f,g]$ it is possible to derive more quantities with this invariance property. One combination of this type which will be of interest later is

$$(5.1) \qquad \Theta[f,g] = \frac{f''}{f'} - \frac{2f'}{f-g} = \frac{1}{2}\,\frac{\Phi'[f,g]}{\Phi[f,g]} + \Psi[f,g] \quad .$$

In the following theorem we shall prove that with some restrictions on the functions $f(z)$ and $g(z)$, every combination of $f(z)$ and $g(z)$ with the desired invariance property can be derived from $\Phi[f,g]$ and $\Theta[f,g]$.

Denote by $RC(D)$ the restricted class in D (see [7], p. 159), namely the class of functions $\{f(z)\}$ which are meromorphic in D with simple poles at most and which satisfy $f'(z) \neq 0$ for all $z \in D$. Note that if f belongs to $RC(D)$ so does Tf.

Theorem 3.

Let $f(z) \in RC(D)$, and let $g(z)$ be a meromorphic function in D such that

$$f(z) \neq g(z), \quad z \in D. \tag{5.2}$$

Let $E[f(z),g(z)] = E(f(z),\ldots,f^{(n)}(z),g(z),\ldots,g^{(n)}(z))$ be a combination of $f(z)$ and $g(z)$ and their derivatives up to order n. If $E(f(z),g(z)]$ remains invariant under the transformaton $f \to Tf$, $g \to Tg$, i.e.,

$$E[Tf(z),Tg(z)] = E[f(z),g(z)] = I(z) \tag{5.3}$$

where T is defined by (1.5), then $E[f(z),g(z)]$ may be derived from $\Phi[f(z),g(z)] = \varphi(z)$ and $\Theta[f(z),g(z)] = \theta(z)$, and

$$I(z) = E[f(z),g(z)] = E^*[\varphi(z),\theta(z)] \tag{5.4}$$

where E^* is a combination of $\varphi(z)$ and $\theta(z)$ and their derivatives up to order $n-1$.

Proof.

Let $z_o \in D$. Without loss of generality we may assume that

$f(z_o) = 0$, $f'(z_o) = 1$, $f''(z_o) = 0$, since this situation may be achieved by means of a transformation $f \to Tf$, $g \to Tg$, [2, Th. 2] which, according to (5.3), leaves $I(z)$ unchanged. It follows now from (5.2) that $g(z_o) = \gamma \neq 0$. If $\gamma \neq \infty$, then by applying the transformaton $f \to [1-\gamma^{-1}f]$, $g \to [1-\gamma^{-1}g]$, we obtain

$$(5.5) \qquad f(z_o) = 1, \quad g(z_o) = 0, \quad f'(z_o) \neq 0, \quad f''(z_o) = 0.$$

Setting now $z = z_o$ in (5.1) and (1.6), it follows from (5.5) that

$$(5.6) \qquad f'(z_o) = -\frac{\theta(z_o)}{2} \neq 0, \quad g'(z_o) = -\frac{2\varphi(z_o)}{\theta(z_o)} .$$

Differentiation of (1.6) and (5.1) gives us

$$(5.7) \qquad \varphi^{(m)}(z) = \frac{g^{(m+1)}(z)f'(z)}{[f(z)-g(z)]^2} + \frac{M_m[f(z),g(z)]}{[f(z)-g(z)]^{m+2}}, \quad m=0,1,2,\ldots,$$

and

$$(5.8) \qquad \theta^{(m)}(z) = \frac{f^{(m+2)}(z)}{f'(z)} + \frac{N_m[f(z),g(z)]}{[f(z)-g(z)]^{m+1}[f'(z)]^{m+1}}, \quad m=1,2,\ldots,$$

where M_m and N_m are polynomials of $f(z)$, $f'(z_o),\ldots,f^{(m+1)}(z)$, and $g(z),g'(z),\ldots,g^{(m)}(z)$. By ellimination and induction it follows now from (5.5), (5.6), (5.7) and (5.8) that

$$(5.9) \qquad g^{(m+1)}(z_o) = -\frac{2\varphi^{(m)}(z_o)}{\theta(z_o)} + R_m[\theta(z_o),\ldots,\theta^{(m)}(z_o),\varphi(z_o),\ldots,\varphi^{(m-1)}(z)]$$

$$m=0,1,2,\ldots,$$

and

$$(5.10) \qquad f^{(m+2)}(z_o) = -\frac{2\theta^{(m)}(z_o)}{\theta(z_o)} + \tilde{R}_m[\theta(z_o),\ldots,\theta^{(m-1)}(z_o),\varphi(z_o),\ldots,\varphi^{(m-1)}(z_o)]$$

$$m=1,2,\ldots,$$

where R_m and $\tilde{R}_m$ are rational functions whose denominators are powers of $\theta(z_o)$. Insertion of (5.6),(5.9) and (5.10) in $E[f(z),g(z)]$ yields

$$(5.11)\quad I(z_o)= E[f(z_o),g(z_o)]= E^*[\theta(z_o),\ldots,\theta^{(n-1)}(z_o),\varphi(z_o),\ldots,\varphi^{(n-1)}(z_o)].$$

In case we have $f(z_o) = 0$, $f'(z_o) = 1$, $f''(z_o) = 0$, $g(z_o)= \infty$ for $z_o \epsilon D$, then by applying the transformation $f(z) \to [1-f(z)]^{-1}$ $g(z) \to [1-g(z)]^{-1}$ we obtain

$$(5.12)\qquad f(z_o) = 1,\ f'(z_o) = 1,\ f''(z_o) = 2,\ g(z_o) = 0.$$

Setting now $z = z_o$ in (5.1) and (1.6) we obtain according to (5.12)

$$(5.6')\qquad \theta(z_o) = 0, \quad \varphi(z_o) = g'(z_o).$$

The derivatives of $f(z)$ and $g(z)$ at the point $z = z_o$, may be eliminated successively from (5.7) and (5.8) as before. This leads us now to

$$(5.9')\qquad g^{(m+1)}(z_o)= \varphi^{(m)}(z_o)+ \tilde{M}_m[\theta(z_o),\varphi(z_o)], \quad m=1,2,\ldots,$$

and

$$(5.10')\qquad f^{(m+2)}(z_o)= \theta^{(m)}(z)+ \tilde{N}_m[\theta(z_o),\varphi(z_o)], \quad m=1,2,\ldots,$$

where $\tilde{M}_m$ and $\tilde{N}_m$ are polynomials of the arguments $\theta^{(s)}(z_o)$ and $\varphi^{(s)}(z_o)$ $s = 0,1,\ldots,n-1$. Insertion of (5.12),(5.6'), (5.9') and (5.10') in $E[f(z),g(z)]$ yields a relation of the type (5.11).

Remark.

It is easily confirmed that for $f(z)$ and $g(z)$ satisfying the assumptions of Theorem 3, $\varphi(z) = \Phi[f(z),g(z)]$ and $\theta(z) = [f(z),g(z)]$ are regular functions in D. Moreover, $\varphi(z) \neq 0$ for $z \in D$, if and only if in addition to the assumptions of the theorem we have $g(z) \in RC(D)$. For $f(z), g(z) \in RC(D)$ satisfying (5.2), the function $\psi(z) = \Psi[f(z),g(z)]$ is also regular in D and a theorem similar to Theorem 3 may be established with $\theta[f,g]$ replaced by $\Psi[f,g]$.

6. A subfamily of 'relatively schlicht' functions.

For the applications it is useful to consider only a subfamily of functions of the type (4.1); namely:

$$(6.1) \qquad g_j(z,a) = \frac{u_j(z)}{v_j(z)}, \quad g_k(z,a) = \frac{u_k(z)}{v_k(z)}, \quad j \neq k,$$

where $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are linearly independent solutions of (1.1), satisfying

$$(6.2) \quad u_i(a) = v_i(a) = 0, \quad i \neq j,k \quad i = 1,2,\ldots,n, \ a \in D.$$

Before taking the problem of establishing relations between the functions (6.1) and the coefficients $p_{ik}(z)$ of (1.1), we first make the following remarks.

(i) As laready discussed in Section 4, there exists exactly two linearly independent solutions satisfying (6.2). Therefore any other solution of (1.1) which satisfy $y_i(a) = 0$ $i \neq j,k$, $i = 1,\ldots,n-2$ is a linear combination of $\underset{\sim}{u}$ and $\underset{\sim}{v}$. Hence, replacement of $\underset{\sim}{u}$ and $\underset{\sim}{v}$ by another set of two linearly independent solutions $\underset{\sim}{y},\underset{\sim}{w}$ satisfying $y_i(a) = w_i(a) = 0$ $i \neq j,k$ $i = 1,2,\ldots,n$, results in a transformation $g_t(z,a) \rightarrow Tg_t(z,a)$ $t = j,k$, where T is defined by (1.5). It follows that the relations between the functions (6.1) and the coefficients $p_{ik}(z)$ must stay invariant under the transformaton $g_t \rightarrow Tg_t$ $t = j,k$.

(ii) Since the transformation (3.3) leaves the functions (6.1) unchanged we may assume that $p_{ii}(z) \equiv 0$ $i = 1,2,\ldots,n$. In this case the coefficients $p_{ik}(z)$ can be determined by the functions (6.1) only up to a relation of the type (3.5).

Theorem 4.

Let $p_{ik}(z)$ $i,k = 1,2,\ldots,n$ *be regular functions in* D *and assume*

(3.3) $$p_{ii}(z) \equiv 0 \cdot i = 1,2,\ldots,n.$$

Let the functions $g_j(z,a)$ *and* $g_k(z,a)$ *be defined by* (6.1) *where* $\underset{\sim}{u}$ *and* $\underset{\sim}{v}$ *are linearly independent solutions of* (1.1) *satisfying* (6.2). *If*

(6.3) $$\varphi_{jk}(z,a) = \Phi[g_j(z,a)\ ,g_k(z,a)] = \frac{g_j' g_k'}{(g_j - g_k)^2}$$

and

(6.4) $$\theta_{jk}(z,a) = \Theta[g_j(z,a), g_k(z,a)] = \frac{g_j''}{g_j'} - \frac{2g_j'}{g_j - g_k}$$

where $g_t' = \frac{d}{dz}[g_t(z,a)]$, $t = j,k$ *then*

(6.5) $$\varphi_{jk}(a,a) = -p_{jk}(a)p_{kj}(a), \quad j \neq k, \quad j,k = 1,2,\ldots,n, \ a \in D$$

and if $p_{jk}(a) \neq 0$, *then*

(6.6) $$\theta_{jk}(a,a) = \frac{p_{jk}'(a)}{p_{jk}(a)} + \frac{\sum_{i=1}^{n} p_{ji}(a)p_{ik}(a)}{p_{jk}(a)}, \ j \neq k, \quad j,k=1,2,\ldots,n, \ a \in D.$$

Proof.

Let $\underset{\sim}{u}(z)$ and $\underset{\sim}{v}(z)$ satisfy

(6.7) $$u_i(a) = \delta_{ik}, \ v_i(a) = \delta_{ij}, \quad j \neq k, \quad i=1,2,\ldots,n, \quad 1 \leq j,k \leq n.$$

According to (1.1) and (6.1) we have

(6.8) $$g_t'(z,a) = \frac{\sum_{i=1}^{n} p_{ti}(z)\,[u_i(z)v_t(z) - u_t(z)v_i(z)]}{v_t^2(z)}.$$

Therefore,

$$(6.9) \qquad \varphi_{jk}(z,a) = \frac{\sum_{i=1}^{n} p_{ji}[u_i v_j - u_j v_i] \sum_{s=1}^{n} p_{ks}[u_s v_k - u_k v_s]}{[u_j v_k - u_k v_j]^2},$$

and (6.5) follows now from (6.9) and (6.7). By setting $t=j$ and $z=a$ in (6.8) we obtain $g_j^!(a,a) = p_{jk}(a)$. Hence if $p_{jk}(a) \neq 0$ for $a \in D$, $g_j(z,a)$ belongs to the restricted class of functions in some neighborhood $N(a) \subset D$ of the point a. Obviously both $g_j(z,a)$ and $g_k(z,a)$ are meromorphic functions in D. So, we conclude now that $\theta_{jk}(z,a)$ is regular in $N(a)$. By differentiating (6.8) and using (6.7) we obtain (6.6).

Since any solution of (1.1) which satisfies $y_i(a) = 0$ $i \neq j,k$ $i=1,2,\ldots,n$, is a linear combination of the normalized solutions $\underset{\sim}{u}(z)$ and $\underset{\sim}{v}(z)$ which satisfy (6.7), a different choice of the two solutions would replace g_t by Tg_t, $(t=j,k)$ where T is of the form (1.5). But $\varphi(z,a)$ and $\theta(z,a)$ are not affected by this transformation, hence (6.5) and (6.6) hold for any choice of the solutions $\underset{\sim}{u}(z)$ and $\underset{\sim}{v}(z)$ regardless of the normalization (6.7).

<u>Remarks</u>.

1. Note that (6.5) holds even without the assumption (3.3), but in this case $p_{ii}(z)$ are not determined by the functions (6.1).

2. If $p_{jk}(z) \neq 0$ for all $z \in D$, $j \neq k$, $j,k=1,2,\ldots,n$, then (6.5) and (6.6) are the 'fundamental relations' between the functions $g_j(z,a)$ and $g_k(z,a)$ and the coefficients $p_{jk}(z)$ of (1.1).

7. Necessary conditions for disconjugancy in the unit disk.

Theorem 5.

Let $p_{jk}(z)$ $j,k=1,2,\ldots,n$ be regular for $|z| < 1$. If the system (1.1) is disconjugate in $|z| < 1$, then

(7.1) $$|p_{jk}(z)p_{kj}(z) \leq \frac{1}{(1-|z|^2)^2}, \quad |z|< 1.$$

Proof.

By Theorem 2 disconjugancy of (1.1) in $|z|< 1$ implies the 'relatively schlichtness' in $|z|< 1$ of every pair of functions $g_j(z)$ and $g_k(z)$ defined by (4.1). In particular $g_j(z,a)$ and $g_k(z,a)$ defined by (6.1) are 'relatively schlicht'. Applying (1.10), it follows that

(7.2) $$|\varphi_{jk}(z,a)| = |\Phi[g_j(z,a),g_k(z,a)]| \leq \frac{1}{(1-|z|^2)^2}, \quad |z|< 1$$

holds for every $j,k=1,2,\ldots,n$ $j\neq k$, and any $|a|< 1$. Setting $z=a$ in (7.2) we obtain by (6.5)

$$|p_{jk}(a)p_{kj}(a)| = |\varphi_{jk}(a,a)| \leq \frac{1}{(1-|a|^2)^2}, \quad |a|< 1.$$

We add the following remarks.

(i) Since $\Theta[g_j(z,a),\ g_k(z,a)]$ cannot be bounded without the further assumption that $g_j(z,a)$ is univalent in z for $|z|< 1$, (6.6) does not yield a necessary condition for disconjugancy. Moreover, in order to obtain a bound for $\Psi[g_j(z,a),g_k(z,a)]$, one has to assume that both $g_j(z,a)$ and $g_k(z,a)$ are univalent in $|z|< 1$, besides being 'relatively schlicht' there [6, Th. 7.2].

(ii) Let $\pi_{ik}(\zeta)$ $i,k=1,2,\ldots,n$ be regular in the domain Δ, and consider the differential system

$$(7.3) \qquad \underset{\sim}{\omega}'(\zeta) = \Pi(\zeta)\,\underset{\sim}{\omega}(\zeta)$$

where $\underset{\sim}{\omega}(\zeta) = [\omega_1(\zeta), \omega_2(\zeta), \ldots, \omega_n(\zeta)]$ and $\Pi(\zeta) = [\pi_{ik}(\zeta)]_1^n$. If Δ is conformally equivalent to D, i.e., if there exists an one-to-one regular function $\zeta(z)$ which maps D onto Δ, then (7.3) may be transformed by $y_j(z) = \omega_j[\zeta(z)]$ $j=1,\ldots,n$ into the system (1.1) and

$$(7.4) \qquad p_{jk}(z)p_{kj}(z) = \pi_{jk}[\zeta(z)]\pi_{kj}[\zeta(z)]\left(\frac{d\zeta}{dz}\right)^2$$

holds. Furthermore, (7.3) is disconjugate in Δ if and only if the transformed system (1.1) is disconjugate in D. Thus, in view of (7.4), Theorem 5 yields a necessary condition for disconjugancy in any domain Δ which is conformally equivalent to the unit disk.

We conclude this section with the following corollary.

<u>Let</u> $f_{ik}(z)$ $i,k=1,2,\ldots,n$ <u>be regular functions in the unit disk</u> D, <u>such that</u> $f_{in}(z) \equiv 1$ $i=1,\ldots,n$, <u>and</u> $\det[f_{ik}(z)]_1^n \neq 0$ <u>for</u> $z \in D$. <u>Let</u> $H_i(z;a_1,\ldots,a_{n-1})$ <u>be defined as in</u> (2.4), <u>and denote by</u> $D_i(a_1,\ldots,a_{n-1})$ <u>the image of</u> D <u>given by</u> $H_i(z;a_1,\ldots,a_{n-1})$. <u>If</u>

$$(2.5) \qquad \bigcap_{i=1}^{n} D_i(a_1,\ldots,a_{n-1}) = \emptyset$$

<u>for every choice of the constants</u> $a_1,\ldots,a_{n-1}$, <u>such that</u> $0 < \sum_{k=1}^{n-1} |a_k| < \infty$, <u>then</u>

$$(7.5) \qquad |B_{ij}(z)B_{ji}(z)| \leq \frac{1}{(1-|z|^2)^2}, \quad i \neq j, \quad i,j=1,\ldots,n, \quad |z|<1,$$

<u>where</u> $B_{ij}(z)$ <u>are</u> <u>defined</u> <u>by</u> (3.10).

<u>Proof</u>.

By Theorem 1, (2.5) implies the disconjugancy of the corresponding system (1.1). According to (3.11) and (6.5) the result follows.

<u>Remarks</u>. (i) (7.5) is a generalization of (1.10) for the case $n > 2$. (ii) Since $B_{ij}(z)B_{ji}(z)$ remains invariant when $f_{ik}(z)$ is subject to a transformation of the type (3.1), our result may be generalized to meromorphic functions $F_{ik}(z)$, $i,k=1,\ldots,n$, which are obtained from $f_{ik}(z)$ by means of (3.1).

8. Disfocality of n-th order differential equations.

In the special case where

$$(8.1) \qquad P(z) = \begin{bmatrix} 0 & 1 & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & . & 0 & 1 \\ -q_n & -q_{n-1} & . & . & . & . & . & -q_2 & -q_1 \end{bmatrix}$$

the column vector $\underset{\sim}{y}(z) = [y_1(z),\dots,y_n(z)]$ of (1.1) becomes $[w(z),w'(z),\dots,w^{(n-1)}(z)]$ and (1.1) is equivalent to the differential equation

$$(8.2) \qquad w^{(n)}(z) + q_1(z)w^{(n-1)}(z) + \dots + q_n(z)w(z) = 0.$$

In this case disconjugancy of (1.1) in D is equivalent to disfocality of (8.2) in the same domain D. (8.2) is called disfocal in D if for every choice of n (not necessarily distinct) points $z_1,\dots,z_n$ of D, the only solution of (8.2) satisfying $w(z_1) = w'(z_2) = \dots w^{(n-1)}(z_n) = 0$, is the trivial one $w(z) \equiv 0$. (See [6]).

Let $q_k(z)$ $k=1,2,\dots,n$ be regular functions in $|z|< 1$. If (8.2) is disfocal in $|z|< 1$, it follows from (6.5) and (8.1) that

$$(8.3) \qquad |q_2(z)| \leq \frac{1}{(1-|z|^2)^2}, \qquad |z|< 1.$$

But (6.5) does not yield bounds for the other coefficients of (8.2), since by (8.1) $p_{in}(z) \equiv 0$ for $i=1,2,\dots,n-2$. Yet such bounds may be obtained by slight modifications of Theorem 4 and 5.

Theorem 6.

Let $q_k(z)$ $k=1,2,\ldots,n$ be regular in the domain D, and let $u(z)$ and $v(z)$ be linearly independent solutions of (8.2) which satisfy

$$(8.4)\quad u^{(s)}(a) = v^{(s)}(a) = 0,\ s=0,1,\ldots,n-1,\ s\neq j-1,j \quad 1 \le j \le n-1,\ a\in D.$$

Let

$$(8.5)\quad g_j(z,a) = \frac{u^{(j-1)}(z)}{v^{(j-1)}(z)}, \quad g_{j+1}(z,a) = \frac{u^{(j)}(z)}{v^{(j)}(z)}, \quad j=1,2,\ldots,n-1.$$

If

$$(8.6)\quad \varphi_{j,j+1}(z,a) = \Phi[g_j(z,a), g_{j+1}(z,a)] = \frac{g_j' g_{j+1}'}{(g_j - g_{j+1})^2} \quad j=1,2,\ldots,n-1$$

and

$$(8.7)\quad \theta_{n-1,n}(z,a) = \Theta[g_{n-1}(z,a), g_n(z,a)] = \frac{g_{n-1}''}{g_{n-1}'} - \frac{2g_{n-1}'}{g_{n-1} - g_n}$$

then

$$\left.\begin{aligned} &(8.8)\quad \varphi_{j,j+1}(a,a) = \varphi_{j,j+1}'(a,a) = \ldots = \varphi_{j,j+1}^{(n-j-2)}(a,a) = 0, \\ &(8.9)\quad \varphi_{j,j+1}^{(n-j-1)}(a,a) = q_{n-j+1}(a) \end{aligned}\right\} \quad j=1,2,\ldots,n-1$$

and

$$(8.10)\qquad \theta_{n-1,n}(a,a) = -q_1(a).$$

All derivatives are with respect to z.

Proof.

Since (8.6) and (8.7) remain invariant under the transformation $g_t \to Tg_t$ $t=j,j+1$, where T is given by (1.5), we may assume that

$$(8.11)\qquad u(z)= w_j(z), \quad v(z)= w_{j+1}(z), \quad 1 \le j \le n-1$$

where $w_t(z)$ $t=1,2,\ldots,n$ is a fundamental set of solutions of (8.2) which satisfy

$$w_t^{(s-1)}(a) = \delta_{st}, \quad s,t=1,2,\ldots,n. \tag{8.12}$$

This assumption results in simplification of the calculations. According to (8.5) and (8.11) we obtain now

$$g_j'(z,a) = \frac{w_j^{(j)}(z)w_{j+1}^{(j-1)}(z) - w_j^{(j-1)}(z)w_{j+1}^{(j)}(z)}{[w_{j+1}^{(j-1)}(z)]^2} = \frac{L_j(z)}{[w_{j+1}^{(j-1)}(z)]^2}$$

$$g_{j+1}'(z,a) = \frac{w_j^{(j+1)}(z)w_{j+1}^{(j)}(z) - w_j^{(j)}(z)w_{j+1}^{(j+1)}(z)}{[w_{j+1}^{(j)}(z)]^2} = \frac{K_j(z)}{[w_{j+1}^{(j)}(z)]^2} .$$

Hence

$$\varphi_{j,j+1}(z,a) = \frac{K_j(z)}{L_j(z)} . \tag{8.13}$$

By (8.12) we obtain for $z=a$

$$L_j(a) = -1, \; K_j(a) = K_j'(a) = \ldots = K_j^{(n-j-2)}(a) = 0, \; j=1,2,\ldots,n-1 \tag{8.14}$$

and

$$K_j^{(n-j-1)}(a) = w_j^{(n)}(a) = -q_{n-j+1}(a), \qquad j=1,2,\ldots,n-1. \tag{8.15}$$

(8.8) and (8.9) follow now from (8.13), (8.14) and (8.15).

In a similar way, it is easily verified that

$$\theta_{n-1,n}(z,a) = \frac{L_{n-1}'(z)}{L_{n-1}(z)} .$$

Setting $z=a$, (8.10) follows.

We apply now Theorem 6 in order to obtain necessary conditions for disfocality of (8.2) in the unit disk.

Theorem 7.

Let $q_k(z)$ $k=1,2,\ldots,n$ be regular in the unit disk. If equation (8.2) is disfocal in $|z|<1$, then

$$|q_k(z)| \le \frac{A_k}{(1-|z|^2)^k} \qquad k=2,3,\ldots,n, \quad |z|<1 \tag{8.16}$$

where

$$A_2 = 1, \quad A_k = (k-2)!\left(\frac{k+2}{4}\right)^2\left(\frac{k+2}{k-2}\right)^{\frac{k-2}{2}}, \qquad k=3,4,\ldots,n. \tag{8.17}$$

We require the following elementary result for the proof of Theorem 7.

Lemma 4.

Let $h_k(z)$, $k=1,2,\ldots,$ be a regular function in $|z|<1$. If

$$|h_k(z)| \le \frac{1}{(1-|z|^2)^k}, \qquad |z|<1, \tag{8.18}$$

then

$$|h_k^{(s)}(z)| \le \frac{C(s,k)}{(1-|z|^2)^{s+k}}, \quad |z|<1, \quad s=1,2,\ldots, \tag{8.19}$$

where $C(s,k)$ are constants depending only on s and k.

Proof.

Let $h_k(z) = \sum_{j=0}^{\infty} b_j z^j$, then by Cauchy inequality

$$|b_j| \le r^{-j}M(r), \quad M(r) = \max_{|z|=r<1} |h_k(z)|.$$

By (8.18) $M(r) \le (1-r^2)^{-k}$, therefore

$$|b_j| \le \min_{0<r<1} r^{-j}(1-r^2)^{-k} = m(j,k) = \left(\frac{2k+j}{2k}\right)^k\left(\frac{2k+j}{j}\right)^{j/2}, \quad j=1,2,\ldots \tag{8.20}$$

Set

$$(8.21)\qquad \eta_k(\zeta) = h_k[z(\zeta)]\left(\frac{dz}{d\zeta}\right)^k, \quad z(\zeta) = \frac{\zeta+a}{1+\bar{a}\zeta}, \quad |a|< 1 .$$

$z(\zeta)$ is a mapping of $|\zeta|< 1$ onto $|z|< 1$, and therefore $\eta_k(\zeta) = \sum_{j=0}^{\infty} \beta_j \zeta^j$ is regular in $|\zeta|< 1$. Moreover, since

$$\left|\frac{dz}{d\zeta}\right| = \frac{1-|z|^2}{1-|\zeta|^2}$$

it follows from (8.18) that

$$(8.22)\qquad |\eta_k(\zeta)| \le \frac{1}{(1-|\zeta|^2)^k}, \quad |\zeta|< 1.$$

Consequently

$$(8.23)\qquad |\beta_j| \le m(j,k), \quad j=1,2,\ldots$$

Differentiation of (8.21) leads us to

$$(8.24)\qquad h_k'(z) = \eta_k'(\zeta)\left(\frac{d\zeta}{dz}\right)^{k+1} + k\eta_k(\zeta)\left(\frac{d\zeta}{dz}\right)^{k-1}\frac{d^2\zeta}{dz^2} .$$

It is easily confirmed that

$$|\zeta''(z)| \le \frac{2|z|}{(1-|z|^2)^2}, \quad |z|< 1,$$

and by setting now $\zeta=0$ in (8.24) we obtain

$$(8.25)\qquad |h_k'(a)| \le \frac{|\eta_k'(0)|+ 2k|a||\eta_k(0)|}{(1-|a|^2)^{k+1}} \le \frac{m(1,k)+ 2k}{(1-|a|^2)^{k+1}} = \frac{C(1,k)}{(1-|a|^2)^{k+1}}.$$

To obtain a bound for $|h_k''(z)|$, one can either apply (8.19) to $h_k'(z)$ or differentiate (8.21) twice. Higher derivatives may be obtained in a similar way.

<u>Remark</u>.

If

$$(8.26)\qquad h_k(a) = h_k'(a) = \ldots = h_k^{(s-1)}(a) = 0, \quad s=1,2,\ldots,$$

then for $z=a$ we have

(8.27) $$|h_k^{(s)}(a)| \leq \frac{|\eta_k^{(s)}(0)|}{(1-|a|^2)^{s+k}} \leq \frac{s!m(s,k)}{(1-|a|^2)^{s+k}} .$$

Proof of Theorem 7.

Since (8.2) is disfocal in $|z|< 1$, it follows from Theorem 2 (and may easily be verified directly) that for every $1 \leq j \leq n-1$ and any $|a|< 1$, the functions $g_j(z,a)$ and $g_{j+1}(z,a)$, defined by (8.5), are 'relatively schlicht' in $|z|< 1$. Consequently,

(8.28) $$|\varphi_{j,j+1}(z,a)| = |\Phi[g_j(z,a),g_{j+1}(z,a)]| \leq \frac{1}{(1-|z|^2)^2}, \quad |z|< 1.$$

We utilize now the relations between the functions $\varphi_{j,j+1}$ and the coefficients q_{n-j+1}, established in Theorem 6. For $j=n-1$, it follows immediately from (8.9) and (8.28) that

$$|q_2(a)| = |\varphi_{n-1,n}(a,a)| \leq \frac{1}{(1-|a|^2)^2}, \quad |a|< 1.$$

For $1 \leq j \leq n-2$ we apply Lemma 4 to $\varphi_{j,j+1}(z,a)$ with $k=2$ and $s=n-j-1$. By (8.9) and (8.19) we conclude that

$$|q_{n-j+1}(a)| = |\varphi_{j,j+1}^{(n-j-1)}(a,a)| \leq \frac{A_{n-j+1}}{(1-|z|^2)^{n-j+1}}, \quad j=1,2,\ldots,n-2.$$

Moreover, according to (8.8) and to the remark following Lemma 4,

$$A_{n-j+1} \leq (n-j-1)!m(n-j-1,2) = (n-j-1)!\left(\frac{n-j+3}{4}\right)^2\left(\frac{n-j+3}{n-j-1}\right)^{\frac{n-j-1}{2}},$$

which completes the proof of the theorem.

We add the following remarks:

(i) (8.10) cannot be utilized to yield a bound for $|q_1(z)|$, since a bound for $\theta_{n-1,n}(z,a)$ may be obtained only if

$g_{n-1}(z,a)$ is univalent in $|z| < 1$, which is more than we can conclude from our assumptions.

(ii) The technique of differentiating the functions φ, may also be applied in the general case when the matrix $P(z)$ does not take the special form (8.1). Assume now that (1.1) is disconjugate in $|z|< 1$ and that (3.3) holds. By differentiating (6.9) once and setting $z=a$, we obtain

$$(8.29)\quad \varphi'_{jk}(a,a) = -p'_{jk}(a)p_{kj}(a) - p_{jk}(a)p'_{kj}(a) - \sum_{i=1}^{n}[p_{ji}(a)p_{ik}(a)p_{kj}(a) + p_{ki}(a)p_{ij}(a)p_{jk}(a)].$$

According to (7.1) and (7.2) we may apply Lemma 4 to $p_{jk}(z)p_{kj}(z)$ as well as to $\varphi_{jk}(z,a)$. It follows now from (8.19) that

$$|\varphi'_{jk}(a,a)| \leq \frac{C(1,2)}{(1-|a|^2)^3}, \quad |a|< 1$$

$$|p'_{jk}(a)p_{kj}(a) + p_{jk}(a)p'_{kj}(a)| \leq \frac{C(1,2)}{(1-|a|^2)^3}, \quad |a|< 1$$

which by (8.29) yields

$$(8.30)\quad \left|\sum_{i=1}^{n}[p_{jk}(a)p_{ik}(a)p_{ki}(a) + p_{ki}(a)p_{ij}(a)p_{jk}(a)]\right| \leq \frac{2C(1,2)}{(1-|a|^2)^3}, \quad |a|< 1.$$

For $n=3$, $j=1$, $k=2$ (8.30) reduces to

$$|\det[P(a)]| \leq \frac{2C(1,2)}{(1-|a|^2)^3}, \quad |a|< 1.$$

By taking the second derivative of (6.9) at the point $z=a$, it is possible to obtain sums of products of 4 coefficients of the matrix $P(z)$ $(n \geq 4)$, and similar results for higher derivatives. The actual calculation is somewhat cumbersome.

We end with the following corollary for second order equations.

If $q_2(z)$ is regular in $|z|< 1$ and if the differential equation

(8.31) $$w''(z) + q_2(z)w(z) = 0$$

is disfocal in $|z|< 1$, then it is also disconjugate in $|z|< 1$. We recall that a second-order differential equation is called disconjugate in a domain D, if the only solution that vanishes twice in D is the trivial one. As for the proof of the corollary, since (8.31) is disfocal in $|z|< 1$, it follows from (8.16) that

$$|q_2(z)| \leq \frac{1}{(1-|z|^2)^2}, \quad |z|< 1$$

which is sufficient to guarantee the disconjugancy of (8.31) in $|z|< 1$. (See [4]).

We note that this result holds only if $q_1(z) \equiv 0$ and is not true in the general case of second order differential equations of the type (8.2). Considering the differential equation

$$y''(z) - (m+1)y'(z) + my(z) = 0, \quad m > 1$$

London and Schwarz [3] showed that, in general, disfocality neither implies disconjugancy nor is implied by it.

In view of the fact that disconjugancy of (8.31) is equivalent to univalence of $f(z) = \frac{w_1(z)}{w_2(z)}$, where $w_1(z)$ and $w_2(z)$ are linearly independent solutions of (8.31), our last corollary may be stated as a univalence criterion.

Theorem 8.

Denote by D the disk $|z-b|< R$, $0 < R < \infty$, and let $f(z)$

be a meromorphic function in D. *If*

$$(8.32)\qquad f(z_1) - \frac{2[f'(z_1)]^2}{f''(z_1)} \neq f(z_2)$$

for every pair of points (not necessarily distinct) $z_1, z_2 \epsilon D$, *then* $f(z)$ *is univalent in* D *and*

$$|\{f(z),z\}| \leq \frac{2}{(R^2-|z-b|^2)^2}\ , \qquad z\epsilon D$$

where

$$\{f(z),z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left[\frac{f''(z)}{f'(z)}\right]^2$$

is the Schwarzian derivative.

Proof.

Without loss of generality we may assume that D is the unit disk, since this situation may be achieved by means of a transformation $\zeta(z) = \frac{z-b}{R}$, which does not violate (8.32).

Consider now the second order differential equation

$$(8.33)\qquad w''(z) + q_1(z)w'(z) + q_2(z)w(z) = 0.$$

According to (8.9) and (8.10) we have

$$-q_1(z) = \Theta[f(z),g(z)],\quad q_2(z) = \Phi[f(z),g(z)]$$

where

$$(8.34)\qquad f(z) = \frac{w_1(z)}{w_2(z)}\ ,\quad g(z) = \frac{w_1'(z)}{w_2'(z)}$$

and $w_1(z)$ and $w_2(z)$ are linearly independent solutions of (8.33). If $q_1(z) \equiv 0$, it follows from (5.1) that

$$(8.35)\qquad g(z) = f(z) - \frac{2[f'(z)]^2}{f''(z)}$$

and

$$\Phi[f(z),g(z)] = \frac{1}{2}\{f(z),z\}.$$

In view of (8.35), formula (8.32) takes the form $g(z_1) \neq f(z_2)$, which by (8.34) is equivalent to the disfocality of the differential equation

$$w''(z) + \frac{1}{2}\{f(z),z\}w(z) = 0. \tag{8.36}$$

By Theorem 6, disfocality of (8.36) in the unit disk implies

$$|\{f(z),z\}| \leq \frac{2}{(1-|z|^2)^2}, \quad |z| < 1, \tag{8.37}$$

which is a sufficient condition for disconjugancy of (8.36) in $|z| < 1$. Since disconjugancy of (8.36) is equivalent to the univalence of $f(z)$ [4], our proof is accomplished.

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