On Internal Interactions
and the Concept of Thermal Isolation

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1. Introduction

Recently GURTIN and WILLIAMS [1] established an axiomatic foundation for the thermodynamics of stationary continua, complementing an earlier work on the mechanics of continua by NOLL [1].[1] The present paper originated a a proposal for a slight restatement of the second law as given by Gurtin and Willaims to a form which seems more in fitting with the usual verbal statements of classical thermodynamics. The difference between this and the previous statement is significant only when the body involved suffers internal radiation, that is, radiation between particles of the body.

In order to examine this question properly it was necessary to treat more carefully than was done by either Noll or Gurtin and Williams the characteristics of the internal interactions of continua; that is, quantities such as internal radiation or mutual body forces. Accordingly Sections 2 and 3 of this paper consist of an examination of internal interactions. I consider a modification of the basic assumptions used by Gurtin and Williams and Noll to require that the heat flux (force, entropy flux, ...) to a portion of the body from another be small not only when the volume of the affected element is small but that it should also be small when the volume of the effective element is small. Making an assumption of this sort I proceed to obtain the results of Noll and Gurtin and Williams and further deduce (Section 3)

[1] Cf. the more general treatment which combines mechanical and thermodynamic theories by GURTIN, NOLL and WILLIAMS. See also the extension of the work of Gurtin and Williams by FISHER and LEITMAN, and the kinematical theory of NOLL [2].

that the effect on an element $\mathcal{A}$ in the body due to a disjoint element $\mathcal{C}$ can be written in terms of a density r as

$$\int_{\mathcal{A}}\int_{\mathcal{C}} r\,dV\,dV .$$

The treatment of the other effects, the area-continuous effect and the external effect is slightly more general than that in the previous works.

In Section 4 I return to the main topic of the paper, the statement of the second law of thermodynamics for continua. The basic statement of a bound on entropy production is inarguable; the distinction between the statement here and that of Gurtin and Williams lies in the definition of what is meant by thermal isolation. I propose here that the entropy flux into an element $\mathcal{A}$ from an element $\mathcal{B}$ be required to vanish when the heat flux into each element $\hat{\mathcal{A}}$ of $\mathcal{A}$ from each element $\hat{\mathcal{B}}$ of $\mathcal{B}$ is zero; Gurtin and Williams require this vanish when just the net heat flux into each $\hat{\mathcal{A}}$ from $\mathcal{B}$ is zero. With the revised statement it is clear that the conductive transfer and the external radiative transfer are governed by exactly the same relations established by Gurtin and Williams; for internal radiation I show that there exists a temperature function θ such that the internal radiation of entropy is given by

$$\int_{\mathcal{A}}\int_{\mathcal{C}} \frac{r}{\theta}\,dV\,dV$$

where r is the density of internal radiation. I then examine the consequences of this for the local form of the second law.

One should note that the treatment is sufficiently general that it applies not only to the theory of a single continuous body but also to theories including interacting, diffusing collections of continua (cf. GURTIN, NOLL and WILLIAMS).

2. Internal Interactions; Continuity Assumptions

The system we consider is a continuous material body $\mathcal{B}$ and a collection of subbodies $\mathfrak{M}^{\mathcal{B}}$ of $\mathcal{B}$. We shall for convenience regard $\mathcal{B}$ as a subset of three-dimensional Euclidean space: the more general case where $\mathcal{B}$ is a differentiable manifold is easily obtained.[1] The subbodies of $\mathcal{B}$ are regular closed subsets of $\mathcal{B}$ (regular in the sense of KELLOGG) and $\mathfrak{M}^{\mathcal{B}}$ is presumed a (closed) Boolean algebra under the operations

$$\mathcal{A} \vee \mathcal{C} = \mathcal{A} \cup \mathcal{C}$$

$$\mathcal{A} \wedge \mathcal{C} = \overline{\widehat{\mathcal{A} \cap \mathcal{C}}}$$

(the closure is actually of no significance). Note also that

$$\mathcal{A} \setminus \mathcal{C} = \overline{\widehat{\mathcal{A} - \mathcal{C}}} .$$

We suppose that $\mathfrak{M}^{\mathcal{B}}$ has the properties described by GURTIN and WILLIAMS [1]. The axioms they take are reproduced in the Appendix: the important features are that the sets of $\mathfrak{M}^{\mathcal{B}}$ are numerous enough to generate the Borel sets $\mathbb{B}(\mathcal{B})$ of $\mathcal{B}$ and that they can serve to define in the limit any regular surface element. We adjoin to $\mathfrak{M}^{\mathcal{B}}$ an unstructured set $\mathcal{B}^e$, the exterior of $\mathcal{B}$, and denote by $\mathfrak{M}$ the material universe for $\mathcal{B}$, the set of all finite unions from $\{\mathfrak{M}^{\mathcal{B}}, \mathcal{B}^e\}$. We write $\mathcal{A}^b$ for $\mathcal{B} \setminus \mathcal{A}$,

[1] Regarding $\mathcal{B}$ as a compact differentiable manifold of class two diffeomorphic to a region in Euclidean space, one need make only the following alterations to the theory: change the axioms on $\mathfrak{M}^{\mathcal{B}}$ (Appendix) by requiring e.g. instead of $\underset{\sim}{4}$ that for some chart $\underset{\sim}{\chi}$, the image under $\underset{\sim}{\chi}^{-1}$ of any parallelipiped be in $\mathfrak{M}^{\mathcal{B}}$, etc.; induce "volume" and "area" measures in $\mathcal{B}$ via $\underset{\sim}{\chi}$ and verify that the measures induced by various charts are Lipschitz-equivalent, so that the assumptions below are independent of the choice of $\underset{\sim}{\chi}$. The entire theory can then be carried over unaltered.

A Lebesgue area measure.[1] We here wish to make a stronger assumption: that $H(\mathcal{G},\mathcal{D})$ not only tend to zero as the volume of $\mathcal{G}$ and the area of $\mathcal{G}\cap\mathcal{D}$ become small but also that it tends to zero with the volume of $\mathcal{D}$. (In physical terms, e.g., we wish to assume that the radiation into $\mathcal{G}$ from $\mathcal{D}$ can be non-zero only if $\mathcal{D}$ has a finite volume.) We in fact assume more than this: we suppose that the first bound as above tends to zero with the product $V(\mathcal{G})V(\mathcal{D})$ and that it does so in a uniform manner, which is expressed below in terms of sums over collections of pairs in $h^{\mathcal{B}}$. Thus we assume:

I. <u>There exists a scalar</u> β <u>and for each</u> $(\mathcal{G},\mathcal{D})\in h$ <u>a scalar</u> $\alpha(\mathcal{G},\mathcal{D})$ <u>such that</u>

$$|H(\mathcal{G},\mathcal{D})| \leq \alpha(\mathcal{G},\mathcal{D}) + \beta A(\mathcal{G}\cap\partial\mathcal{D}).$$

α <u>is such that there exists a scalar</u> $\overline{\alpha}$ <u>with</u>

$$0 \leq \alpha(\mathcal{G},\mathcal{D}) \leq \overline{\alpha}V(\mathcal{G}) \quad \text{for all} \quad (\mathcal{G},\mathcal{D}) \quad \text{in} \quad h$$

<u>and such that</u>

$$\sum^{N} \alpha(\mathcal{G}_i,\mathcal{D}_i) \to 0 \quad \text{as} \quad \sum^{N} V(\mathcal{G}_i)V(\mathcal{D}_i) \to 0$$

<u>for all collections</u> $\{(\mathcal{G}_i,\mathcal{D}_i)\}^N$ <u>in</u> $h^{\mathcal{B}}$.

This condition is implied by the much simpler assumption of <u>Lipschitz continuity</u>: $\exists$ $\overline{\alpha},\beta$ such that

$$|H(\mathcal{G},\mathcal{D})| \leq \overline{\alpha}V(\mathcal{G})V(\mathcal{D}) + \beta A(\mathcal{G}\cap\partial\mathcal{D}^e).$$

if $\mathcal{G},\mathcal{D}$ are separate subbodies and

[1] Note that in light of the lemma below this continuity assumption suffices to guarantee that $H(\cdot,\mathcal{D})$ can be extended to $\mathcal{B}(\mathcal{D}^b)$ and there satisfy the same bounds. Hence this theorem guarantees such an assumption is equivalent to the starting assumptions on heat flux used by GURTIN and WILLIAMS [1].

similar argument based on the volume bound implies that this limit is the same regardless of the choice of the sequence $\{\mathfrak{A}_n\}$. We shall now establish that it depends only upon $\mathcal{S}_{\mathfrak{A}}$. Suppose that $\mathcal{S}_{\mathfrak{A}} = \mathcal{S}_{\mathfrak{A}*}$. It is a consequence of the properties of $\mathfrak{M}^{\mathfrak{B}}$ that $\mathcal{S}_{\mathfrak{A}} = \mathcal{S}_{\mathfrak{A}\wedge\mathfrak{A}*}$ in these circumstances. The fact that the limiting value is independent of choice of sequence then implies for sequences $\{\mathfrak{A}_n\}$, $\{\mathfrak{A}_n^*\}$

$$\lim_{n\to\infty} \alpha(\mathfrak{A}_n) = \lim_{n\to\infty} \alpha(\mathfrak{A}_n \wedge \mathfrak{A}_n^*) = \lim_{n\to\infty} \alpha(\mathfrak{A}_n^*) .$$

Thus we can unambiguously define

$$\tilde{\alpha}(\mathcal{S}_{\mathfrak{A}}) = \lim_{n\to\infty} \alpha(\mathfrak{A}_n), \quad \{\mathfrak{A}_n\} \text{ in } \mathfrak{A} \text{ as above.}$$

Then we set

$$\bar{\alpha}(\mathfrak{A}) = \alpha(\mathfrak{A}) - \tilde{\alpha}(\mathcal{S}_{\mathfrak{A}}) = \lim_{n\to\infty} \alpha(\mathfrak{A} \setminus \mathfrak{A}_n) .$$

It should cause only slight confusion if we also write

$$\tilde{\alpha}(\mathfrak{A}) \stackrel{\text{def}}{=} \tilde{\alpha}(\mathcal{S}_{\mathfrak{A}}) .$$

On the set $\mathfrak{M}^{\partial\mathfrak{B}} = \{\mathfrak{A}\cap\partial\mathfrak{B} \mid \mathfrak{A}\in\mathfrak{M}^{\mathfrak{B}}\}$ [1] we induce a Boolean structure from that on $\mathfrak{M}^{\mathfrak{B}}$. Then $\bar{\alpha}$ on $\mathfrak{M}^{\mathfrak{B}}$ and $\tilde{\alpha}$ on $\mathfrak{M}^{\partial\mathfrak{B}}$ are separately additive. We prove this for $\tilde{\alpha}$ as follows: if $\mathcal{S}_{\mathfrak{A}}, \mathcal{S}_{\mathfrak{D}}$ are separate, we can suppose $\mathfrak{A}, \mathfrak{D}$ are also. $\mathcal{S}_{\mathfrak{A}\cup\mathfrak{D}} = \mathcal{S}_{\mathfrak{A}} \cup \mathcal{S}_{\mathfrak{D}}$ which means

$$\tilde{\alpha}(\mathcal{S}_{\mathfrak{A}} \cup \mathcal{S}_{\mathfrak{D}}) = \tilde{\alpha}(\mathcal{S}_{\mathfrak{A}\cup\mathfrak{D}}) = \lim_{n\to\infty} \alpha(\mathfrak{C}_n)$$

[1] Cf. the discussion of such collections as $\mathfrak{M}^{\partial\mathfrak{B}}$ by FISHER and LEITMAN.

where $\{\mathcal{C}_n\}$ is taken in $\mathcal{A}\cup\mathcal{D}$ as above. This says

$$\tilde{\alpha}(\mathcal{S}_{\mathcal{A}}\cup \mathcal{S}_{\mathcal{D}}) = \lim_{n\to\infty} (\alpha(\mathcal{C}_n\wedge \mathcal{A}) + \alpha(\mathcal{C}_n\wedge \mathcal{D}))$$

$$= \tilde{\alpha}(\mathcal{S}_{\mathcal{A}}) + \tilde{\alpha}(\mathcal{S}_{\mathcal{D}})$$

since $\mathcal{S}_{\mathcal{B}}\subset \mathcal{C}_n\wedge \mathcal{D}$, $\mathcal{S}_{\mathcal{A}} \subset \mathcal{C}_n\wedge \mathcal{A}$ for each n and the sequences clearly satisfy the other requirements. This implies that both $\tilde{\alpha}$ on $\mathfrak{M}^{\mathcal{B}}$ and $\bar{\alpha}$ are separately additive. From the construction of $\tilde{\alpha},\bar{\alpha}$ it is clear that

$$|\tilde{\alpha}(\mathcal{S}_{\mathcal{A}})| \le dA(\mathcal{S}_{\mathcal{A}})$$

$$|\bar{\alpha}(\mathcal{A})| \le cV(\mathcal{A}).$$

GURTIN and WILLIAMS [1, Appendix] prove that a separately additive function on $\mathfrak{M}^{\mathcal{B}}$ satisfying the latter condition has a unique extension to a countably additive measure on $\mathbb{B}(\mathcal{B})$, the extension having the same bound. Since we shall have occasion to refer to this later we briefly outline their procedure. Define the field consisting of all half-open parallelepipeds and finite unions of such parallepipeds; we denote by $\mathbb{F}_{\mathcal{B}}$ the field consisting of all intersections of $\mathcal{B}$ with the original field. Then the closure of the interior of any element of $\mathbb{F}_{\mathcal{B}}$ is in $\mathfrak{M}^{\mathcal{B}}$ (axiom 4 in the Appendix). We define on $\mathbb{F}_{\mathcal{B}}$ an additive set function by taking the value for $A\in\mathbb{F}_{\mathcal{B}}$ that the original function gave to $\overline{\mathring{A}}$. The extension to $\mathbb{B}(\mathcal{B})$ then follows via a standard argument since $\mathbb{F}_{\mathcal{B}}$ generates $\mathbb{B}(\mathcal{B})$; the bound is not difficult to establish. Thus we can extend $\bar{\alpha}$ to $\mathbb{B}(\mathcal{B})$.

It is not difficult to see that the same procedure can be applied to $\tilde{\alpha}$ on $\mathfrak{m}^{\partial\mathfrak{B}}$: $\mathbb{F}_{\mathfrak{B}} \cap \partial\mathfrak{B}$ generates $B(\partial\mathfrak{B})$ and the closure of the (relative) interior of any element of $\mathbb{F}_{\mathfrak{B}} \cap \partial\mathfrak{B}$ is a member of $\mathfrak{m}^{\partial\mathfrak{B}}$. We note that if $\tilde{\alpha}$ (a third usage) is so defined on $\mathbb{B}(\partial\mathfrak{B})$ we may define (a fourth)

$$\tilde{\alpha}(\mathcal{P}) = \tilde{\alpha}(\mathcal{P} \cap \partial\mathfrak{B})$$

on $\mathbb{B}(\mathfrak{B})$. Then we define for any $\mathcal{P}$

$$\alpha(\mathcal{P}) = \tilde{\alpha}(\mathcal{P}) + \bar{\alpha}(\mathcal{P})$$

It clearly is an extension of the original α.

The uniqueness is trivial: if α^* is any measure extending α we define $\tilde{\alpha}^*$ and $\bar{\alpha}^*$ exactly as we defined $\tilde{\alpha}, \bar{\alpha}$: since these agree with $\tilde{\alpha}, \bar{\alpha}$ respectively on the appropriate sets we have that they are unique and obey the Lipschitz bounds. These bounds imply that $\alpha^* = \bar{\alpha}^* + \tilde{\alpha}^*$ is a Lebesgue decomposition with respect to volume (which is unique).□

We note that we have proved more than stated: the extended measure can be decomposed uniquely into a measure on $\mathfrak{B}$ and a measure on $\partial\mathfrak{B}$.

Applying the lemma to the function $H(\cdot, \mathcal{D})$ (replacing $\mathfrak{B}$ by $\mathcal{D}^b$) one obtains the first half of

Proposition 1: <u>For each</u> $\mathcal{D} \in \mathfrak{m}$ <u>there exists a unique pair of measures</u> $R(\cdot, \mathcal{D})$ <u>and</u> $Q(\cdot, \mathcal{D})$ <u>on</u> $\mathbb{B}(\mathcal{D}^b)$ <u>such that</u>

$$|Q(\mathcal{P}, \mathcal{D})| \leq \beta A(\mathcal{P} \cap \partial\mathcal{D})$$

$$|R(\mathcal{P}, \mathcal{D})| \leq \bar{\alpha} V(\mathcal{P})$$

<u>for all</u> $\mathcal{P}\in\mathcal{B}(\mathcal{D}^b)$, <u>and for each</u> $\mathcal{G}\in\mathfrak{m}^{\mathcal{B}}$ <u>separate from</u> $\mathcal{D}$

$$H(\mathcal{G},\mathcal{D}) = Q(\mathcal{G},\mathcal{D}) + R(\mathcal{G},\mathcal{D}).$$

<u>Moreover</u> R <u>satisfies, for all</u> $\{(\mathcal{G}_i,\mathcal{D}_i)\}^N$ <u>in</u> $\mathfrak{n}^{\mathcal{B}}$,

$$\left|\sum^N R(\mathcal{G}_i,\mathcal{D}_i)\right| \to 0 \quad \underline{\text{as}} \quad \sum^N V(\mathcal{G}_i)V(\mathcal{D}_i)\to 0.$$

<u>Proof</u>: As noted we have already proved the first half of this Proposition. For the remainder recall how R is defined in terms of H from the previous proof and note

$$\left|\sum^N R(\mathcal{G}_i,\mathcal{D}_i)\right| = \lim_{n\to\infty}\left|\sum^N H(\mathcal{G}_i\setminus(\mathcal{G}_i)_n,\mathcal{D}_i)\right|$$

$$\leq \limsup_{n\to\infty} \sum^N \alpha(\mathcal{G}_i\setminus(\mathcal{G}_i)_n,\mathcal{D}_i)\ .$$

But $\sum^N V(\mathcal{G}_i\setminus(\mathcal{G}_i)_n)V(\mathcal{D}_i) \leq \sum^N V(\mathcal{G}_i)V(\mathcal{D}_i)$ so that if the latter is small then the limit superior above is small; hence R satisfies the volume-continuity condition.□

The function $Q(\cdot,\mathcal{D})$ can be shown to depend only upon the (oriented) surface $\partial\mathcal{D}$ in question, so that a function $Q(\mathcal{S})$ can be defined consistently for any surface $\mathcal{S}$ which is part of the surface of a subbody. Finally for any such surface $\mathcal{S}$ we have a function $q_{\mathcal{S}}$ on $\mathcal{S}$ such that for any $\mathcal{D}$ with $\mathcal{S}$ a positively-oriented segment of $\partial\mathcal{D}$

$$Q(\mathcal{S},\mathcal{D}) = Q(\mathcal{S}) = \int_{\mathcal{S}} q_{\mathcal{S}}\,dA.$$

For details of this procedure one is referred to NOLL [1] and GURTIN and WILLIAMS [1].

Regarding R, it is clear that for any $\mathcal{P}\in\mathbf{B}(\mathcal{B})$ if $\mathcal{D}_1,\mathcal{D}_2\in\mathfrak{m}$ are separate and $\mathcal{P}\subset(\mathcal{D}_1\cup\mathcal{D}_2)^b$ then

$$R(\mathcal{P},\mathcal{D}_1\cup\mathcal{D}_2) = R(\mathcal{P},\mathcal{D}_1) + R(\mathcal{P},\mathcal{D}_2).$$

In particular R on $\mathfrak{h}$ is bi-separately-additive, so that if $\mathcal{B}^e\subset\mathcal{D}$ we write

$$R(\mathcal{C},\mathcal{D}) = R(\mathcal{C},\mathcal{D}\wedge\mathcal{B}) + R(\mathcal{C},\mathcal{B}^e).$$

The function $R(\cdot,\mathcal{B}^e)$, since $\mathcal{B}^e$ is unstructured, is clearly volume-continuous on $B(\mathcal{B})$. Thus we deduce the existence of $r_e\colon \mathcal{B}\to\mathbb{R}$ such that

$$R(\mathcal{C},\mathcal{B}^e) = \int_{\mathcal{C}} r_e\, dV.$$

This completes the theory for the <u>external</u> effect $R(\cdot,\mathcal{B}^e)$. Henceforth then we need concern ourselves only with the internal interaction, the map $(\mathcal{C},\mathcal{D})\mapsto R(\mathcal{C},\mathcal{D})$ on the set $\mathfrak{h}^{\mathcal{B}}$ of separate subbodies.

We could go on to extend the function $R(\mathcal{P},\cdot)$ for any $\mathcal{P}\in\mathbf{B}(\mathcal{B})$ to a class of Borel sets but this is inconvenient and unnecessary; we really need only the existence of R on $\mathfrak{h}^{\mathcal{B}}$ and the volume-continuity on $\mathfrak{h}^{\mathcal{B}}$ to establish the desired result. We first note the useful and easily proved

<u>Corollary</u>: <u>If</u> H <u>satisfies the Lipschitz condition then for all</u> $(\mathcal{C},\mathcal{D})\in\mathfrak{h}^{\mathcal{B}}$

$$|R(\mathcal{C},\mathcal{D})| \le \bar{\alpha}V(\mathcal{C})V(\mathcal{D}).$$

Now we shall extend the function $R(\mathfrak{G},\mathfrak{D})$ to (some of) $B(\mathfrak{B}\times\mathfrak{B})$. Let us fix separate subbodies $\mathfrak{G},\mathfrak{C}$, denote subsets of of $\mathfrak{G}$ by $\hat{\mathfrak{G}}$ (subbodies) A, subsets of $\mathfrak{C}$ by $\hat{\mathfrak{C}}$,C.

Proposition 2: There exists a unique countably-additive volume-continuous measure λ on $\mathbb{B}(\mathfrak{G}\times\mathfrak{C})$ such that for all $\hat{\mathfrak{G}},\hat{\mathfrak{C}}$

$$\lambda(\hat{\mathfrak{G}}\times\hat{\mathfrak{C}}) = R(\hat{\mathfrak{G}},\hat{\mathfrak{C}}).$$

Proof: Let us recall that $\mathbb{F}_{\mathfrak{G}}$ is the intersection with $\mathfrak{G}$ of the collection of unions of half-open parallelepipeds. We define a function R on $\mathbb{F}_{\mathfrak{G}}\times\mathbb{F}_{\mathfrak{C}}$ in the expected way:

$$R(A,C) = R(\overline{\overset{\circ}{A}},\overline{\overset{\circ}{C}})$$

($\overline{\overset{\circ}{A}},\overline{\overset{\circ}{C}}$ are subbodies). It is not difficult to verify that R is bi-additive, and that

$$\sum^{N} R(A_i,C_i) \to 0 \quad \text{as} \quad \sum^{N} V(A_i)V(C_i) \to 0.$$

Now let Σ denote the set of all finite disjoint unions of sets of the form $A\times C$ (of course the $A\times C$'s include the collection of half-open 6-intervals intersected with $\mathfrak{B}\times\mathfrak{B}$). Σ is a field and its generated σ-field is $\mathbb{B}(\mathfrak{G},\mathfrak{C})$. Following exactly the technique of construction of product measures (cf. HAHN and ROSENTHAL pp. 223-227) one can construct a unique additive map $\lambda: \Sigma \to R$ which agrees with R on sets $A\times C$; of course

$$\lambda\left(\bigcup^{N} A_i\times C_i\right) = \sum^{N} R(A_i,C_i).$$

Clearly λ is volume-continuous on Σ (we use "volume" and the symbol V for both the usual volume in 3-space and its product with itself) so that it is regular. Since $\mathcal{G}\times\mathcal{C}$ is compact and λ bounded it follows that λ has a unique countably-additive extension to $\mathbb{B}(\mathcal{G}\times\mathcal{C})$ (e.g. DUNFORD and SCHWARTZ p. 138) which we also denote λ. Since λ is volume-continuous on Σ it follows that it is volume-continuous on $\mathbb{B}(\mathcal{G}\times\mathcal{C})$ (e.g. DUNFORD and SCHWARTZ, p. 171).

We must now verify that λ agrees with R on sets $\hat{\mathcal{G}}\times\hat{\mathcal{C}}$. We note that for all $B = \bigcup^{N} A_i \times C_i \in \Sigma$

$$|R(\hat{\mathcal{G}},\hat{\mathcal{C}}) - \lambda(\hat{\mathcal{G}}\times\hat{\mathcal{C}})| \leq |R(\hat{\mathcal{G}},\hat{\mathcal{C}}) - \sum^{N} R(\bar{\mathring{A}}_i,\bar{\mathring{\mathcal{C}}}_i)| + |\lambda(B) - \lambda(\hat{\mathcal{G}}\times\hat{\mathcal{C}})|.$$

It is then easy to see that we can choose B such that both

$$|\lambda(B) - \lambda(\hat{\mathcal{G}}\times\hat{\mathcal{C}})|$$

and

$$|V(B\,\Delta\,\hat{\mathcal{G}}\ \hat{\mathcal{C}})|$$

are arbitrarily small (Δ denotes the symmetric difference). But we can clearly write

$$\begin{aligned}|R(\hat{\mathcal{G}},\hat{\mathcal{C}}) - \sum^{N} R(\bar{\mathring{A}}_i,\bar{\mathring{C}}_i)| = |\sum_i^N \sum_j^N (&R(\hat{\mathcal{G}}\setminus\bar{\mathring{A}}_i,\hat{\mathcal{C}}) \\ &- R(\bar{\mathring{A}}_i\setminus\hat{\mathcal{G}},\bar{\mathring{\mathcal{C}}}_i) \\ &+ R(\hat{\mathcal{G}}\wedge\bar{\mathring{A}}_i,\hat{\mathcal{C}}\setminus\bar{\mathring{\mathcal{C}}}_j) \\ &- R(\bar{\mathring{A}}_i\setminus\hat{\mathcal{G}},\bar{\mathring{\mathcal{C}}}_j\wedge\hat{\mathcal{C}})|.\end{aligned}$$

The volume of this miscellany of sets is exactly that of

$B \Delta \hat{\mathcal{G}} \times \hat{\mathcal{C}}$; hence the entire quantity can be made arbitrarily small by taking this volume small. Thus we conclude $|R(\hat{\mathcal{G}},\hat{\mathcal{C}}) - \lambda(\hat{\mathcal{G}} \times \hat{\mathcal{C}})|$ arbitrarily small whence the desired conclusion: $\lambda(\hat{\mathcal{G}} \times \hat{\mathcal{C}}) = R(\hat{\mathcal{G}},\hat{\mathcal{C}})$.

The uniqueness of the extension is easily shown since $\lambda(A \times C) = \lambda(\overline{\overset{\circ}{A}} \times \overline{\overset{\circ}{C}})$ for all such extensions.□

We shall when necessary write λ as $\lambda_{\mathcal{G},\mathcal{C}}$ to emphasize its dependence on $(\mathcal{G},\mathcal{C})$; there exists $r_{\mathcal{G},\mathcal{C}}: \mathcal{G} \times \mathcal{C} \to \mathbb{R}$ such that

$$\lambda_{\mathcal{G},\mathcal{C}}(A) = \int_A r_{\mathcal{G},\mathcal{C}}\, dV.$$

<u>Corollary</u>: <u>If</u> H <u>is Lipschitz continuous then</u> $\lambda, r_{\mathcal{G},\mathcal{C}}$ <u>obeys</u>

$$|\lambda(A)| \leq \bar{\alpha}\, V(A), \quad \underline{\text{all}} \quad A \in \mathbb{B}(\mathcal{G} \times \mathcal{C})$$

$$|r_{\mathcal{G},\mathcal{C}}| \leq \bar{\alpha}, \qquad V - \underline{\text{a.e.}}.$$

That λ satisfies this bound on Σ is trivially true; the fact that it is true on $\mathbb{B}(\mathcal{G} \times \mathcal{C})$ follows as in GURTIN and WILLIAMS [1, Appendix] and the bound on $r_{\mathcal{G},\mathcal{C}}$ is then obvious.

Central to the remainder of our work is the

<u>Lemma</u>: <u>If</u> $\mathcal{G},\mathcal{C}$, $\hat{\mathcal{G}},\hat{\mathcal{C}}$ <u>are subbodies</u>, $\mathcal{G},\mathcal{C}$ <u>separate and</u> $\hat{\mathcal{G}} \subset \mathcal{G}$, $\hat{\mathcal{C}} \subset \mathcal{C}$ <u>then</u>

$$r_{\hat{\mathcal{G}},\hat{\mathcal{C}}} = r_{\mathcal{G},\mathcal{C}}\big|_{\hat{\mathcal{G}} \times \hat{\mathcal{C}}}, \quad V - \underline{\text{a.e.}}$$

<u>Proof</u>: Let $\hat{\Sigma}$ be the field defined by $\hat{\mathcal{G}},\hat{\mathcal{C}}$. For any $\mathcal{G}^*,\mathcal{C}^*$ with $\mathcal{G}^* \subset \hat{\mathcal{G}}$, $\mathcal{C}^* \subset \hat{\mathcal{C}}$

$$\lambda_{\mathcal{G},\mathcal{C}}(\mathcal{G}^*,\mathcal{C}^*) = R(\mathcal{G}^*,\mathcal{C}^*) = \lambda_{\hat{\mathcal{G}},\hat{\mathcal{C}}}(\mathcal{G}^*,\mathcal{C}^*).$$

It follows that $\lambda_{\mathcal{G},\mathcal{C}}$ and $\lambda_{\hat{\mathcal{G}},\hat{\mathcal{C}}}$ agree on $\hat{\Sigma}$ and hence on $\mathbb{B}(\hat{\mathcal{G}}\times\hat{\mathcal{C}})$. This means in turn that $r_{\mathcal{G},\mathcal{C}} = r_{\hat{\mathcal{G}},\hat{\mathcal{C}}}$ V-a.e. on $\hat{\mathcal{G}}\times\hat{\mathcal{C}}$. □

We wish to construct an integrable function which will yield $R(\mathcal{G},\mathcal{C})$ for <u>any</u> separate subbodies, $\mathcal{G},\mathcal{C}$. The above result makes it clear that we may expect to do so. The easiest way to define this function is by means of taking successively refined sequences of partitions of $\mathcal{B}$.

If Γ denotes any partition of $\mathcal{B}$ into subbodies, i.e. $\Gamma = \{\mathcal{G}_i\}^N$ where $\mathcal{G}_i$ are separate and sum to $\mathcal{B}$, we define an integrable function r_Γ on $\mathcal{B}\times\mathcal{B}$ as

$$r_\Gamma = \sum_{i\neq j}^{N} \chi_{\mathcal{G}_i\times\mathcal{G}_j} r_{ij}$$

where $r_{ij} = r_{\mathcal{G}_i,\mathcal{G}_j}$, and χ_A is the characteristic function for the set A.

<u>Lemma</u>: <u>For any partion</u> $\Gamma = \{\mathcal{G}_i\}^N$ if $\mathcal{G} = \bigcup_{\Lambda} \mathcal{G}_i$, $\mathcal{C} = \bigcup_{\Lambda'} \mathcal{G}_i$ <u>are separate</u>

$$R(\mathcal{G},\mathcal{C}) = \int_{\mathcal{G}\times\mathcal{C}} r_\Gamma dV.$$

<u>Proof</u>: We have

$$R(\mathcal{G},\mathcal{C}) = \int_{\mathcal{G}\times\mathcal{C}} r_{\mathcal{G},\mathcal{C}} dV.$$

But $\mathcal{G}\times\mathcal{C} = \bigcup_{\substack{i\in\Lambda \\ j\in\Lambda'}} \mathcal{G}_i \times \mathcal{G}_j$ and on $\mathcal{G}_i\times\mathcal{G}_j$, $r_{\mathcal{G},\mathcal{C}} = r_{\mathcal{G}_i,\mathcal{G}_j} = r_\Gamma|_{\mathcal{G}_i\times\mathcal{G}_j}$ by the definition of r_Γ; this yields the desired result. □

These two lemmas suffice to establish the desired result as follows: Consider a sequence $\{\Gamma_n\}$ of partitions, each a refinement of the previous one, such that

$$\operatorname{diam} \mathfrak{a}_i^{(n)} \leq 1/n$$

for all $\mathfrak{a}_i^{(n)} \in \Gamma_n$ (it is clear that there exists at least one such sequence). We write $r_n = r_{\Gamma_n}$ and note that (cf. Figure 1) r_n and r_{n+1} agree almost everywhere in $\bigcup_{i \neq j}^{N_n} \mathfrak{a}_i^{(n)} \times \mathfrak{a}_j^{(n)}$, or, less precisely, outside the set

$$A_n = \{(x,y) \in \mathfrak{B} \times \mathfrak{B} \,\big|\, |x-y| < 1/n\}$$

(cf. Figure 2). This creates a very simple sequence of functions $\{r_n\}$. If B_n is the null set of $\mathfrak{B} \times \mathfrak{B} - A_n$ on which r_n fails to equal r_{n+1} it is clear that outside $\bigcup_{p=n}^{\infty} B_p \cup A_n$ one has $r_{n+i} = r_n$ for all integers i. Hence it is clear that

$$r(x,y) = \lim_{n \to \infty} r_n(x,y)$$

exists for almost every $(x,y) \in \mathfrak{B} \times \mathfrak{B}$ - the excluded set is $\{(x,x) \,|\, x \in \mathfrak{B}\} \cup \bigcup^{\infty} B_n$. We set $r=0$ on this set; since r is the point-wise almost-everywhere limit of a sequence of integrable functions it is measurable. It is in fact, integrable on any set included in a set of the form $\bigcup_{i \neq j}^{N_n} \mathfrak{a}_i^{(n)} \times \mathfrak{a}_j^{(n)}$ or, less precisely, any set at a finite distance from $\{(x,x) \,|\, x \in \mathfrak{B}\}$.

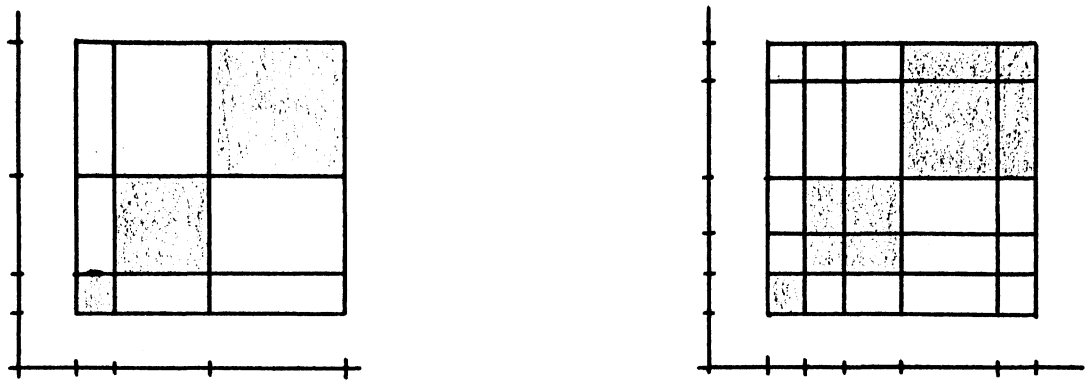
It remains only to show that we obtain the same function regardless of our choice of partitions. If $\{\Gamma_n\}$, $\{r_n\}$ and $\{\hat{\Gamma}_n\}$, $\{\hat{r}_n\}$ are sequences yielding $r, \hat{r}$ respectively we denote by Γ_n^* the common refinement of Γ_n and $\hat{\Gamma}_n$. If r_n^* is the function defined by Γ_n^* it is clear that

$$r_n = r_n^* = \hat{r}_n \quad \text{a.e. on} \quad \mathcal{B}\times\mathcal{B} - A_n,$$

from the definition of Γ_n^* and the lemma above. Hence

$$r = \hat{r} \quad V\text{-a.e.}$$

Considering the way r is defined it is clear that given any $\mathcal{G},\mathcal{C}$ in $\mathcal{H}^{\mathcal{B}}$ the integral of r over $\mathcal{G}\times\mathcal{C}$ is $R(\mathcal{G},\mathcal{C})$, for we can always find a sequence of partitions such that $\mathcal{G}$ and $\mathcal{C}$ are both finite unions of collections from partitions in the sequence. We summarize our results in the next section.



$n^{\underline{th}}$ partition

$n+1^{\underline{st}}$ partition

Figure 1

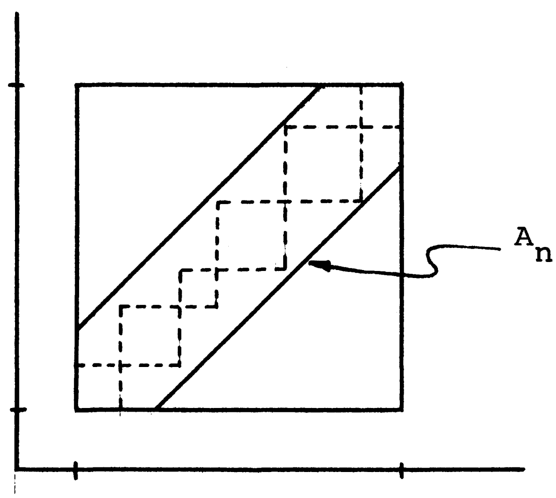


Figure 2

3. Representation for R

We have proved in Section 2

Theorem 1: There exists a unique (V-a.e.) measurable function $r\colon \mathcal{B}\times\mathcal{B} \to \mathbb{R}$ such that for any $\mathcal{A},\mathcal{C}\in\mathfrak{M}^{\mathcal{B}}$

$$R(\mathcal{A},\mathcal{C}) = \int_{\mathcal{A}\times\mathcal{C}} r \, dV.$$

In fact r is integrable on any set $A \subset \mathcal{B}\times\mathcal{B}$ which fails to intersect the interior of some set of the form $\bigcup^{N} \mathcal{A}_i \times \mathcal{A}_i$, $\mathcal{A}_i \in \mathfrak{M}^{\mathcal{B}}$.

Of course we can use this to define $R(P,Q)$ for any Borel sets which are separated by an element of $\mathfrak{M}^{\mathcal{B}}$.

In the case that $|R(\mathcal{A},\mathcal{C})| \leq \bar{\alpha}\, V(\mathcal{A})V(\mathcal{C})$ then all of the functions used in constructing r: $r_{\mathcal{A},\mathcal{C}}, r_{\Gamma}, r_n$, are bounded by $\bar{\alpha}$ so that r is also. Hence the

Corollary: If H is Lipschitz continuous then r is integrable on $\mathcal{B}\times\mathcal{B}$ and $|r| \leq \bar{\alpha}$ V-a.e.

Combining Theorem 1 with the earlier results we obtain, if $\mathcal{B}^e \subset \mathcal{D}$,

$$R(\mathcal{A},\mathcal{D}) = \int_{\mathcal{A}\times\mathcal{D}_1} r\,dV + \int_{\mathcal{A}} r_e\,dV$$

where $\mathcal{D}_1 = \mathcal{D} \setminus \mathcal{B}^e$, so that

$$H(\mathcal{A},\mathcal{D}) = \int_{\mathcal{A}\cap\partial\mathcal{D}} q_{\partial\mathcal{D}}\,dA + \int_{\mathcal{A}\times\mathcal{D}_1} r\,dV + \int_{\mathcal{A}} r_e\,dV.$$

(To fix terminology we note that if H denotes heat flux then $q_{\mathcal{S}}$ describes the heat conduction through $\mathcal{S}$, r the internal radiation, r_e the external radiation or heat supply. If H denotes forces then these are respectively the contact force across $\mathcal{S}$, the mutual body force and the external body force .)

It may often happen that the original function H is balanced, i.e. that for each $\mathcal{A}, \mathcal{C}$ in $\mathfrak{h}^{\mathcal{B}}$

$$H(\mathcal{A},\mathcal{C}) = -H(\mathcal{C},\mathcal{A}) .$$

This balance is, for example, true for body forces as a consequence of the law of balance of momentum (NOLL [1, p.275]) and for heat flux in an immobile body the first law of thermodynamics requires it hold (GURTIN and WILLIAMS [1, p. 9]).[1] If H is balanced, then it is easy to show that R also is balanced. In this case the previous analysis is greatly simplified. The easiest procedure is to start at the beginning and define a new function $\overline{R}$ on all of $\mathfrak{m}^{\mathcal{B}} \times \mathfrak{m}^{\mathcal{B}}$ by

$$\overline{R}(\mathcal{A},\mathcal{C}) = R(\mathcal{A} \setminus \mathcal{C},\mathcal{C}) + R(\mathcal{A} \wedge \mathcal{C},\mathcal{C} \setminus \mathcal{A}) .$$

It is clear that $\overline{R}(\mathcal{A},\mathcal{C}) = R(\mathcal{A},\mathcal{C})$ if $\mathcal{A}$ and $\mathcal{C}$ are separate and that $\overline{R}(\mathcal{A},\mathcal{A}) = 0$. A short but messy calculation serves to verify that $\overline{R}$ is balanced (on all of $\mathfrak{m}^{\mathcal{B}} \times \mathfrak{m}^{\mathcal{B}}$).

Lemma: $\overline{R}$ is separately bi-additive and for any $\{(\mathcal{A}_i,\mathcal{D}_i)\}^N$ in $\mathfrak{m}^{\mathcal{B}} \times \mathfrak{m}^{\mathcal{B}}$

$$\sum^N \overline{R}(\mathcal{A}_i,\mathcal{D}_i) \to 0 \quad \text{as} \quad \sum^N V(\mathcal{A}_i)V(\mathcal{D}_i) \to 0$$

Proof: Let $\mathcal{A},\mathcal{C},\mathcal{D}$ be any subbodies, with $\mathcal{A}$ and $\mathcal{D}$ separate. Then

---

[1] In general balance of heat flux is not to be expected; in the presence of internal radiation the entropy flux is balanced only in pathological cases.

$$\begin{aligned}\overline{R}(\mathcal{A}\cup\mathcal{B},\mathcal{C}) &= R((\mathcal{A}\cup\mathcal{B})\setminus\mathcal{C},\mathcal{C}) + R((\mathcal{A}\cup\mathcal{B})\wedge\mathcal{C},\mathcal{C}\setminus(\mathcal{A}\cup\mathcal{B}))\\ &= R(\mathcal{A}\setminus\mathcal{C},\mathcal{C}) + R(\mathcal{B}\setminus\mathcal{C},\mathcal{C}) + R(\mathcal{A}\wedge\mathcal{C},\mathcal{C}\setminus(\mathcal{A}\cup\mathcal{B}))\\ &\quad + R(\mathcal{B}\wedge\mathcal{C},\mathcal{C}\setminus(\mathcal{A}\cup\mathcal{B}))\\ &= \overline{R}(\mathcal{A},\mathcal{C}) + \overline{R}(\mathcal{B},\mathcal{C}) - R(\mathcal{A}\wedge\mathcal{C},\mathcal{B}\wedge\mathcal{C})\\ &\quad - R(\mathcal{B}\wedge\mathcal{C},\mathcal{A}\wedge\mathcal{C}) = \overline{R}(\mathcal{A},\mathcal{C}) + \overline{R}(\mathcal{B},\mathcal{C})\end{aligned}$$

since R is balanced. Similarly $\overline{R}(\mathcal{C},\mathcal{A}\cup\mathcal{B}) = \overline{R}(\mathcal{C},\mathcal{A}) + \overline{R}(\mathcal{C},\mathcal{B})$.

We need only note, to establish the volume continuity, that $\sum^N V(\mathcal{A}_i\setminus\mathcal{C}_i)V(\mathcal{C}_i) \leq \sum^N V(\mathcal{A}_i)V(\mathcal{C}_i)$ and $\sum^N V(\mathcal{A}_i\wedge \mathcal{C}_i)V(\mathcal{C}_i\setminus\mathcal{A}_i) \leq \sum^N V(\mathcal{A}_i)V(\mathcal{C}_i)$. □

<u>Theorem 2</u>: <u>Let</u> H <u>be balanced</u>. <u>Then there exists a unique</u> (V-<u>a</u>.<u>e</u>.) <u>integrable function</u> $r\colon \mathcal{B}\times\mathcal{B} \to \mathbb{R}$ <u>such that for each</u> $(\mathcal{A},\mathcal{B})\in\mathfrak{h}^{\mathcal{B}}$

$$R(\mathcal{A},\mathcal{B}) = \int_{\mathcal{A}\times\mathcal{B}} r\, dV.$$

<u>Moreover for</u> V-<u>almost every</u> (x,y), $r(x,y) = -r(y,x)$.

<u>Proof</u>: The technique of extending $\overline{R}$ is exactly the same as that for R but simpler, since we do not have to contend with the problem of separation. Thus $\overline{R}$ can be extended to a unique volume-continuous function on $\mathbb{B}(\mathcal{B}\times\mathcal{B})$. This yields the existence of r (the uniqueness is clear since there is only one way to construct $\overline{R}$ from R). Since $\overline{R}$ is balanced

$$\int_{\mathcal{A}\times\mathcal{C}} r\, dV = -\int_{\mathcal{C}\times\mathcal{A}} \hat{r}\, dV$$

for all $\mathcal{A},\mathcal{C}\in\mathfrak{m}^{\mathcal{B}}$; $\mathfrak{m}^{\mathcal{B}}$ is sufficiently rich in sets that this guarantees the essential anti-symmetry of r. □

A result due to GURTIN and WILLIAMS [1, p. 93] says that if H is balanced there exists a function $\hat{r}$ on $\mathcal{B}$ such that for all $\mathcal{G} \in \mathfrak{m}^{\mathcal{B}}$

$$R(\mathcal{G}, \mathcal{G}^e) = \int_{\mathcal{G}} \hat{r}\, dV.$$

Noting that $R(\mathcal{G},\mathcal{G}^e) = R(\mathcal{G},\mathcal{G}^b) + R(\mathcal{G},\mathcal{B}^e)$ one has for almost every $x \in \mathcal{B}$

$$\hat{r}(x) = r_e(x) + \int_{\mathcal{B}} r(x,y)\, dV_y$$

with r as in Theorem 2. This is easily proven by noting $\int_{\mathcal{G}\times\mathcal{G}} r\, dV = 0$.

Finally let us note that even if H is not balanced we are still often interested in evaluating $\frac{1}{V(\mathcal{G})} \int_{\mathcal{G}\times\mathcal{G}^b} r\, dV$ as $V(\mathcal{G}) \to 0$ in order to derive local forms of the various balance laws. If r is integrable over $\mathcal{B}\times\mathcal{B}$ we can write

$$\int_{\mathcal{G}\times\mathcal{G}^b} r dV = \int_{\mathcal{G}\times\mathcal{B}} r dV - \int_{\mathcal{G}\times\mathcal{G}} r dV$$

$$= \int_{\mathcal{G}}\int_{\mathcal{B}} r(x,y)\, dV_y dV_x - \int_{\mathcal{G}\times\mathcal{G}} r dV.$$

Since $r(x,\cdot)$ for almost every x is integrable we obtain

Proposition 3: *If* r *is integrable then for* V-*a.e.* x *and any sequence* $\{\mathcal{G}_n\}$ *in* $\mathfrak{m}^{\mathcal{B}}$ *converging to* $\{x\}$

$$\lim_{n\to\infty} \frac{1}{V(\mathcal{G}_n)} R(\mathcal{G}_n, \mathcal{G}_n^e) = \int_{\mathcal{B}} r(x,y)\, dV_y + r_e(x).$$

Remark. We may replace the assumption I by the stronger assumption that for some $\gamma \in R$, for all $(\mathcal{G},\mathcal{D}) \in \mathfrak{h}^{\mathcal{B}}$

$$|H(\mathfrak{a},\mathfrak{D})| \leq \gamma\left(\iint_{\mathfrak{a}\,\mathfrak{D}} \frac{1}{|x-y|^{3-\delta}}\, dV_y dV_x + A(\mathfrak{a}\cap\partial\mathfrak{D})\right)$$

and for all $\mathfrak{a}\in\mathfrak{m}^{\mathfrak{B}}$

$$|H(\mathfrak{a},\mathfrak{B}^e)| \leq \gamma(V(\mathfrak{a}) + A(\mathfrak{a}\cap\partial\mathfrak{B})).$$

In this case one obtains $\underset{\sim}{I}$ as a consequence and in addition that r is essentially bounded by the function $(x,y)\mapsto \frac{\gamma}{|x-y|^{3-\delta}}$ which implies r integrable. The motivation for this is that most internal effects are usually assumed to obey a reciprocal distance law; the $3-\delta$ power limitation ensures integrability.

4. The Second Law and Internal Radiation

We now consider the modification of the second law of thermodynamics previously mentioned. We will omit discussion of the physical motivations for the basic definitions of entropy, heat flux and entropy flux, referring the reader to GURTIN and WILLIAMS [1] for such a discussion. We only note here that the results of this section apply not only to rigid heat conductors but to general continua since mechanical interactions do not occur in the second law - the only change is that one should then consider $\mathfrak{B}$ a manifold, and as pointed out previously this introduces only notational inconveniences. We do <u>not</u> assume the heat flux balanced.

We suppose that the <u>rate of change of entropy</u> of any sub-body $\mathfrak{A}$ is given by $\dot{S}(\mathfrak{A})$, where $\dot{S}$ is a real-valued function on $\mathfrak{M}^{\mathfrak{B}}$ and that the <u>heat flux</u> and <u>entropy flux</u> from $\mathfrak{D}\in\mathfrak{M}$ into $\mathfrak{A}\in\mathfrak{M}^{\mathfrak{B}}$ are given by $H(\mathfrak{A},\mathfrak{D})$ and $M(\mathfrak{A},\mathfrak{D})$ respectively, where H and M are both bi-separately-additive and obey condition $\underset{\sim}{I}$ of Section 2.

As pointed out by Gurtin and Williams the statement of the second law (for non-isolated systems) actually consists of two statements: first that the production of entropy of the system (growth of "internal" entropy less the net influx of entropy) is non-negative and second that entropy is not transferred into a thermally isolated system. The first is simple to formulate; for the second one, one must examine what is meant by thermal isolation.

For a continuous body it seems clear that the first should be (GURTIN and WILLIAMS [1]):

T1. For all $\mathcal{A} \in \mathfrak{m}^{\mathcal{B}}$

$$\dot{S}(\mathcal{A}) \geq M(\mathcal{A},\mathcal{A}^e).$$

Gurtin and Williams take as the second part

T2. If $\mathcal{D} \in \mathfrak{m}$, $\mathcal{P} \in B(\mathcal{D}^b)$ are such that

$$H(\mathcal{P}',\mathcal{D}) = 0 \quad \text{for all} \quad \mathcal{P}' \subset \mathcal{P} \quad \text{then}$$
$$M(\mathcal{P},\mathcal{D}) = 0.$$

This statement is concise and mathematically convenient. On physical grounds however it is more acceptable to restate this in terms only of elements of $\mathfrak{m}$; T2 is equivalent to the statement:

For each ϵ there exists a δ such that if $(\mathcal{A},\mathcal{D}) \in \mathfrak{h}$ and $|H(\hat{\mathcal{A}},\mathcal{D})| < \delta$ for each subbody $\hat{\mathcal{A}} \subset \mathcal{A}$ then $|M(\mathcal{A},\mathcal{D})| < \epsilon$.

Thus this requires that as the amount of heat accepted from $\mathcal{D}$ by each subbody in $\mathcal{A}$ decreases to zero so does the entropy accepted by $\mathcal{A}$ from $\mathcal{D}$. In particular, if there is no heat flux into any such element from $\mathcal{D}$ then there is no entropy flux into $\mathcal{A}$ from $\mathcal{D}$.

Such a statement may be too restrictive, however. Thus one has the possibility that $\mathcal{D}$ can be written as the separate union $\mathcal{D}_1 \cup \mathcal{D}_2$ where each of $\mathcal{D}_1, \mathcal{D}_2$ has considerable intercourse with each subbody $\hat{\mathcal{A}}$ in $\mathcal{A}$, but $H(\hat{\mathcal{A}},\mathcal{D}_1) = -H(\hat{\mathcal{A}},\mathcal{D}_2)$ so that $H(\hat{\mathcal{A}},\mathcal{D}) = 0$ for all $\hat{\mathcal{A}} \subset \mathcal{A}$ and thus $M(\mathcal{A},\mathcal{D}) = 0$. This means that T2 requires the entropy flux vanish even though there is a significant amount of *thermal interaction* between $\mathcal{A}$ and $\mathcal{B}$. It seems more consistent with the verbal statement of the second law to assume instead that the entropy flux from $\mathcal{B}$ into $\mathcal{A}$ be required to vanish only when there is no such thermal interaction, i.e. only when there is no heat flux into any subbody in $\mathcal{A}$ from *any subelement* of $\mathcal{B}$.

To formalize this we must unfortunately have recourse, as in **I**, to ensure a certain degree of uniformity by making a statement in terms of finite collections in $\mathfrak{n}$. Thus we propose

T2*. *For each* ϵ *there exists* δ *such that if* $\{(\mathcal{A}_i,\mathcal{B}_i)\}\in\mathfrak{n}$ *satisfy*

$$\sum^{N} |H(\hat{\mathcal{A}}_i,\hat{\mathcal{B}}_i)| < \delta$$

for all $\hat{\mathcal{A}}_i \subset \mathcal{A}_i$ *in* $\mathfrak{m}^{\mathcal{B}}$, $\hat{\mathcal{B}}_i \subset \mathcal{B}_i$ *in* $\mathfrak{m}$ *then*

$$\left|\sum^{N} M(\mathcal{A}_i,\mathcal{B}_i)\right| < \epsilon$$

Of course if there is no internal radiation, i.e. if $H(\mathcal{A},\mathcal{B}) = 0$ for all *disjoint* pairs of subbodies then it follows that $H(\hat{\mathcal{A}}_i,\hat{\mathcal{B}}_i) = H(\tilde{\mathcal{A}}_i,\mathcal{B}_i)$ for some $\tilde{\mathcal{A}}_i$ (since $\hat{\mathcal{A}}_i,\hat{\mathcal{B}}_i$ have contact only through $\mathcal{A}_i \cap \partial\mathcal{B}_i$ and $\mathcal{B}^e$ has no subelements), and thus **T2*** is only a slightly strengthened form of **T2**.

This remark indicates also that the distinction between **T2** and **T2*** must have effect only upon the internal radiation and not upon the conductive (area continuous) and external radiative heat transfers. Proposition 4 below verifies this, and we thereafter turn to the question of consequences of **T2*** upon the internal radiation.

Suppose that the decompositions of H and M are

$$H(\mathcal{A},\mathcal{D}) = R(\mathcal{A},\mathcal{D}) + Q(\mathcal{A}\cap\partial\mathcal{D})$$

$$M(\mathcal{A},\mathcal{D}) = K(\mathcal{A},\mathcal{D}) + J(\mathcal{A}\cap\partial\mathcal{D}).$$

We will call a surface $\mathcal{S}$ in $\mathcal{B}$ a material surface if it is of the form $\partial\mathcal{A}\cap\mathcal{D}$ for some $\mathcal{A},\mathcal{D}\in\mathfrak{m}$. The following does not use the fact that the functions can be extended to Borel sets.

Proposition 4: T2* is equivalent to the two statements:

i) for each ϵ there exists δ such that $\sum^N |R(\hat{\mathcal{A}}_i,\hat{\mathcal{D}}_i)| < \delta$ for each $\hat{\mathcal{A}}_i \subset \mathcal{A}_i$ in $\mathfrak{m}^{\mathcal{B}}$, $\hat{\mathcal{D}}_i \subset \mathcal{D}_i$ in $\mathfrak{m}$ implies $|\sum^N K(\mathcal{A}_i,\mathcal{D}_i)| < \epsilon$;

ii) for each ϵ there exists δ such that $\sum^N |Q(\hat{\mathcal{S}}_i)| < \delta$ for each set of material surfaces $\hat{\mathcal{S}}_i \subset \mathcal{S}_i$ implies $|\sum^N J(\mathcal{S}_i)| < \epsilon$.

Proof: Let i), ii) hold. For any given ϵ let δ satisfy both of i), ii). Suppose $\{(\mathcal{A}_i,\mathcal{D}_i)\}^N \in \mathfrak{n}$ are such that $\sum^N |H(\hat{\mathcal{A}}_i,\hat{\mathcal{D}}_i)| < \delta$ for all $\hat{\mathcal{A}}_i \subset \mathcal{A}_i, \hat{\mathcal{D}}_i \subset \mathcal{D}_i$. Then it must also be be true that $\sum^N |R(\hat{\mathcal{A}}_i,\hat{\mathcal{D}}_i)| < \delta$ and $\sum^N |Q(\hat{\mathcal{S}}_i)| < \delta$ for all $\hat{\mathcal{S}}_i \subset \mathcal{A}_i \cap \partial\mathcal{D}_i$. To show this for R: suppose there exist $\{\hat{\mathcal{A}}_i,\hat{\mathcal{D}}_i\}^N$ such that the sum exceeds δ, say by an amount λ. For each i we can choose $\tilde{\mathcal{A}}_i \subset \hat{\mathcal{A}}_i$ such that $\tilde{\mathcal{A}}_i$ is disjoint from $\mathcal{D}_i$ and $|H(\tilde{\mathcal{A}}_i,\hat{\mathcal{D}}_i) - R(\hat{\mathcal{A}}_i,\hat{\mathcal{D}}_i)| < \frac{\lambda}{N}$ (recall how R is defined in terms of H). Thus $|\sum^N |H(\tilde{\mathcal{A}}_i,\hat{\mathcal{D}}_i)| - \sum^N |R(\hat{\mathcal{A}}_i,\hat{\mathcal{D}}_i)|| < \lambda$, so $\sum^N |H(\tilde{\mathcal{A}}_i,\hat{\mathcal{D}}_i)| > \delta$, a contradiction. One can argue in a similar manner to establish the bound on Q. Thus i) and ii) imply

$$\left|\sum^N M(\mathcal{G}_i,\mathcal{D}_i)\right| \le \left|\sum^N K(\mathcal{G}_i,\mathcal{D}_i)\right| + \left|\sum^N J(\mathcal{G}_i \cap \partial\mathcal{D}_i)\right| < 2\epsilon$$

whence T2*.

Conversely suppose T2*. Let $\{\mathcal{S}_i\}^N$ be any collection of material surfaces. We can choose $\{(\mathcal{G}_i,\mathcal{D}_i)\}^N$ in $\mathfrak{h}$ such that $\mathcal{G}_i \cap \partial\mathcal{D}_i = \mathcal{S}_i$ for each i and we can further assure their volume sufficiently small that both $\sum^N |R(\hat{\mathcal{G}}_i,\hat{\mathcal{D}}_i)|$ and $\sum^N K(\hat{\mathcal{G}}_i,\hat{\mathcal{D}}_i)$ for all $\hat{\mathcal{G}}_i \subset \mathcal{G}_i$, $\hat{\mathcal{D}}_i \subset \mathcal{D}_i$ are arbitrarily small, since both R and K obey the volume-continuity condition of Section 2. (We need in fact only the bounds $|R(\mathcal{G},\mathcal{D})| \le \bar{\alpha}V(\mathcal{G})$, $|K(\mathcal{G},\mathcal{D})| \le \bar{\alpha}V(\mathcal{G})$.) Then for all $\hat{\mathcal{G}}_i \subset \mathcal{G}_i$, $\hat{\mathcal{D}}_i \subset \mathcal{D}_i$ we note

$$\sum^N |H(\hat{\mathcal{G}}_i,\hat{\mathcal{D}}_i)| \le \sum^N |Q(\hat{\mathcal{S}}_i)| + \sum^N |R(\hat{\mathcal{G}}_i,\hat{\mathcal{D}}_i)|$$

when $\hat{\mathcal{S}}_i = \hat{\mathcal{G}}_i \cap \partial\hat{\mathcal{D}}_i \subset \mathcal{S}_i$.

Now suppose $\epsilon > 0$ is prescribed. We choose δ as given by T2* and suppose that $\{\mathcal{S}_i\}^N$ are such that $\sum^N |Q(\hat{\mathcal{S}}_i)| < \delta$ for all $\hat{\mathcal{S}}_i \subset \mathcal{S}_i$. Then we can choose $\{(\mathcal{G}_i,\mathcal{D}_i)\}^N$ as above and by taking them of sufficiently small volume assure

$$\sum^N |H(\hat{\mathcal{G}}_i,\hat{\mathcal{D}}_i)| < \delta$$

for all $\hat{\mathcal{G}}_i \subset \mathcal{G}_i$, $\hat{\mathcal{D}}_i \subset \mathcal{D}_i$. Thus from T2*

$$\epsilon > \left|\sum^N M(\mathcal{G}_i,\mathcal{D}_i)\right| \ge \left|\sum^N J(\mathcal{S}_i)\right| - \left|\sum^N K(\mathcal{G}_i,\mathcal{D}_i)\right|$$

and if the $(\mathcal{G}_i,\mathcal{D}_i)$ have been properly chosen we conclude

$$\left|\sum^N J(\mathcal{S}_i)\right| < \epsilon,$$

establishing ii).

Condition i) follows by an analogous argument which we spare the reader. □

An immediate consequence of this Proposition is that as measures on $\mathbb{B}(\mathcal{B})$ J is absolutely continuous with respect to Q and $K(\cdot,\mathcal{B}^e)$ absolutely continuous with respect to $R(\cdot,\mathcal{B}^e)$. (Of course i),ii) are in fact stronger than necessary for these conclusions.) Thus we can deduce the existence of functions θ_e and (for each material surface $\mathcal{S}$) $\varphi_{\mathcal{S}}$ on $\mathcal{B}$ and $\mathcal{S}$ respectively such that

$$J(\mathcal{S}) = \int_{\mathcal{S}} \frac{q_{\mathcal{S}}}{\varphi_{\mathcal{S}}} \, dA$$

$$K(\mathcal{A},\mathcal{B}^e) = \int_{\mathcal{A}} \frac{r_e}{\theta_e} \, dV.$$

We have established in this way the existence of <u>conductive temperature functions</u> $\varphi_{\mathcal{S}}$ and an <u>external radiation temperature</u> θ_e [1]. Conditions which lead to the existence of a single conductive temperature field φ and which lead to the conclusion that $\theta_e = \varphi$ in the absence of internal radiation are given by GURTIN and WILLIAMS [1, Section VIII] and [2].

We now immediately establish the main result of this section.

<u>Theorem 3</u>: <u>There exists a function θ on $\mathcal{B}\times\mathcal{B}$ such that $\frac{r}{\theta}$ is measurable and for each pair of separate subbodies</u>

$$K(\mathcal{A},\mathcal{C}) = \int_{\mathcal{A}\times\mathcal{C}} \frac{r}{\theta} \, dV.$$

---

[1] CHEN and GURTIN call θ_e the thermodynamic temperature.

<u>In fact</u> $\frac{r}{\theta}$ <u>is integrable over any set in</u> $\mathfrak{B}\times\mathfrak{B}$ <u>not intersecting the interior of some set of the form</u> $\bigcup^{N} \mathfrak{a}_i \times \mathfrak{a}_i$, $\mathfrak{a}_i \in \mathfrak{M}^{\mathfrak{B}}$.

<u>Proof</u>: Recall the nomenclature of Section 2 (Proposition 2). Consider first any fixed $(\mathfrak{a},\mathfrak{C}) \in \mathfrak{h}^{\mathfrak{B}}$. Let λ_R, λ_K denote the measures induced on $\mathbb{B}(\mathfrak{a}\times\mathfrak{C})$ by R and K respectively.

We now show that λ_K is absolutely continuous with respect to λ_R on the field Σ, which implies that this is also true on $\mathbb{B}(\mathfrak{a}\times\mathfrak{C})$.

Suppose $B \in \Sigma$ is given as $B = \bigcup^{N} A_i \times C_i$ where $A_i \in \mathbb{F}_{\mathfrak{a}}$, $C_i \in \mathbb{F}_{\mathfrak{C}}$, and that $|\lambda_R|(B) < \delta$ (where $|\lambda_R|$ is the total variation of the measure λ_R). For any $\mathfrak{a}_i \subset \overline{\mathring{A}}_i$, $\mathfrak{C}_i \subset \overline{\mathring{C}}_i$ clearly

$$\sum^{N} |R(\mathfrak{a}_i, \mathfrak{C}_i)| = \sum^{N} |\lambda_R(\mathfrak{a}_i \times \mathfrak{C}_i)| \leq |\lambda_R|(B) < \delta.$$

Hence, if δ is properly chosen we can ensure that

$$|\lambda_K(B)| = |\sum^{N} K(\overline{\mathring{A}}_i, \overline{\mathring{C}}_i)|$$

is arbitrarily small. Thus there exists $\theta_{\mathfrak{a},\mathfrak{C}}$ such that

$$\lambda_K(A) = \int_A \frac{r}{\theta_{\mathfrak{a},\mathfrak{C}}} \, dV$$

whenever $A \subset \mathfrak{a}\times\mathfrak{C}$. We extend θ to all of $\mathfrak{B}\times\mathfrak{B}$ exactly as in the material preceeding Theorem 1.□

<u>Remark</u>: It is clear that the assumption <u>T2*</u> is stronger than is necessary to establish Theorem 3; it was chosen as the physically most appealing of several unappealing possible assumptions. The weakest possible assumption would be: <u>for each</u>

ϵ <u>there exists</u> δ <u>such that</u> $\sum^{M} |H(\hat{\mathcal{G}}_i, \hat{\mathcal{C}}_i)| < \delta$ <u>for all</u> $\{(\hat{\mathcal{G}}_i, \hat{\mathcal{C}}_i)\}^M$ <u>with</u> $\bigcup^{M} \hat{\mathcal{G}}_i \times \hat{\mathcal{C}}_i \subset \bigcup^{N} \mathcal{G}_i \times \mathcal{C}_i$ <u>implies</u> $|\sum^{N} M(\mathcal{G}_i, \mathcal{C}_i)| < \epsilon$.

Note that θ is defined only λ_R-almost everywhere, and may be infinite-valued. If M satisfies the volume Lipschitz-condition then $\frac{r}{\theta}$ is V-essentially bounded and θ must be λ_R-essentially non-zero.

As pointed out previously it is unreasonable to require that K should be balanced. Hence it cannot be true in general that $\mathcal{G} \to K(\mathcal{G}, \mathcal{G}^e)$ can be expressed in terms of a single density. However we may use Proposition 3 to verify that <u>if</u> $\frac{r}{\theta}$ <u>is integrable then for almost every</u> x <u>in</u> $\mathcal{B}$

$$\lim_{\mathcal{G} \to \{x\}} \frac{1}{V(\mathcal{G})} K(\mathcal{G}, \mathcal{G}^e) = \int_{\mathcal{B}} \frac{r(x,y)}{\theta(x,y)} \, dV_y + \frac{r_e(x)}{\theta_e(x)} \; .$$

Now let us consider an interesting side-light of this result. Let us suppose that $\frac{r}{\theta}$ is integrable on $\mathcal{B} \times \mathcal{B}$, that $J(\mathcal{S}) = \int_{\mathcal{S}} \underset{\sim}{j} \cdot \underset{\sim}{n} \, dA$ where $\underset{\sim}{j}$ is a class C^1 vector field (cf. GURTIN and WILLIAMS [1, pp. 106-110]) and that $\lim_{\mathcal{G} \to \{x\}} \frac{1}{V(\mathcal{G})} \dot{S}(\mathcal{G}) = \dot{s}(x)$ exists with $\dot{s}$ an integrable function. Then we derive the <u>local form of the second law</u>: for V-almost every x

$$\dot{s}(x) \geq \operatorname{div} \underset{\sim}{j}(x) + \frac{r_e(x)}{\theta_e(x)} + \int_{\mathcal{B}} \frac{r(x,y)}{\theta(x,y)} \, dV_y .$$

Now let us integrate this over a subbody $\mathcal{G}$; we obtain

$$\int_{\mathcal{G}} \dot{s} \, dV \geq J(\partial \mathcal{G}) + K(\mathcal{G}, \mathcal{B}^e) + K(\mathcal{G}, \mathcal{G}^b) + \int_{\mathcal{G} \times \mathcal{G}} \frac{r}{\theta} \, dV$$

or

or

$$\int_{\mathcal{A}} \dot{s}\, dV - \int_{\mathcal{A}\times\mathcal{A}} \frac{r}{\theta}\, dV \geq J(\partial\mathcal{A}) + K(\mathcal{A},\mathcal{A}^e) = M(\mathcal{A},\mathcal{A}^e).$$

Thus $\int_{\mathcal{A}} \dot{s}\, dV - \int_{\mathcal{A}\times\mathcal{A}} \frac{r}{\theta}\, dV$ is an upper-bound for $M(\mathcal{A},\mathcal{A}^e)$. This strongly suggests - although it by no means proves - that $\mathcal{A} \to \dot{S}(\mathcal{A})$ should not be required to be separately additive but that it should take the form

$$\dot{S}(\mathcal{A}) = \int_{\mathcal{A}} \dot{s}\, dV + \dot{S}_b(\mathcal{A})$$

where $\dot{S}_b(\mathcal{A}) = o(V(\mathcal{A}))$ as $V(\mathcal{A}) \to 0$ and

$$\dot{S}_b(\mathcal{A}) \geq - \int_{\mathcal{A}\times\mathcal{A}} \frac{r}{\theta}\, dV.$$

Compare this to the presentation of the second law - including a second law for interactions - as presented by GURTIN, NOLL and WILLIAMS.

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Appendix

The axioms which characterize $\mathfrak{M}^{\mathfrak{B}}$ are given by GURTIN and WILLIAMS [1] as

1. Each element of $\mathfrak{M}^{\mathfrak{B}}$ is a regular region.
2. $\mathfrak{A}\in\mathfrak{M}^{\mathfrak{B}}$ implies $\mathfrak{A}^b\in\mathfrak{M}^{\mathfrak{B}}$.
3. $\mathfrak{A},\mathfrak{C}\in\mathfrak{M}^{\mathfrak{B}}$ implies $\mathfrak{A}\cup\mathfrak{C}\in\mathfrak{M}^{\mathfrak{B}}$.
4. If $\mathfrak{C}$ is a solid circular cylinder or a solid prism then $\mathfrak{C}\cap\mathfrak{B}\in\mathfrak{M}^{\mathfrak{B}}$.
5. If $\mathfrak{A}\in\mathfrak{M}^{\mathfrak{B}}$ and $\mathfrak{S}$ is any regular surface included in $\partial\mathfrak{A}$ there exists a monotone sequence $\{\mathfrak{A}_n\}$ in $\mathfrak{M}^{\mathfrak{B}}$, $\mathfrak{A}_n\subset\mathfrak{A}$ such that $\overset{\infty}{\cap}\,\mathfrak{A}_n = \mathfrak{S}$.
6. If $\mathfrak{A}\in\mathfrak{M}^{\mathfrak{B}}$ and $\underset{\sim}{a}$ any vector then $\mathfrak{A}+\underset{\sim}{a}$ contained in $\mathfrak{M}^{\mathfrak{B}}$ implies $\mathfrak{A}+\underset{\sim}{a}\in\mathfrak{M}^{\mathfrak{B}}$.

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