SUBDIRECT IRREDUCIBILITY AND EQUATIONAL COMPACTNESS IN UNARY ALGEBRAS <A;f>

by

Gunter H. Wenzel

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SUBDIRECT IRREDUCIBILITY AND EQUATIONAL COMPACTNESS IN UNARY ALGEBRAS <A;f>.

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In [6] M. Yoeli characterizes the subdirectly irreducible unary algebras $G = \langle A;f\rangle$ with a finite carrier-set A, a unary operation f and a connected f-graph (mentioning its significance for the problem of synthesizing automata by parallel composition). We intend to give a different proof based on a very simple criterion for subdirect irreducibility for a more general result (namely the characterization of all irreducible unary algebras) in §1. We couple this with a few simple remarks which seem nevertheless of some independent interest. In §2 we study equationally compact unary algebras $G = \langle A;f\rangle$ and give a complete characterization of them. Finally, in §3 we show that every equationally compact algebra $G = \langle A;f\rangle$ is a retract of its Stone-Čech-compactification (in case of algebras $G = \langle A;f\rangle$ with $f^n(x) \neq x$ for all n and x this follows, of course, from the fact that the Stone-Čech compactification βG of G is an elementary extension of G; see [10]). The results in §2 and §3 contain all the answers concerning unary algebras that are usually asked in this line of questioning for specific classes of universal algebras (see [4],[10],[12],[14]). I want to express my appreciation of numerous stimulating discussions with Dr. R. Alo.

We use the standard terminology (see, e.g., [3],[12],[15]) and assume familiarity with basic results (see [2],[3],[12]).

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We recall that $G = \langle A;f\rangle$ is called "connected" if for any two elements $a,b \in A$ there exist $n,m \in N_0 = N \cup \{0\}$ (N = set of natural numbers) such that $f^n(a) = f^m(b)$. Each unary algebra $G = \langle A;f\rangle$ is in a unique way the disjoint union of connected unary algebras $G_i = \langle A_i;f_i\rangle$, $i \in I$, called the "connected components of G". Finally, $\cup$ and $\cap$ denote the set-theoretic operations "union" and "intersection" ($\dot{\cup}$ means disjoint union), $\vee$ and $\wedge$ denote the lattice-theoretical operations "cup" and "meet". A universal algebra $G = \langle A;f\rangle$ is then subdirectly irreducible if $\wedge(\theta_i; i \in I) = \omega$ always implies $\theta_{i_0} = \omega$ for some $i_0 \in I$ (θ_i are congruence relations on G, ω = identity relation).

§1. Subdirect irreducibility.

As well-known, an arbitrary universal algebra G of type r is subdirectly irreducible if and only if it has only one element (i.e. $G = 1_\tau$) or the congruence-lattice C(G) = <C(G); V,A> is atomic and has exactly one atom (see [1],[2] and [3]). One also knows that if a,b are two different elements in A there exists a congruence relation $*/r_{a_5D}$, e C(G) which is maximal with respect to the property a ^ b(t/j$_{a,D}$,) and there exists a congruence relation $9_{a,ID}$, £ C(G) wjiich is minimal with respect to the property a s b(8$_{a,D}$.). Thus, since 0(0$_{a,y\,JD}$, *a / b and (a,b) € A_τ x A) = co, we conclude that the subdirect irreducibility of G]4 1 implies the existence of two different elements a,b € A such that $0_{a,D}$, = 60 (= identity relation) • On the other hand, since the congruence lattice of the factor algebra G/0$_{a,iD}$, is atomic with the unique atom */)$_{a,ID}$ ^ V 0$_{a,D}$ T ^ / 0$_{a,ID}$ ^ in case G / 1_x we conclude that the existence of two different elements a,b e A such that if)$_{a,D}$, = co implies the subdirect irreducibility of G. We state this simple observation in the next remark.

Remark 1: If G 4 1_T is a universal algebra then it is subdirectly irreducible if and only if ib$_{a,ID}$, = & for some a,b e A.

We could immediately apply the above remark to determine the subdirectly irreducible ones among the unary algebras G = <A;f> but prefer to mention first anothqr simple observation on subdirect irreducibility in unary algebras:

Remark 2; Every subalgebra IB of a subdirectly irreducible unary algebra G = <A;F> (F = set of unary operations) is subdirectly irreducible.

The proof of the last remark is quite clear since every congruence 0 *e* *C(H)* can be extended to 0' *e* C(G) by specifying that a^ = a_2(0') holds if and only if a_i = a_2(A) or a_i = a_2#

Thus, if 6 is the unique atom in the atomic lattice C(G) then 6_0 (= 6 restricted to B) is the unique atom in the atomic lattice C(6) unless |B| = l(in which case the matter is even simpler).

To attack the problem of characterizing the subdirectly irreducible unary algebras G = <A;f> we prove the following fundamental lemma which reduces the problem to the case of connected unary algebras.

Lemma 1: The unary algebra G = <A;f> is subdirectly irreducible if and only if either G is connected and irreducible or G is the disjoint union of G. and G_2 where G_1 is connected and irreducible and $|A_2|$ - X.

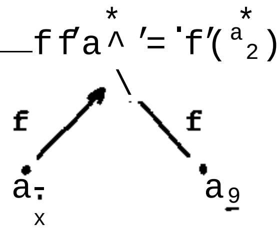
~~proof~~: Whenever ft is a subalgebra of G we know that f)

(defined by x = y(0_) if and only if x,y € B or x = y) is a congruence on G. Thus, if G would have more than two connected components G^ say G^G^, . . . then 0 A_{UA} fl 9 A_y A.0 P A.UA, = co together with 0_i ∵ $_D$ / co for i / j implies that G is not subdirectly irreducible. Thus, G = G_1 or Q = G_1 U G_2 where G_1,G_2 are connected components. If C = G_1 U G_2 and $|A_1|$,$|A_2|$ ^ 2 we get the contradiction 0_{A_1} n A_{A_2} = w, Q_{A_1} ^ ^, 0_A / 0% Hence, say, $|A_2|$ = 1. The remainder of the proof is clear. q. e. d.

We proceed to derive another simple observation after which the main-result has the character of an easy corollary:

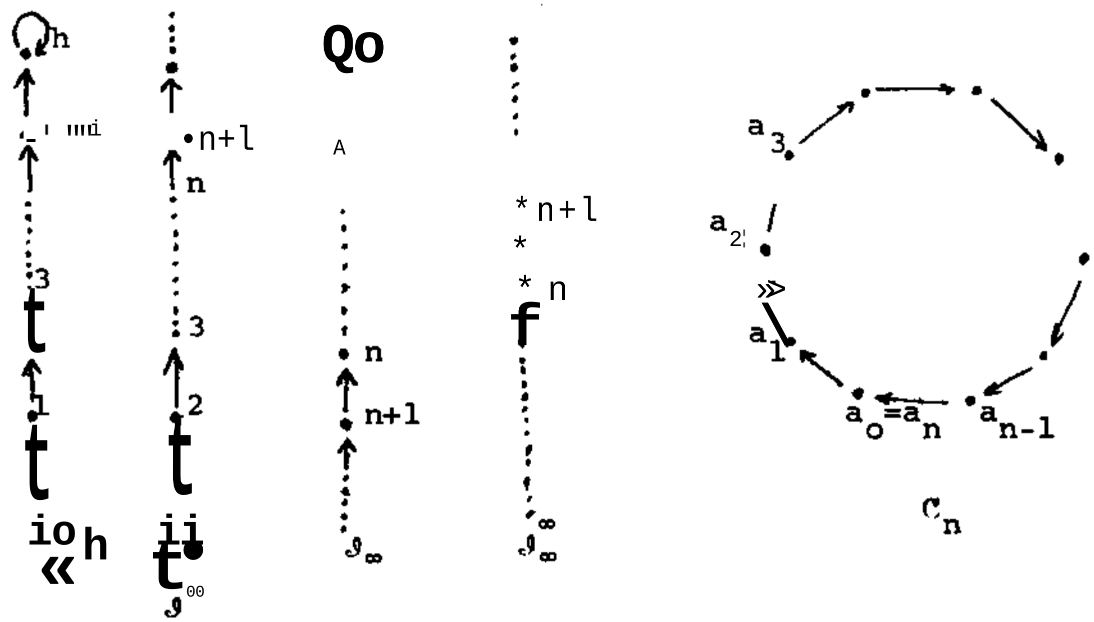
Lemma 2: If $a_1 \neq a_2$ are two different elements in the unary algebra $G = \langle A;f\rangle$ such that $f(a_1) = f(a_2)$ and $|\{f(a_1), a_1, a_2\}| = 3$ then G is not subdirectly irreducible.

proof:



We can assume without loss of generality that $a_1 \notin [a_2]$ ($\langle [a_2];f\rangle$ is the subalgebra generated by a_2). Since $x \equiv y(\Theta_{a_1,a_2})$ is equivalent to $x = y$ or $\{x,y\} = \{a_1,a_2\}$ we conclude that $\Theta_{a,b} = \omega$ implies $\{a,b\} = \{a_1,a_2\}$, i.e. $\Theta_{a_1,a_2} = \omega$; this is impossible since $a_1 \ldots$ and $\ldots$. Thus, $\Theta_{a,b} \neq \omega$ for all $a \neq b \in A$ which (in view of remark 1) settles the matter. q. e. d.

Thus, if we want to find the subdirectly irreducible connected algebras $G = \langle A;f\rangle$ we are left with the following possibilities (we describe the structure by the associated f-graphs):



$j^n(j_\infty)$ is subdirectly irreducible since $\Theta_{n,n-1}(\Theta_{0,1}) = \omega$ in case $h \geq 1$.

j^∞ is not subdirectly irreducible since for arbitrary $n \neq m$, $n,m \geq 0$, the relation $n \equiv m(\Theta_{[\max(n,m)]})$ is true and $\Theta_{[\max(n,m)]} \neq \omega$; thus, $\Theta_{n,m} \neq \omega$.

j^∞_∞ is not subdirectly irreducible since j^∞ is (up to isomorphism) a subalgebra of j^∞_∞ (we use remark 2).

Every congruence on C_n is (as easily seen) of the form Θ_m (m divides n) defined by $a_i \equiv a_j(\Theta_m)$ if and only if $m / i-j$. Thus $\Theta_{a_i,a_j} = \omega$ is equivalent to $a_i \equiv a_j(\Theta_m)$ for all divisors m of n; i.e. $\Theta_{a_i,a_j} = \omega$ is equivalent to the divisibility of $i - j$ by all divisors m of n. Since this is equivalent to $i \equiv j(\mathrm{mod}\ n)$ or $n = p^k$ (p = prime number, $k \in N_0$) we conclude that $\Theta_{a_i,a_j} = \omega$ for $a_i \neq a_j$ can hold if and only if $n = p^k$. Hence, C_{p^k} are exactly all subdirectly irreducible algebras of the form C_n.

We sum up our results in the following theorem:

Theorem 1: The (up to isomorphism) only subdirectly irreducible algebras of type $\tau = \langle 1 \rangle$ are C_g^h $(h \geq 1)$, j_∞, C_{p^k} (where $k \in N_0$ and p is a prime number) and $j \cup 1_\tau$, $j_\infty \cup 1_\tau$, $j_{p^k} \cup 1_\tau$.

We realize that all subdirectly irreducible algebras but j_∞ and $j_\infty \cup 1_\tau$ are finite, hence equationally compact (for the concepts see, e.g. [12]). As we will see in the next section, $j^\infty \cup 1_\tau$ is also equationally compact while j^∞ is not. We therefore turn our attention now to the characterization of the equationally compact algebras $G = \langle A;f \rangle$.

§2. Equationally compact algebras G = <A;f>

To state and prove our main-result it proves useful to enrich our language by a few suggestive concepts? An element x e A is called stagnant if f(x) = x; st(G) denotes the set of stagnant elements in G. If n *e* N_o and a € A then the n-periphery $^IV_;(^a)$ of a in G is defined by $n_r(a)$ = (b;b *e* A, $f^n(b)$ = a and $f^{n"1}(b)$ *4* a] (in case n = 0, the last condition becomes void). Thus, a *e* ^(a) is equivalent to <[a] ;f> ≃ C_n for n J> *1,* *he* Ou(a) is equivalent to b = a, and a *e* $l_G(^a)$ is equivalent to a *e* st(G). The following observation is equally clear: If n,m *e* *N,* *n* ≮ m and b € A, then b *e* *Vj(^{a}) fi ^{It}Vi(^{a}) i^s equivalent to m - n J> 2, $f^n(b)$ = a and <[a];f> ≃ c_d with 2 ^ d_5d/m-n.

We denote by Ii the language of first order logic with identity and countably many variables x_k of type r=<l> associated with the class of all unary algebras G = <A;f>, We use the well-known concept of satisfiability of a formula 0 *e* L (see, e.g., [11]) and recall that the algebra 8 = <B;f> is an elementary extension of G = <A;f> if and only if an arbitrary $\bar{a}$ *e* A^{a} satisfies an arbitrary formula $ e L in G if and only if it satisfies the same formula in *H.* We then have the following lemma:

Lemma 3: If B is an elementary extension of G = <A;f> then we have the following relationships:

(1) st(G) = <*f)* is equivalent to st(<R) = *(f*>

(2) For every a *e* *A,* ^{n}Q(^{a}) = ^ is equivalent to n_{fi}(a) = <*f*>.

(3) B contains a subalgebra isomorphic to $r»_n$ if and only if G contains such an algebra.

proof;

(1) follows from the fact that (3x)(f(x) = x) is a sentence in L which is true in G if and only if it is true in 8.

(3) follows for the same reason from the sentence $(3x)(f^n(x) =$ x A $f^{n""1}(x)$ / x) . To see (2) we take the formula $\$ = ({}^{ax}{}_Q)$ $(f^n(x\ ) = x_n$ A $f^{n"1}(x\ )$ ^ x.). If we assume that n^a) *j4 <b* then $\overset{OX}{a}$ = (a^a^a,. . . . .) *e* $\overset{O}{A}$ ° satisfies \$ in B; hence $\overline{a}$ satisfies \$ in G, i.e. there exists a_1 *e* A such that $f^n(a_1)$ = a, $f^{n-1}(a_1)$ ^ a. In short: $r_u(a)$ ^ (J). q. e. d.

Lemma 4: C_n is retract of every extension B = <B;f> that contains no subalgebras (isomorphic to) c_m unless n divides m.

proof; We prove the result for convenience's sake for n = m. The general case can be easily adjusted. We assume as before that $C_n = [a_o, a_1 .5.\#., a_n$^j with $f(a_1) = a_{1+1}$ (all indices are determined modulo n). Evidently we can define homomorphism componentwise. If B is connected then for every b e B there exists a unique smallest m(b) e N_o such that $f^{m(b)}(b)$ e C_{n9} say $f^{m(b)}(b) = a_{1(b)}$. We define <p(b) = $a_{1(b)-m(b)}$. and easily verify that *<p* : B -• C_n is a retraction. If B is not connected and B_1 is a connected component disjoint from B then we have two possibilities; Either iU_1 contains a subalgebra $C^*_n \cong f_n$ in which case we have a retraction $<p_1$: 8_1 -• e^s_n and an isomorphism

$\varphi: C^{g}_{n} \to C_{n}$, i.e. a homomorphism $\psi \circ \varphi_{g}$ or B_1 contains no subalgebra C_m, $m \geq 1$. In the latter case we pick some $b_0 \in B_1$ and map it via φ to $a_0 \in C_n$. Then we have for every $b \in B$ a unique $m(b) \in N_0$ such that $f^{m(b)}(b) \in [b_0]$, say $f^{m(b)}(b) = f^{k(b)}(b_0)$. We complete the definition of φ by requiring that $\varphi(b) = a_{k(b)-m(b)}$ (where the index of a is again determined modulo n). It is again easy to check that $\varphi: B_1 \to C_n$ defines a homomorphism. q. e. d

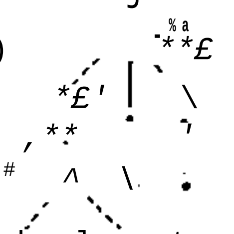
We should remark that to construct φ in the above lemma we in effect applied an algorithm due to Novotny [9].

To derive the next crucial lemma we again facilitate matters by suitable concepts: The element $a \in A$ is an element of <u>order n</u> ($n \in N$) if (1) $f^m(a) \neq a$ for all $m \in N$ and (2) $n(a)$ contains a <u>minimal element</u> b, i.e. there is no $c \in A$ such that $b = f(c)$. The element $a \in A$ is an element of <u>infinite order</u> if (1) $f^m(a) \neq a$ for all $m \in N$ and (2) there is an infinite chain $a_0, a_1, a_2, \ldots, a_i, \ldots$ such that $a = a_0$, $a_m = f(a_{m+1})$ and $a_i \neq a_j$ for $i \neq j$.

<u>Lemma 5</u>: If $C = \langle A;f\rangle$ is a unary algebra with at least one subalgebra (isomorphic to) C_n for some $n \in N$ in which every element whose finite orders approach infinity is of infinite order, then C is equationally compact.

<u>proof</u>: We know from a result by Węglorz [12] that a universal algebra is equationally compact if and only if it is a retract of every elementary extension. So let $B = \langle B;f\rangle$ be an elementary

extension of G and let us prove the existence of some retraction (pi B -> A. We know from lemma 3 that cyclic subalgebras C_n of (B which are contained in B\A have isomorphic copies contained in A. We therefore conclude with lemma 4 that every connected component $_i of 8 which is disjoint from G can be mapped into G via a homomorphism <p.$_i$ So let [[G]] = <[[A]];f> be the subalgebra of *H* each of whose connected components intersects A. Then $^n f_t(^a)$ ^ ^{n}rrcii^^{a}^ holds true for every a e A and n *e* N_Q; therefore,, [[A]] = U(n_{fi}(a);n *e* N_o a e A). Really a bit more is true: If we call an element a *e* A a <u>branch-element</u> if l_B(a)\ l_G(a) / </ j and call U(n_{fi} (a) \ *r*^ (a) ; n *e* N) ^ T. r. *r:* -^ , ^ the ~~branch of~~ a, say br(a) (see diagram)^ then [[A]] =



U(br(a); a runs through all branch-elements a *e* A) U A. We, of course, define $<p_A$ = identity and fix now an arbitrary branch-element a € A. Our aim is to define a homomorphism $<p_a$: A U br(a) -• A such that (p |$_A$ = <pL = id. There are three possibilities:

(1) <[a];f > = c_m for some m e N, say [a] = C^ = f$^{a==a}$$_o$^^{a}i^ • • • ^^{a}$_m$. If then b *e* br(a) H n_H(a) then we define θ_a (b) = a_{m-n} (indices: recall, count modulo m). (2) f (a) ^ a for all m ^> 1, but there exists some n *e* N such that n^(a) *j4 (f>* while k^(a) = *d>* for all k ^ n+1. Then, by lemma 3, n_r(a) / (^); say z_o *e* n_G(a). In this case we define cp_a on br(a) as follows: If 0 <£ k <; n and b € br(a) n $k_{o ft}$(a) then, o_a (b) = $f^{n"k}$(z_o): (3) For every

n there exists $n_o > n$ such that a is of order n_o in G.

In this case our assumption implies that a is of infinite order; i.e. there exists a sequence $a = a^0, a^1, \ldots, a^n, \ldots$ of elements in A such that $a^n = f(a^{n+1})$ (equivalently, $a^n \in np(a)$). We then map every $b \in br(a) \cap n_o(a)$ via φ_a to $\varphi_a(b) = a_n$. Thus, in each of the possible three cases we defined φ_a on $A \cup br(a)$ as homomorphsim into A which is the identity on A. If we do this for every branch-element $a \in A$ in the manner indicated then it is a matter of simple verification that the locally defined homomorphisms $\varphi_a : A \cup br(a) \to A$ patch up to a retraction $\varphi : [A] \to A$. q. e. d.

Since the proof would proceed in a quite analogous fashion we only state the following analogous result:

Lemma 6; If $G = \langle A;f \rangle$ is a unary algebra with a subalgebra (isomorphic to) j^* in which every element whose finite orders approach infinity is of infinite order, then G is equationally compact.

An analysis of the proof of lemma 5 brings to light the fact that the meat of the matter consisted in showing that an algebra $G = \langle A;f \rangle$ is retract of every extension $ft = \langle B;f \rangle$ which essentially underlies the conditions of lemma 3 (which, in view of that lemma, is granted by elementary extensions). We state that observation in the following remark.

Remark 3: If $G = \langle A;f \rangle$ contains some ^'^ J> 1* then it is retract of every extension $ft = \langle B;f \rangle$ with the following properties:

(1) $n_{\omega}(a) = \wedge$ is equivalent to $\wedge(a) = \omega_0$ for all $a \in A$ and $n \in N$.

(2) $n_m \subseteq B$ implies the existence of $e_n \subseteq Q$ such that n divides m.

(3) Every element $a \in A$ whose finite orders approach infinity is of infinite order.

We now have collected what is the meat of the following characterization-theorem:

Theorem 2; The unary algebra $G = \langle A;f\rangle$ is equationally compact if and only if

(1) For every $a \in A$, lim(n;n = order of a) $= \omega_0$ implies that a is of infinite order.

(2) G contains either some subalgebra r_n $(n \geq 1)$ or the subalgebra j^{∞}.

proof: Lemmas 5 and 6 state that (1) and (2) imply the equational compactness of G. Vice versa, let G be equationally compact, then lim(n;n = order of a) $= \omega_0$ implies that the infinite set $T = \{a = f(x_1), x_1 = f(x_2), \ldots, x_n = f(x_{n+1}), \ldots\}$ of equations is finitely solvable, hence solvable. Thus, a is of infinite order verifying (1). To verify (2) assume that G contains no cyclic subalgebra r_n. Then $|\{f^n(a), f^{n-1}(a), \ldots, f(a), a\}| = n+1$ for every $n \in N$ and $a \in A$. Thus, the infinite system of equations $T = \{x_0 = f(x_1), x_1 = f(x_2), \ldots, x_n = f(x_{n+1}), \ldots\}$ is finitely solvable, hence solvable. If $(a_0, a_1, \ldots, a_n, \ldots) \in A$ is a solution of V then evidently $\langle\{a_n; n \in N_0\} \cup \{f^n(a_0); n \in N_0\}; f\rangle$

$\cong j^*_\infty$ is a subalgebra of Q. q. e. d

We now turn our attention to the obvious next question in this line of investigation: Is "Mycielski's conjecture" (see [8], [12]) true in the class K(<1»? We will give an affirmative answer in the next section. In other words: We will show that every equationally compact unary algebra G = <A;f> is the algebraic retract of some topologically compact Hausdorff algebra B = <B;f>.

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§3. On the Stone-Čech Compactification of $G = \langle A;f\rangle$.

If $G = \langle A;f\rangle$ is a unary algebra, A is endowed with the discrete topology and pA is the Stone-Čech compactification of the topological space A then $f\colon A \to A$ is a continuous mapping and has, thus, a unique extension $f\colon pA \to pA$. The unary algebra $\beta G = \langle pA;f\rangle$ is then called the Stone-Čech compactification of the algebra G (Evidently we can construct pG for arbitrary unary algebras $\langle B;F\rangle$ in the same fashion). If G has no cyclic subalgebra C_n then $f^n(x) \neq x$ for all $n \geq 1$ and $x \in A$.

It was shown in Pacholsky and Weglorz [10] that the Stone-Čech compactification pG of such G is always an elementary extension of G. Hence if G, in addition, is equationally compact then it is a retract of pG. But even if the equationally compact G has cyclic subalgebras C_n there exists a retraction $\varphi\colon pG \to G$. To establish the result we need the following decisive lemma:

Lemma 7: If $G = \langle A;f\rangle$ is a unary algebra and C_m is a subalgebra of pG then there exists a subalgebra C_n of G such that n divides m.

proof; Let us make a few preliminary remarks: If $y \in pA \setminus A$ then $y = \lim_{d \in D} a_d$ where $(a_d, q^*D>J\geq)$ is a net in A (i.e., D is a directed partially ordered set, shortly directed poset, with respect to $\geq$, and $a_d \in A$ for every $d \in D$). Since A is a dense, discrete subspace of pA we know that $A_1, A_2 \subseteq A$ and $A_1 \cap A_2 = \emptyset$ implies $\bar{A}_1 \cap \bar{A}_2 = \emptyset$ (where $\bar{A}_i$ is the closure of A_i in pA), for $A_1 \subseteq A$ is an open and closed set in A, thus $\bar{A}$ and $\overline{A \setminus A_1}$

are complementary open and closed sets in βA (see, e.g., [5], chapter 6.9). Thus, if $A = \{a_d; d \in D\}$ and $y = \lim_{d \in D} a_d$ such that $A \cap f^m(A) = \emptyset$ for some $m \geq 1$ [of course, $f^m(A) = \{f^m(a_d); a_d \in A\}$], then $A \cap f^m(A) = \emptyset$ implies that $y \notin f^m(A)$, i.e. $y \neq f^m(y) = \lim_{d \in D} f^m(a_d)$.

So assume that $C_d \notin G$ for every divisor d of some $m \in N$ and let $x \in \beta A \setminus A$. We then have to show that $f^m(x) \neq x$ to end our proof. Let $G_i, i \in I$, be the connected components of G. The carrier set of every G_i, since, by assumption, it has no stagnant element, can be represented as $A_i = A_i^1 \cup A_i^2 \cup A_i^3$ such that $A_i^j \cap A_i^k = \emptyset$ for $j \neq k$ and $f(A_i^j) \cap A_i^j = \emptyset$, $i,j = 1,2,3$. If G_i is in the class $\mathcal{U}$ of unary algebras without cyclic subalgebra then this is a lemma by Ryll-Nardzewski (see [10]). If, on the other hand, G_i has a cyclic subalgebra $C_{n(i)}$, $n(i) \geq 2$, say $C_{n(i)} = \{a_0, a_1, \ldots, a_{n(i)-1}\}$ then we first subdivide $C_{n(i)}$ as follows: If $n(i)$ is even, we take $C^1_{n(i)} = \{a_0, a_2, \ldots, a_{n(i)-2}\}$, $C^2_{n(i)} = \{a_1, a_3, \ldots, a_{n(i)-1}\}$, $C^3_{n(i)} = \emptyset$; if n is odd, we take $C^1_{n(i)} = \{a_0, a_2, \ldots, a_{n(i)-3}\}$, $C^2_{n(i)} = \{a_1, a_3, \ldots, a_{n(i)-2}\}$, $C^3_{n(i)} = \{a_{n(i)-1}\}$. In either of the two cases we define $A_i^j = C^j_{n(i)} \cup \{a; a \in A_i \setminus C_{n(i)}$ and $a \in f^{-(2k)n(i)}(c)$ for some $c \in C^j_{n(i)}\}$, $j = 1,2,3$. It is an easy matter to check that A_i^j, $j = 1,2,3$, thus defined, satisfies the conditions stated at the beginning. Thus, the carrier set A of G satsifies $A = A^1 \cup A^2 \cup A^3$, $A^1 \cap A^2 = A^1 \cap A^3 = A^2 \cap A^3 = \emptyset$, $A^1 \cap f^m(A^1) = A^2 \cap f^m(A^2) = A^3 \cap f^m(A^3) = \emptyset$ if $A^j = \bigcup(A_i^j; i \in I)$, $j = 1,2,3$. Hence, $\beta A =$

$\bar{A}^1 \cup \bar{A}^2 \cup \bar{A}^3$, and we can now assume that $x \in \bar{A}^j$ for some $1 \le j \le 3$. In other words: $x = \lim_{d \in D} a_d$ where $(a_d, D, \ge)$ is a net in A^j. Since $\bar{A}^j \cap f^m(\bar{A}^j) = \emptyset$ we conclude that $f^m(x) = \lim_{d \in D} f^m(a_d) \in \beta A \setminus \bar{A}^j$, i.e. $f^m(x) \ne x$. q. e. d.

Appendix: The triple-division of A which we used in the last proof was stated as Ryll-Nardzewski's lemma in [10] for the case $G \in K_\omega$. It should be noted that in case $G \in K$ there are actually already two subsets A_1, A_2 such that $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$ and $f(A_1) \cap A_1 = f(A_2) \cap A_2 = \emptyset$. The proof, of course, remains elementary.

We can now prove our last result:

Theorem 3: If $G = \langle A; f \rangle$ is an equationally compact algebra then it is the algebraic retract of its Stone-Čech compactification βG.

proof: We can assume that G contains some cyclic algebra C_d, $d \ne 1$. In light of the last lemma, remark 3 and theorem 2 we only need to show that $\pi_n(a) = \emptyset$ is equivalent to $\pi_n^{\beta}(a) = \emptyset$ for every $a \in A$ and $n \in N$. So let $x \in (\beta A \setminus A) \cap \pi_n^{\beta}(a)$ for some $a \in A$. Then there is a net $(a_d, D, \ge)$ in A such that $x = \lim_{d \in D} a_d$. Hence, $a = f^n(x) = \lim_{d \in D} f^n(a_d)$. Since $a \in A$ is an isolated point in βA there exists $d_0 \in D$ such that $f^n(a_d) = a$ for all $d \ge d_0$. This settles the matter if at least one of the a_d is in $\pi_n(a)$; this is guaranteed unless a is a stagnant element. If a were a stagnant element and none of the a_d was in $\pi_n(a)$ we would conclude that $f^{n-1}(a_d) = a$ for all $d \ge d_0$, i.e. $a = \lim_{d \in D} f^{n-1}(a_d) = f^{n-1}(\lim_{d \in D} a_d) = f^{n-1}(x) \ne a$. This contradiction finishes the proof. q. e. d.

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