ON THE USE OF SYMMETRY TO SIMPLIFY THE CONSTITUTIVE EQUATIONS OF ISOTROPIC MATERIALS WITH MEMORY

by

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Abstract

This article is concerned with general, compressible, isotropic materials, solid or fluid, characterized by functionals which give the stress when the history of the strain is specified. It is shown that for certain broad classes of motions the requirements of material symmetry and frame-indifference greatly simplify the form of constitutive equations. These simplifications are derived without invoking integral expansions or other special hypotheses of smoothness for material response. Among the motions considered in detail are those which are locally equivalent to pure extensions and sheared extensions.

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Preface

A simple material is one for which the stress at each time t is determined by a functional $\mathcal{S}_{\mathcal{R}}$ of the history $\underset{\sim}{F}^t$ of the deformation gradient $\underset{\sim}{F}$ up to time t:

$$\underset{\sim}{S}(t) = \mathcal{S}_{\mathcal{R}}(\underset{\sim}{F}^t). \tag{1}$$

The functional $\mathcal{S}_{\mathcal{R}}$ generally depends on the choice of the reference configuration $\mathcal{R}$. If, starting from some particular reference configuration, it is found that the group of changes of reference which leave $\mathcal{S}_{\mathcal{R}}$ invariant contains all orthogonal transformations, then the simple material is said to be isotropic.# If $\mathcal{S}_{\mathcal{R}}$ contains all unimodular

#In an early paper on materials with memory, Green and Rivlin (1957) discuss isotropic materials assuming that the constitutive functional can be expressed as a sum of iterated integrals.

transformations, then the material is a simple fluid.##

##These definitions, due to Noll (1958), will be explained in greater detail in Section 2.

For certain classes of motions the constitutive equation (1) can be simplified by making direct use of the symmetry of the material and without calling upon integral expansions or other special assumptions of smoothness for $\mathcal{S}_{\mathcal{R}}$. It is known that this is the case for particular motions of incompressible fluids.[#] In this essay I derive reduced forms

[#] Coleman and Noll (1959 a & b, 1961, 1962).

of (1) which are valid for general isotropic materials, whether solid or fluid, and which hold for broad classes of motions. I then show that in particular kinematical situations use of these reduced forms can greatly simplify the dynamical equations.

No assumption of incompressibility is made in this article; the further reductions yielded by such an assumption will be apparent to the reader.

Chapter I of the article is an introduction to the general theory of isotropic simple materials. The new reduced forms derived there, in Section 2, are valid for arbitrary motions. Chapter II is concerned with simplifications of (1) holding in motions for which the right principal directions of stretch are constant in time at each particle, although these directions of stretch may vary from particle to particle and the amounts of stretch may vary in time. Such motions, called <u>extensions</u>, include as special cases the inflation and stretch of a circular tube, the inflation of a spherical shell, and the bending of a block into a cylindrical wedge. In Chapter III I derive reduced forms of (1) appropriate for <u>sheared extensions</u>, a class of motions containing shearing motions and motions of extension as special cases.

I. General Properties of Isotropic Materials

1. Definitions

A motion of a body $\mathcal{B}$ is described by expressing the position $\underset{\sim}{x}$ at time t of a particle X as a function $\underset{\sim}{\chi}$ of t and the position $\underset{\sim}{X}$ occupied by X in some fixed reference configuration $\mathcal{R}$ of $\mathcal{B}$:

$$\underset{\sim}{x} = \underset{\sim}{\chi}(\underset{\sim}{X},t). \tag{1.1}$$

The gradient $\underset{\sim}{F}$ of the function $\underset{\sim}{\chi}$ with respect to $\underset{\sim}{X}$ is called the <u>deformation gradient</u>:

$$\underset{\sim}{F} = \nabla_{\underset{\sim}{X}}\underset{\sim}{\chi}(\underset{\sim}{X},t). \tag{1.2}$$

Of course, for a given motion, $\underset{\sim}{F}$ depends not only on X and t but also on the choice of the reference configuration $\mathcal{R}$. For the same motion, particle, and time, the deformation gradient $\underset{\sim}{F}'$ relative to some other fixed reference configuration $\mathcal{R}'$ is

$$\underset{\sim}{F}' = \underset{\sim}{F}\underset{\sim}{G}^{-1}, \qquad \text{i.e.} \quad \underset{\sim}{F} = \underset{\sim}{F}'\underset{\sim}{G}. \tag{1.3}$$

Here $\underset{\sim}{G}$ is the deformation gradient at X of $\mathcal{R}'$ relative to $\mathcal{R}$; to indicate this we may use the notation

$$\mathcal{R}' = \underset{\sim}{G}\mathcal{R}. \tag{1.4}$$

It is assumed that deformation gradients are non-singular. Hence, in (1.2) and (1.4) we have det $\underset{\sim}{F} \neq 0$ and det $\underset{\sim}{G} \neq 0$.

The <u>history of the deformation gradient</u> at X up to time t is the function $\underset{\sim}{F}^t$ defined by

$$\underset{\sim}{F}^t(s) = \underset{\sim}{F}(X, t-s), \qquad 0 \leq s < \infty; \tag{1.5}$$

thus $\underset{\sim}{F}^t$ is a member of the class $\underline{\underline{F}}^*$ of all functions which map the positive real axis $[0,\infty)$ into the set $\underline{\underline{F}}$ of non-singular linear transformations.

As stated in the introduction, for a <u>simple material</u> the stress $\underset{\sim}{S}$ at a particle is given by a functional $\mathcal{S}_{\mathcal{R}}$ of the history $\underset{\sim}{F}^t$ of the deformation gradient at that particle:

$$\underset{\sim}{S}(t) = \mathcal{S}_{\mathcal{R}}(\underset{\sim}{F}^t). \tag{1.6}$$

$\mathcal{S}_{\mathcal{R}}$ is called a <u>constitutive functional</u>; its domain of definition is $\underline{\underline{F}}^*$. In (1.6) I have not rendered explicit the understood dependence of $\underset{\sim}{S}$ and $\underset{\sim}{F}^t$ on X. Constitutive functionals are affected by the choice of reference configuration. ~~Indeed,~~ Since the motion determines the stress independently of the reference configuration, it follows from (1.3) that if $\mathcal{R}' = \underset{\sim}{G}\mathcal{R}$, then

$$\mathcal{S}_{\mathcal{R}'}(\underset{\sim}{F}^*) = \mathcal{S}_{\mathcal{R}}(\underset{\sim}{F}^*\underset{\sim}{G}) \tag{1.7}$$

for all $\underset{\sim}{F}^*$ in $\underline{\underline{F}}^*$.

In general, the functional $\mathcal{S}_{\mathcal{R}}$ depends on the particle X under consideration. Let us here assume, however, that there exist reference configurations $\mathcal{R}$ of $\mathcal{B}$ which make $\mathcal{S}_{\mathcal{R}}$ independent of X, i.e. that the body $\mathcal{B}$ is <u>materially homogeneous</u>; these configurations $\mathcal{R}$ are called <u>homogeneous configurations</u>. It is easily seen that if $\mathcal{R}$ is a homogeneous configuration of $\mathcal{B}$ and if $\mathbf{G}$ in (1.4) is independent of X, then $\mathcal{R}'$ is also a homogeneous configuration of $\mathcal{B}$. Unless a statement is made to the contrary, whenever we consider reference configurations, let it be understood that they are homogeneous.

Let $\mathbb{Q}^*$ be the class of all functions $\mathbf{Q}^*$ whose values are orthogonal tensors and whose domain is $[0,\infty)$. For simple materials, the <u>principle of material frame-indifference</u># asserts that for each function

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#The principle, ~~in generality~~ as used here, is due to Noll (1958) who called it "objectivity". See also the work of Green and Rivlin (1957) who used a closely related principle of invariance under "rigid rotations".

$\mathbf{Q}^*$ in $\mathbb{Q}^*$, the functional $\mathcal{S}_{\mathcal{R}}$ must obey the following identity in $\mathbf{F}^*$:

$$\mathcal{S}_{\mathcal{R}}(\mathbf{Q}^*\mathbf{F}^*) = \mathbf{Q}^*(0)\,\mathcal{S}_{\mathcal{R}}(\mathbf{F}^*)\,\mathbf{Q}^*(0)^{-1}. \tag{1.8}$$

If $\rho_{\mathcal{R}}$ and $\rho_{\mathcal{R}^\circ}$ are the mass densities in two reference configurations $\mathcal{R}$ and $\mathcal{R}^\circ$ with $\mathcal{R} = \mathbf{G}\mathcal{R}^\circ$, then

$$\frac{1}{\rho_{\mathcal{R}}} = |\det \mathbf{G}|\,\frac{1}{\rho_{\mathcal{R}^\circ}} . \tag{1.9}$$

Tensors $\mathbf{H}$ for which $|\det \mathbf{H}| = 1$ are called <u>unimodular</u>.

The group $\mathcal{J}_{\mathcal{R}}$ of unimodular tensors $\underset{\sim}{H}$ for which

$$\mathcal{S}_{\underset{\sim}{H}\mathcal{R}} = \mathcal{S}_{\mathcal{R}} \tag{1.10}$$

is called the <u>symmetry group</u> of the material comprising $\mathcal{B}$ relative to the reference configuration $\mathcal{R}$.# The elements of $\mathcal{J}_{\mathcal{R}}$ may be interpreted

#Noll (1958) called $\mathcal{J}_{\mathcal{R}}$ the <u>isotropy group</u> relative to $\mathcal{R}$.

as the deformation gradients of those homogeneous changes of reference configuration, starting from $\mathcal{R}$, which are "undetectable" in the sense that they preserve both the mass density and the form of the constitutive functional. It follows from (1.7) that a unimodular tensor $\underset{\sim}{H}$ is in $\mathcal{J}_{\mathcal{R}}$ if and only if $\mathcal{S}_{\mathcal{R}}$ obeys the equation

$$\mathcal{S}_{\mathcal{R}}(\underset{\sim}{F}^{*}\underset{\sim}{H}) = \mathcal{S}_{\mathcal{R}}(\underset{\sim}{F}^{*}) \tag{1.11}$$

for all functions $\underset{\sim}{F}^{*}$ in its domain.

Since $\mathcal{S}_{\mathcal{R}}$ depends on $\mathcal{R}$, so also does $\mathcal{J}_{\mathcal{R}}$. In fact, it is easily shown that if $\mathcal{R} = \underset{\sim}{G}\mathcal{R}^{\circ}$, then $\mathcal{J}_{\mathcal{R}}$ equals the conjugate of $\mathcal{J}_{\mathcal{R}^{\circ}}$ under the tensor $\underset{\sim}{G}$:

$$\mathcal{J}_{\mathcal{R}} = \underset{\sim}{G}\mathcal{J}_{\mathcal{R}^{\circ}}\underset{\sim}{G}^{-1}; \tag{1.12}$$

i.e. $\underset{\sim}{H}$ is in $\mathcal{J}_{\mathcal{R}}$ if and only if $\underset{\sim}{H} = \underset{\sim}{G}\underset{\sim}{H}^{\circ}\underset{\sim}{G}^{-1}$ for some $\underset{\sim}{H}^{\circ}$ in $\mathcal{J}_{\mathcal{R}^{\circ}}$.

The following definitions are due to Noll (1958): If there exists a reference configuration $\mathcal{R}$ of $\mathcal{B}$ such that $\mathcal{J}_{\mathcal{R}}$ contains the full orthogonal group $\underline{\underline{O}}$ as a subgroup, then the material comprising $\mathcal{B}$ is <u>isotropic</u> and $\mathcal{R}$ is said to be an <u>undistorted</u> <u>reference</u> <u>configuration</u> of $\mathcal{B}$. If, for some $\mathcal{R}$, $\mathcal{J}_{\mathcal{R}} = \underline{\underline{O}}$, then the material is an <u>isotropic</u> <u>solid</u> with undistorted reference configuration $\mathcal{R}$. If $\mathcal{J}_{\mathcal{R}}$ is the full unimodular group $\underline{\underline{U}}$, then the material is a <u>fluid</u>. It follows from (1.12) that if $\mathcal{J}_{\mathcal{R}} = \underline{\underline{U}}$ for one reference configuration $\mathcal{R}^{\circ}$, then $\mathcal{J}_{\mathcal{R}} = \underline{\underline{U}}$ for <u>every</u> other reference configuration $\mathcal{R}$; hence, <u>every</u> <u>homogeneous</u> <u>configuration</u> <u>of</u> <u>a</u> <u>fluid</u> <u>is</u> <u>undistorted</u>.

2. General Reduced Forms for Constitutive Functionals

There is a theorem of group theory which states that if a group $\underline{\underline{G}}$ is a subgroup of the unimodular group $\underline{\underline{U}}$ and contains the orthogonal group $\underline{\underline{O}}$ as a subgroup, then either $\underline{\underline{G}} = \underline{\underline{U}}$ or $\underline{\underline{G}} = \underline{\underline{O}}$.# In our present

#See Brauer (1965) and Noll (1965).

terminology this theorem becomes

Lemma 1: An isotropic material is either a solid or a fluid.

Noll## has observed that for a fluid $\mathcal{S}_{\mathcal{R}}$ can depend on the

##(1958, eq. (21.2).)

reference configuration $\mathcal{R}$ only through the mass density $\rho_{\mathcal{R}}$ of $\mathcal{R}$:

$$\underset{\sim}{S}(t) = \mathcal{S}_{\mathcal{R}}(\underset{\sim}{F}^t) = \mathcal{S}(\underset{\sim}{F}^t;\rho_{\mathcal{R}}). \tag{2.1}$$

In words: For each fluid there exists a functional $\mathcal{S}(\cdot;\rho_{\mathcal{R}})$, depending on only the parameter $\rho_{\mathcal{R}}$, which gives the stress when the history of the deformation gradient relative to $\mathcal{R}$ is known.

We have noted that every reference configuration of a fluid is undistorted; such is not the case for isotropic solids; nor does (2.1) hold for arbitrary reference configurations of a solid. I shall show, however, that (2.1) does hold for an isotropic solid provided it is assumed in advance that the configuration $\mathcal{R}$ is undistorted. That is, the response of an isotropic solid to a given deformation history is the same for all undistorted reference configurations having the same density. The proof rests on a lemma of Coleman and Noll (1964) characterizing the strain relating two undistorted reference configurations of an isotropic solid.

A tensor $\underset{\sim}{G}$ of form $\underset{\sim}{G} = \alpha\underset{\sim}{Q}$, with α a scalar and $\underset{\sim}{Q}$ an orthogonal tensor, is called a <u>similarity transformation</u>.

Lemma 2: <u>Let $\mathcal{R}$ and $\mathcal{R}^\circ$, with $\mathcal{R} = \underset{\sim}{G}\mathcal{R}^\circ$, be two reference configurations of an isotropic solid. If $\mathcal{R}^\circ$ is undistorted, then a necessary and sufficient condition that $\mathcal{R}$ also be undistorted is that $\underset{\sim}{G}$ be a similarity transformation</u>.

Proof#: To prove necessity we assume that $\mathcal{J}_{\mathcal{R}^\circ}$ and $\mathcal{J}_{\mathcal{R}}$

#The proof I give here for general isotropic solids is the same as that given by Coleman and Noll (1964, §2) for elastic isotropic solids. See also Truesdell and Noll (1965, §85).

are both equal to the orthogonal group $\underset{=}{Q}$. Then, by (1.12), $\underset{\sim}{G}$ must be such that $\underset{=}{Q} = \underset{\sim}{G}\underset{=}{Q}\underset{\sim}{G}^{-1}$; i.e. $\underset{\sim}{G}$ must belong to the normalizer of $\underset{=}{Q}$ in the full linear group $\underset{=}{F}$. But, the normalizer of $\underset{=}{Q}$ in $\underset{=}{F}$ is precisely the group of all similarity transformations. To prove sufficiency we assume that $\underset{\sim}{G} = \alpha\underset{\sim}{Q}$ with $\underset{\sim}{Q}$ orthogonal. Then, by (1.12)

$$\mathcal{J}_{\mathcal{R}} = \alpha\underset{\sim}{Q}\mathcal{J}_{\mathcal{R}^\circ}(\alpha\underset{\sim}{Q})^{-1} = \underset{\sim}{Q}\mathcal{J}_{\mathcal{R}^\circ}\underset{\sim}{Q}^{-1}, \tag{2.2}$$

and, by hypothesis, $\mathcal{J}_{\mathcal{R}^\circ} = \underset{=}{Q}$. Since the conjugate of $\underset{=}{Q}$ under any orthogonal tensor is again $\underset{=}{Q}$, it follows from (2.2) that $\mathcal{J}_{\mathcal{R}} = \underset{=}{Q}$; q.e.d.

Theorem 1: *For each isotropic material there exists a single functional $\mathfrak{F}$ such that if $\mathbf{F}^t$ is the history of the deformation gradient relative to any undistorted reference configuration $\mathcal{R}$ with density $\rho_{\mathcal{R}}$, then*

$$\mathbf{S}(t) = \mathscr{S}_{\mathcal{R}}(\mathbf{F}^t) = \mathfrak{F}(\nu \mathbf{F}^t) \tag{2.3}$$

where

$$\nu = \sqrt[3]{\frac{1}{\rho_{\mathcal{R}}}}\,. \tag{2.4}$$

In using this theorem it should be borne in mind that for a fluid the reference configuration $\mathcal{R}$ may be any configuration we wish, even the configuration at time t,# while for an isotropic solid the class

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#This is true even if the global configuration at time t is not homogeneous, but, the simplified terminology I use here, assuming as it does that reference configurations are homogeneous, does not permit a demonstration. The point is, however, easily made precise if one uses Noll's (1958) theory of *local* reference configurations.

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of undistorted reference configurations is limited in accordance with Lemma 2.

Proof of Theorem 1: Let $\mathcal{R}^\circ$ be a fixed undistorted reference configuration of the isotropic material. Let $\mathcal{R}$, with $\mathcal{R} = \mathbf{G}\mathcal{R}^\circ$ be another reference configuration of the same material. Clearly, by (1.9), if we put

$$\mathbf{H} = \sqrt[3]{\frac{\rho_{\mathcal{R}}}{\rho_{\mathcal{R}^\circ}}}\,\mathbf{G}, \qquad \text{i.e.} \qquad \mathbf{G} = \mathbf{H}\sqrt[3]{\frac{\rho_{\mathcal{R}^\circ}}{\rho_{\mathcal{R}}}}\,, \tag{2.5}$$

then $\underset{\sim}{H}$ is a unimodular tensor. By Lemma 1, our isotropic material is either a solid or a fluid. Suppose first that it is a solid and that $\mathcal{R}$ is undistorted. Then, by Lemma 2, $\underset{\sim}{G}$ must be a similarity transformation, and $\underset{\sim}{H}$ in (2.5) must be an orthogonal tensor; i.e. $\underset{\sim}{H}$ must be in $\mathcal{J}_{\mathcal{R}^\circ}$. It follows from (1.7), (2.5) and (1.11) that for all $\underset{\sim}{F}^*$ in $\underset{=}{F}^*$

$$\mathcal{S}_{\mathcal{R}}(\underset{\sim}{F}^*) = \mathcal{S}_{\mathcal{R}^\circ}(\underset{\sim}{F}^*\underset{\sim}{G}) = \mathcal{S}_{\mathcal{R}^\circ}\left(\sqrt[3]{\frac{\rho_{\mathcal{R}^\circ}}{\rho_{\mathcal{R}}}}\,\underset{\sim}{F}^*\underset{\sim}{H}\right) = \mathcal{S}_{\mathcal{R}^\circ}\left(\sqrt[3]{\frac{\rho_{\mathcal{R}^\circ}}{\rho_{\mathcal{R}}}}\,\underset{\sim}{F}^*\right). \tag{2.6}$$

Suppose now that the material is a fluid; then, since $\underset{\sim}{H}$ is unimodular, $\underset{\sim}{H}$ is automatically in $\mathcal{J}_{\mathcal{R}}$, and the equations (2.6) hold again. If we now define $\mathcal{F}$ by the equation

$$\mathcal{F}(\underset{\sim}{F}^*) = \mathcal{S}_{\mathcal{R}^\circ}\left(\sqrt[3]{\rho_{\mathcal{R}^\circ}}\,\underset{\sim}{F}^*\right), \tag{2.7}$$

the theorem follows immediately from (2.6).

Corollary to Theorem 1: <u>With reference to an undistorted configuration $\mathcal{R}$ with density $\rho_{\mathcal{R}}$, the constitutive equation of an isotropic material can be written in the form</u>

$$\underset{\sim}{S}(t) = \mathcal{S}(\underset{\sim}{F}^t;\rho_{\mathcal{R}}), \tag{2.8}$$

<u>where, as the notation indicates, $\mathcal{S}(\cdot;\rho_{\mathcal{R}})$ depends on $\mathcal{R}$ only through $\rho_{\mathcal{R}}$.</u>

Since we here have $\mathcal{J}_{\mathcal{R}} = \underset{=}{Q}$, the functional $\mathcal{S}(\cdot;\rho_{\mathcal{R}}) = \mathcal{S}_{\mathcal{R}}(\cdot)$ must obey (1.11) for all $\underset{\sim}{H}$ that are orthogonal. Furthermore, by the principle of material frame-indifference, $\mathcal{S}(\cdot;\rho_{\mathcal{R}})$ must obey (1.8). Combining the two identities so obtained we get

Theorem 2: For each constant orthogonal tensor $\underset{\sim}{P}$ and each function $\underset{\sim}{Q}^*$ in $\underset{=}{Q}^*$, the functional $\mathcal{S}(\cdot;\rho_{\mathcal{R}})$ of (2.8) must obey the following identity in $\underset{\sim}{F}^*$:

$$\mathcal{S}(\underset{\sim}{Q}^*\underset{\sim}{F}^*\underset{\sim}{P};\rho_{\mathcal{R}}) = \underset{\sim}{Q}^*(0)\,\mathcal{S}(\underset{\sim}{F}^*;\rho_{\mathcal{R}})\underset{\sim}{Q}^*(0)^{-1}. \qquad (2.9)$$

Putting $\underset{\sim}{Q}^*(s) \equiv \underset{\sim}{Q}$, a constant, and $\underset{\sim}{P} = \underset{\sim}{Q}$ in (2.9) we obtain the following

Corollary to Theorem 2: The functional $\mathcal{S}(\cdot;\rho_{\mathcal{R}})$ in (2.8) is an isotropic functional; that is, for each constant orthogonal tensor $\underset{\sim}{Q}$, $\mathcal{S}(\cdot;\rho_{\mathcal{R}})$ obeys the identity

$$\mathcal{S}(\underset{\sim}{Q}\underset{\sim}{F}^*\underset{\sim}{Q}^{-1};\rho_{\mathcal{R}}) = \underset{\sim}{Q}\,\mathcal{S}(\underset{\sim}{F}^*;\rho_{\mathcal{R}})\underset{\sim}{Q}^{-1}. \qquad (2.10)$$

The polar decomposition theorem tells us that the (non-singular) deformation gradient tensor $\underset{\sim}{F}$ can be written in two ways as the product of a symmetric positive-definite tensor and an orthogonal tensor $\underset{\sim}{R}$ called the rotation tensor:

$$\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{U} = \underset{\sim}{V}\underset{\sim}{R}. \qquad (2.11)$$

Not only $\underset{\sim}{R}$, but also the right stretch tensor $\underset{\sim}{U}$ and the left stretch *tensor* $\mathbf{V}$ in (2.11) are uniquely determined by $\mathbf{F}$. The following relations are well known:

$$\left.\begin{aligned} \mathbf{V} &= \mathbf{R}\mathbf{U}\mathbf{R}^T, \\ \mathbf{U}^2 &= \mathbf{F}^T\mathbf{F} \stackrel{\text{def}}{=} \mathbf{C}, \\ \mathbf{V}^2 &= \mathbf{F}\mathbf{F}^T \stackrel{\text{def}}{=} \mathbf{B}. \end{aligned}\right\} \tag{2.12}$$

The tensors $\mathbf{C}$ and $\mathbf{B}$ are called, respectively, the *right* and *left Cauchy-Green tensors*.

The functions $\mathbf{R}^t$, $\mathbf{U}^t$, and $\mathbf{C}^t$ defined by

$$\mathbf{R}^t(s) = \mathbf{R}(X,t-s), \qquad \mathbf{U}^t(s) = \mathbf{U}(X,t-s), \qquad \mathbf{C}^t(s) = \mathbf{C}(X,t-s), \qquad 0 \le s < \infty. \tag{2.13}$$

are the *histories* of the rotation tensor, the right stretch tensor and the right Cauchy-Green tensor at X up to time t. It follows from (2.9) with $\mathbf{F}^*(s) = \mathbf{F}^t(s) = \mathbf{R}^t(s)\mathbf{U}^t(s)$, $\mathbf{P} = \mathbf{1}$, and $\mathbf{Q}^*(s) = \mathbf{R}^t(s)^{-1}$ that

$$\mathcal{S}(\mathbf{U}^t;\rho_{\mathcal{R}}) = \mathbf{R}^t(0)^{-1}\mathcal{S}(\mathbf{F}^t;\rho_{\mathcal{R}})\mathbf{R}^t(0). \tag{2.14}$$

In view of (2.8), the equation (2.14) establishes#

#The reader may find it interesting to contrast our present equation (2.15) with Noll's (1959, Theorem 6, p. 219) equations (20.1) and (20.2). In his equation (20.1), $\mathbf{K}_1$ depends in an unspecified manner on the choice of the undistorted reference configuration (which we call $\mathcal{R}$). In our equation (2.15), $\mathcal{S}$ depends on $\mathcal{R}$ only through the density $\rho_{\mathcal{R}}$; the

the nature of this dependence is rendered explicit in (2.18). Noll's equation (20.2) is, in a sense, more general than (2.15), because his intermediate local reference configuration $\hat{M}$ need not be undistorted, but his (20.2) involves a tensorial parameter "T_ℓ" more complicated than the $\rho_{\mathcal{R}}$ appearing in (2.15). These basic differences are related to the fact that the proof of our present Theorem 3 employs the Lemmas 1 and 2, whereas the proof of Noll's Theorem 6 does not.

Theorem 3: The constitutive equation of a simple isotropic material may be written in the form

$$\tilde{S}(t) = \mathcal{S}(U^t;\rho_{\mathcal{R}}), \tag{2.15}$$

where

$$\tilde{S}(t) = R(t)^{-1}S(t)R(t) \tag{2.16}$$

is called the rotated stress, and $R(t)$ and U^t are, respectively, the rotation tensor at time t and the history of the right stretch tensor up to time t, both taken relative to an arbitrary undistorted reference configuration $\mathcal{R}$ with density $\rho_{\mathcal{R}}$.

The functional $\mathcal{S}$ in (2.15) is the same as that in (2.8)-(2.10). It is, furthermore, clear from the relation

$$\mathcal{S}(F^*;\rho_{\mathcal{R}}) = \mathcal{F}(\nu F^*), \qquad \nu = \sqrt[3]{\frac{1}{\rho_{\mathcal{R}}}} \tag{2.17}$$

that (2.15) can be written in the form

$$\tilde{\bar{S}}(t) = \mathcal{F}(\nu U^t). \qquad (2.18)$$

Let us call the tensor $\bar{S}$, defined by

$$\bar{S} = \frac{\rho_{\mathcal{R}}}{\rho} F^{-1} S (F^{-1})^T, \qquad (2.19)$$

the <u>material stress tensor</u>#. If we define the functional $\mathcal{H}$ by

#Truesdell and Noll (1965, §43A, p. 125) refer to $\bar{S}$ as the "second Piola-Kirchoff tensor".

$$\mathcal{H}(U^{*2};\rho_{\mathcal{R}}) = |\det U^*(0)| U^*(0)^{-1} \mathcal{S}(U^*;\rho_{\mathcal{R}}) U^*(0)^{-1}, \qquad (2.20)$$

for all positive $\rho_{\mathcal{R}}$ and all functions U^* in $\mathbb{F}^*$ whose values are positive-definite symmetric tensors, then $\mathcal{H}(\cdot;\rho_{\mathcal{R}})$ is also isotropic, i.e. for each orthogonal tensor Q the identity

$$\mathcal{H}(QC^*Q^{-1};\rho_{\mathcal{R}}) = Q\mathcal{H}(C^*;\rho_{\mathcal{R}})Q^{-1} \qquad (2.21)$$

holds for all functions C^* in the domain of $\mathcal{H}(\cdot;\rho_{\mathcal{R}})$. Furthermore, we have the following useful

Corollary to Theorem 3: <u>The constitutive equation of an isotropic material may be written in the form</u>

$$\bar{S}(t) = \mathcal{H}(C^t;\rho_{\mathcal{R}}), \quad \text{i.e.} \quad S(t) = \frac{\rho}{\rho_{\mathcal{R}}} F \mathcal{H}(C^t;\rho_{\mathcal{R}}) F^T. \qquad (2.22)$$

<u>where $\bar{S}(t)$ is the material stress tensor at time</u> t, <u>and C^t is the history</u>

of the right Cauchy-Green tensor up to time t, *both computed relative to an undistorted reference configuration* $\mathcal{R}$ *with density* $\rho_{\mathcal{R}}$. *The functional* $\mathcal{H}(\cdot\,;\rho_{\mathcal{R}})$, *defined in* (2.20), *is isotropic and depends on* $\mathcal{R}$ *only through* $\rho_{\mathcal{R}}$.

It follows from (2.8), (2.15), and (2.22) that the isotropy of the functionals $\mathcal{S}(\cdot\,;\rho_{\mathcal{R}})$ and $\mathcal{H}(\cdot\,;\rho_{\mathcal{R}})$ may be expressed as follows: *For each constant orthogonal tensor* $\mathbf{Q}$

$$\mathbf{F}^t \to \mathbf{Q}\mathbf{F}^t\mathbf{Q}^{-1} \implies \mathbf{S}(t) \to \mathbf{Q}\mathbf{S}(t)\mathbf{Q}^{-1}, \tag{2.23}$$

$$\mathbf{U}^t \to \mathbf{Q}\mathbf{U}^t\mathbf{Q}^{-1} \implies \tilde{\mathbf{S}}(t) \to \mathbf{Q}\tilde{\mathbf{S}}(t)\mathbf{Q}^{-1}, \tag{2.24}$$

and

$$\mathbf{C}^t \to \mathbf{Q}\mathbf{C}^t\mathbf{Q}^{-1} \implies \bar{\mathbf{S}}(t) \to \mathbf{Q}\bar{\mathbf{S}}(t)\mathbf{Q}^{-1}. \tag{2.25}$$

Here "$A \to B$" means "A replaced by B", and $\implies$, as usual, denotes an implication.

Henceforth, whenever we discuss motions, deformation gradients, or stretch tensors, it is to be understood that the material under consideration is isotropic and that the reference configuration is not only homogeneous, but also undistorted.

II. Motions of Extension

3. Reduced Forms Valid in Motions of Extension

Since the rotation tensor $\underset{\sim}{R}$ in (2.12) is an orthogonal tensor, the right and left stretch tensors $\underset{\sim}{U}$ and $\underset{\sim}{V}$ have the same proper numbers α_i; these numbers are called principal stretch ratios. The tensors $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are, by definition, positive-definite, and, therefore, the principal stretch ratios are always positive. The proper vectors $\underset{\sim}{u}_i$ and $\bar{\underset{\sim}{u}}_i$ of $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are called, respectively, right and left principal directions of stretch. The vectors $\underset{\sim}{u}_i$ and $\bar{\underset{\sim}{u}}_i$ are not, in general, equal; in fact, it follows from (2.12) that

$$\bar{\underset{\sim}{u}}_i(t-s) = \underset{\sim}{R}(t-s)\underset{\sim}{u}_i(t-s), \qquad 0 \leq s < \infty. \tag{3.1}$$

The proper numbers σ_i and proper vectors $\underset{\sim}{s}_i$ of the stress tensor $\underset{\sim}{S}$ are called principal stresses and principal axes of stress. The principal stresses are also the proper numbers of the rotated stress $\tilde{\underset{\sim}{S}} = \underset{\sim}{R}^{-1}\underset{\sim}{S}\underset{\sim}{R}$, while the principal axes of stress at time t are calculated as follows from the proper vectors $\tilde{\underset{\sim}{s}}_i(t)$ of $\tilde{\underset{\sim}{S}}(t)$:

$$\underset{\sim}{s}_i(t) = \underset{\sim}{R}(t)\tilde{\underset{\sim}{s}}_i(t). \tag{3.2}$$

If the right principal directions of stretch are constant in time at the particle X, then the motion of X is an extension.

More precisely, we say that *the history of* X *up to time* t *has been an extension if there exists (at least) one orthonormal basis* $\mathbf{u}_i$, *independent of* s, *such that the matrix of the components of* $\mathbf{U}(t-s)$ *with respect to* $\mathbf{u}_i$ *has the form*

$$[\mathbf{U}(t-s)] = \begin{bmatrix} \alpha_1(t-s) & 0 & 0 \\ 0 & \alpha_2(t-s) & 0 \\ 0 & 0 & \alpha_3(t-s) \end{bmatrix} \tag{3.3}$$

for all s, $0 \leq s \leq \infty$. The vectors $\mathbf{u}_i$ are then obviously the three right principal directions of stretch which have been constant in time up to the present time t, and the numbers $\alpha_i(t-s)$ are the principal stretch ratios.

Note that we here make no hypothesis about the time-dependence of the rotation tensor $\mathbf{R}$; hence, by (3.1), the left principal directions of stretch $\bar{\mathbf{u}}_i$ need not be constant in time in an extension.

It follows from (2.24) that if a particular orthogonal tensor $\mathbf{Q}$ commutes with $\mathbf{U}(t-s)$ for all s then $\mathbf{Q}$ must commute with $\tilde{\mathbf{S}}(t)$:

$$\mathbf{Q}\mathbf{U}^t\mathbf{Q}^{-1} = \mathbf{U}^t \implies \mathbf{Q}\tilde{\mathbf{S}}(t)\mathbf{Q}^{-1} = \tilde{\mathbf{S}}(t).$$

When the motion is an extension, $\mathbf{U}(t-s)$ commutes with all orthogonal tensors

$\underset{\sim}{Q}$ whose components relative to the basis $\underset{\sim}{u}_i$ have a matrix of the form

$$[\underset{\sim}{Q}] = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} . \qquad (3.5)$$

But, an elementary calculation shows that a symmetric tensor $\underset{\sim}{\tilde{S}}(t)$ commutes with all tensors of the type (3.5) only if the components of $\underset{\sim}{\tilde{S}}(t)$ relative to the basis $\underset{\sim}{u}_i$ have the matrix[#]

[#]See, for example, the argument given by Serrin (1959, p. 232).

$$[\underset{\sim}{\tilde{S}}(t)] = \begin{bmatrix} \sigma_1(t) & 0 & 0 \\ 0 & \sigma_2(t) & 0 \\ 0 & 0 & \sigma_3(t) \end{bmatrix} . \qquad (3.6)$$

Hence, in an extension the proper vectors $\underset{\sim}{\tilde{s}}_i$ of $\tilde{S}(t) = \underset{\sim}{R}(t)^{-1}\underset{\sim}{S}(t)\underset{\sim}{R}(t)$ coincide with the constant right principal directions of stretch:

$$\underset{\sim}{\tilde{s}}_i = \underset{\sim}{u}_i . \qquad (3.7)$$

The equations (3.7) and (3.2) yield

$$\underset{\sim}{s}_i(t) = \underset{\sim}{R}(t)\underset{\sim}{u}_i . \qquad (3.8)$$

Putting s = 0 in (3.1) and comparing the result with (3.8), we see that

$$\underset{\sim}{s}_i(t) = \underset{\sim}{\bar{u}}_i(t) . \qquad (3.9)$$

This proves

<u>Theorem</u> 4: <u>If the motion of an isotropic material is an extension, then at each moment the left principal directions of stretch are also principal axes of stress</u>.

<u>Remark</u>: It follows from (2.12) that the proper vectors of $\underset{\sim}{U}$ coincide with those of right Cauchy-Green tensor $\underset{\sim}{C}$ while the proper vectors of $\underset{\sim}{V}$ coincide with those of left Cauchy-Green tensor $\underset{\sim}{B}$. Hence, to see whether the history of a particle X has been an extension, one may compute $\underset{\sim}{F}(t-s)$ at X for all $s \geq 0$ and then observe whether the proper vectors of $\underset{\sim}{C}(t-s) = \underset{\sim}{F}(t-s)^T\underset{\sim}{F}(t-s)$ are independent of s; if they are, and if the material is isotropic, then the proper vectors of $\underset{\sim}{B}(t) = \underset{\sim}{F}(t)\underset{\sim}{F}(t)^T$ will give the principal axes of stress at X.

Since, according to Theorem 3, $\underset{\sim}{\tilde{S}}(t)$ is given by a function of $\mathcal{S}$ of $\rho_{\mathcal{R}}$ and the history $\underset{\sim}{U}^t$, each non-zero component $\sigma_j(t)$ of the matrix (3.6) is given by a function $\mathcal{f}_j$ of $\rho_{\mathcal{R}}$ and the history of the three non-zero components of the matrix (3.3):

$$\left.\begin{aligned} \sigma_1(t) &= \mathcal{f}_1(\alpha_1^t,\alpha_2^t,\alpha_3^t;\rho_{\mathcal{R}}), \\ \sigma_2(t) &= \mathcal{f}_2(\alpha_1^t,\alpha_2^t,\alpha_3^t;\rho_{\mathcal{R}}), \\ \sigma_3(t) &= \mathcal{f}_3(\alpha_1^t,\alpha_2^t,\alpha_3^t;\rho_{\mathcal{R}}), \\ \alpha_i^t(s) &= \alpha_i(t-s), \qquad 0 \leq s < \infty, \qquad i = 1,2,3. \end{aligned}\right\} \tag{3.10}$$

The functionals $\mathcal{f}_j$ do not depend on the orthonormal basis $\underset{\sim}{u}_i$, i.e. on the direction of stretch. To see this let us note that if $\underset{\sim}{U}^t$ and $\underset{\sim}{U}^{t'}$ are the histories of $\underset{\sim}{U}$ occurring in two motions of extension with different right principal directions of stretch $\underset{\sim}{u}_i$, $\underset{\sim}{u}_i'$, but with the same triplet $\alpha_1^t,\alpha_2^t,\alpha_3^t$ of stretch-ratio histories, then $\underset{\sim}{U}^{t'} = \underset{\sim}{Q}\underset{\sim}{U}^t\underset{\sim}{Q}^{-1}$, where $\underset{\sim}{Q}$ is the constant orthogonal tensor obeying $\underset{\sim}{u}_i' = \underset{\sim}{Q}\underset{\sim}{u}_i$, $i = 1,2,3$. It follows from (2.24) that the rotated stresses, $\tilde{\underset{\sim}{S}}(t)$ and $\tilde{\underset{\sim}{S}}'(t)$, corresponding respectively to $\underset{\sim}{U}^t$ and $\underset{\sim}{U}^{t'} = \underset{\sim}{Q}\underset{\sim}{U}^t\underset{\sim}{Q}^{-1}$, must obey the formula $\tilde{\underset{\sim}{S}}'(t) = \underset{\sim}{Q}\tilde{\underset{\sim}{S}}(t)\underset{\sim}{Q}^{-1}$ and hence must have the same proper numbers, which is just another way of stating that the functions $\mathcal{f}_j$ do not depend on the basis $\underset{\sim}{u}_j$. Since the functions $\mathcal{f}_j$ are determined by $\mathcal{S}_{\mathcal{R}}$, they, like $\mathcal{S}_{\mathcal{R}}$, can depend on the choice of the reference configuration $\mathcal{R}$ through only the mass density $\rho_{\mathcal{R}}$ of $\mathcal{R}$.

Because the $\mathcal{f}_j$ are functionals determined by $\mathcal{S}$, i.e. by the isotropic material under consideration, they may be called <u>material functionals</u>.

Let us now take for $\underset{\sim}{Q}$ in (2.24) the orthogonal tensor whose components relative to the basis $\underset{\sim}{u}_i$ have the matrix

$$[\underset{\sim}{Q}] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{3.11}$$

A direct calculation using (3.3) and (3.11) yields

$$[\underset{\sim}{Q}\underset{\sim}{U}(t-s)\underset{\sim}{Q}^{-1}] = \begin{bmatrix} \alpha_2(t-s) & 0 & 0 \\ 0 & \alpha_1(t-s) & 0 \\ 0 & 0 & \alpha_3(t-s) \end{bmatrix}, \tag{3.12}$$

while (3.6) and (3.11) yield

$$[\underset{\sim}{Q}\tilde{\underset{\sim}{S}}(t)\underset{\sim}{Q}^{-1}] = \begin{bmatrix} \sigma_2(t) & 0 & 0 \\ 0 & \sigma_1(t) & 0 \\ 0 & 0 & \sigma_3(t) \end{bmatrix}. \tag{3.13}$$

Of course, by (2.24), $\underset{\sim}{Q}\tilde{\underset{\sim}{S}}(t)\underset{\sim}{Q}^{-1}$ is the rotated stress corresponding to $\underset{\sim}{Q}\underset{\sim}{U}^t\underset{\sim}{Q}^{-1}$. Hence, interchanging the functions α_1^t and α_2^t, with α_3^t held fixed results in an interchange of $\sigma_1(t)$ and $\sigma_2(t)$ with $\sigma_3(t)$ held fixed. Clearly, this is true for all functions α_i^t in the domain of the functionals $\mathscr{f}_j$ if and only if these functionals obey the identities

$$\mathscr{f}_2(\alpha_1^t,\alpha_2^t,\alpha_3^t;\rho_{\mathcal{R}}) = \mathscr{f}_1(\alpha_2^t,\alpha_1^t,\alpha_3^t;\rho_{\mathcal{R}}), \tag{3.14}$$

$$\mathscr{f}_3(\alpha_1^t,\alpha_2^t,\alpha_3^t;\rho_{\mathcal{R}}) = \mathscr{f}_3(\alpha_2^t,\alpha_1^t,\alpha_3^t;\rho_{\mathcal{R}}). \tag{3.15}$$

Similarly, on putting for $\underset{\sim}{Q}$ in (2.24) the orthogonal tensor whose components relative to the basis $\underset{\sim}{u}_i$ are

$$[\underset{\sim}{Q}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{3.16}$$

one easily shows that interchange of α_1^t and α_3^t with α_2^t held fixed implies an interchange of $\sigma_1(t)$ and $\sigma_3(t)$ with $\sigma_2(t)$ held fixed. Hence, the functionals $\mathcal{f}_j$ also obey the identities

$$\mathcal{f}_3(\alpha_1^t,\alpha_2^t,\alpha_3^t;\rho_{\mathcal{R}}) = \mathcal{f}_1(\alpha_3^t,\alpha_2^t,\alpha_1^t;\rho_{\mathcal{R}}), \tag{3.17}$$

$$\mathcal{f}_2(\alpha_1^t,\alpha_2^t,\alpha_3^t;\rho_{\mathcal{R}}) = \mathcal{f}_2(\alpha_3^t,\alpha_2^t,\alpha_1^t;\rho_{\mathcal{R}}). \tag{3.18}$$

If we now put

$$\mathcal{f} = \mathcal{f}_1 , \tag{3.19}$$

the equations (3.10) and the identities (3.14), and (3.17) yield

$$\begin{aligned} \sigma_1(t) &= \mathcal{f}(\alpha_1^t,\alpha_2^t,\alpha_3^t;\rho_{\mathcal{R}}), \\ \sigma_2(t) &= \mathcal{f}(\alpha_2^t,\alpha_1^t,\alpha_3^t;\rho_{\mathcal{R}}), \\ \sigma_3(t) &= \mathcal{f}(\alpha_3^t,\alpha_1^t,\alpha_2^t;\rho_{\mathcal{R}}). \end{aligned} \tag{3.20}$$

On considering (3.14) and (3.18) [or, equivalently, (3.15) and (3.17)] we see that $\mathcal{f}$ must obey the identity

$$\mathcal{f}(\alpha^*,\beta^*,\gamma^*;\rho_{\mathcal{R}}) = \mathcal{f}(\alpha^*,\gamma^*,\beta^*;\rho_{\mathcal{R}}) \tag{3.21}$$

for all triplets of positive functions $\alpha^*,\beta^*,\gamma^*$ on $[0,\infty)$. This proves

Theorem 5: If, relative to the undistorted reference configuration $\mathcal{R}$, the motion of an isotropic material is an extension, then one scalar-valued material functional $\mathcal{f}$, enjoying the symmetry property (3.21) and depending on $\mathcal{R}$ only through the density $\rho_{\mathcal{R}}$, gives, as in (3.20), each principal stress $\sigma_j(t)$ as a function of the histories α_i^t of the principal stretch ratios.

Remark 1: It follows from (2.18) that for each isotropic material there exists a single functional $\hat{\mathcal{f}}$ such that

$$\mathcal{f}(\alpha^*,\beta^*,\gamma^*;\rho_{\mathcal{R}}) = \hat{\mathcal{f}}(\nu\alpha^*,\nu\beta^*,\nu\gamma^*), \qquad \nu = \frac{1}{\sqrt[3]{\rho_{\mathcal{R}}}} . \qquad (3.22)$$

for each triplet of positive functions $\alpha^*,\beta^*,\gamma^*$ on $[0,\infty)$.

An extension is called symmetric if two of the principal stretch ratios, say α_2 and α_3, are always equal. Using the functional $\mathcal{f}$ of Theorem 5, let us define two material functionals $\mathcal{l}$ and $\mathcal{k}$ as follows:

$$\left.\begin{aligned} \mathcal{l}(\alpha^*,\beta^*;\rho_{\mathcal{R}}) &\stackrel{\text{def}}{=} \mathcal{f}(\alpha^*,\beta^*,\beta^*;\rho_{\mathcal{R}}), \\ \mathcal{k}(\alpha^*,\beta^*;\rho_{\mathcal{R}}) &\stackrel{\text{def}}{=} \mathcal{f}(\beta^*,\beta^*,\alpha^*;\rho_{\mathcal{R}}) = \mathcal{f}(\beta^*,\alpha^*,\beta^*;\rho_{\mathcal{R}}). \end{aligned}\right\} \quad (3.23)$$

The following remark is now an obvious corollary to Theorem 5:

Remark 2: In a symmetric extension with $\alpha_2^t(s) \equiv \alpha_3^t(s)$, the principal stresses are given by

$$\left.\begin{aligned} \sigma_1(t) &= \ell(\alpha_1^t,\alpha_2^t;\rho_{\mathcal{R}}), \\ \sigma_2(t) = \sigma_3(t) &= \mathcal{k}(\alpha_1^t,\alpha_2^t;\rho_{\mathcal{R}}). \end{aligned}\right\} \tag{3.24}$$

A planar extension is an extension for which one stretch ratio is always unity. Let us define $\mathcal{m}$ and $\mathcal{n}$ by

$$\left.\begin{aligned} \mathcal{m}(\alpha^*,\beta^*;\rho_{\mathcal{R}}) &\overset{\text{def}}{=\!=} \mathcal{f}(\alpha^*,1^\dagger,\beta^*;\rho_{\mathcal{R}}) = \mathcal{f}(\alpha^*,\beta^*,1^\dagger;\rho_{\mathcal{R}}), \\ \mathcal{n}(\alpha^*,\beta^*;\rho_{\mathcal{R}}) &\overset{\text{def}}{=\!=} \mathcal{f}(1^\dagger,\alpha^*,\beta^*;\rho_{\mathcal{R}}) = \mathcal{f}(1^\dagger,\beta^*,\alpha^*;\rho_{\mathcal{R}}), \end{aligned}\right\} \tag{3.25}$$

where $1^\dagger$ is the (constant) function mapping $[0,\infty)$ onto the number 1. Clearly, another corollary to Theorem 5 is

Remark 3: In a planar extension with $\alpha_2^t(s) \equiv 1$,

$$\begin{aligned} \sigma_1(t) &= \mathcal{m}(\alpha_1^t,\alpha_3^t;\rho_{\mathcal{R}}), \\ \sigma_2(t) &= \mathcal{n}(\alpha_1^t,\alpha_3^t;\rho_{\mathcal{R}}), \\ \sigma_3(t) &= \mathcal{m}(\alpha_3^t,\alpha_1^t;\rho_{\mathcal{R}}). \end{aligned} \tag{3.26}$$

The functional $\mathcal{n}$ appearing here is symmetric in the sense that

$$\mathcal{n}(\alpha^*,\beta^*;\rho_{\mathcal{R}}) = \mathcal{n}(\beta^*,\alpha^*;\rho_{\mathcal{R}}) \tag{3.27}$$

for each pair of positive functions α^*,β^* on $[0,\infty)$.

A <u>longitudinal extension</u> is one for which two stretch ratios are always unity. Of course, such an extension is both planar and symmetric. We may now put

$$\begin{aligned} \mathscr{g}(\alpha^*;\rho_{\mathcal{R}}) &\stackrel{\text{def}}{=\!=} \mathscr{f}(\alpha^*,1^\dagger,1^\dagger;\rho_{\mathcal{R}}) = \mathscr{l}(\alpha^*,1^\dagger;\rho_{\mathcal{R}}) = \mathscr{m}(\alpha^*,1^\dagger;\rho_{\mathcal{R}}), \\ \mathscr{h}(\alpha^*;\rho_{\mathcal{R}}) &\stackrel{\text{def}}{=\!=} \mathscr{f}(1^\dagger,1^\dagger,\alpha^*;\rho_{\mathcal{R}}) = \mathscr{k}(\alpha^*,1^\dagger;\rho_{\mathcal{R}}) = \mathscr{n}(\alpha^*,1^\dagger;\rho_{\mathcal{R}}), \end{aligned} \tag{3.28}$$

and observe that Theorem 5 yields

Remark 4: In a longitudinal extension with $\alpha_2^t(s) \equiv \alpha_3^t(s) \equiv 1$,

$$\begin{aligned} \sigma_1(t) &= \mathscr{g}(\alpha_1^t;\rho_{\mathcal{R}}), \\ \sigma_2(t) = \sigma_3(t) &= \mathscr{h}(\alpha_1^t;\rho_{\mathcal{R}}). \end{aligned} \tag{3.29}$$

Let us define a material functional $\mathscr{j}$ by

$$\mathscr{j}(\upsilon^*) = \hat{\mathscr{f}}\left(\sqrt[3]{\upsilon^*}, \sqrt[3]{\upsilon^*}, \sqrt[3]{\upsilon^*}\right), \tag{3.30}$$

where υ^* is an arbitrary positive function on $[0,\infty)$. Clearly, by (3.22) and Theorem 5, we can assert

Remark 5: In a motion for which $\underset{\sim}{F}$ is always a similarity transformation, i.e. a motion for which $\underset{\sim}{U}(t-s) \equiv \underset{\sim}{V}(t-s) \equiv \alpha(t-s)\underset{\sim}{1}$ for all s and hence $\alpha_1^t(s) \equiv \alpha_2^t(s) \equiv \alpha_3^t(s) \equiv \sqrt[3]{\dfrac{\rho_{\mathcal{R}}}{\rho(t-s)}}$, the stress is a hydrostatic pressure,

$$\sigma_1(t) = \sigma_2(t) = \sigma_3(t) = -\,p(t), \tag{3.31}$$

given by

$$p(t) = \mathcal{j}(\upsilon^t), \qquad \upsilon^t(s) \equiv \rho(t-s)^{-1}. \tag{3.32}$$

Here υ^t is the history of the specific volume up to time t. The material functional $\mathcal{j}$ does not depend on a choice of reference configuration.

4. Rectilinear Extensions

In this and the following two sections we employ orthogonal coordinate systems to discuss global motions of isotropic bodies. The coordinates x^1, x^2, x^3 of a particle X at time t are expressed as functions of t and the coordinates X^1, X^2, X^3 of X in an undistorted homogeneous reference configuration $\mathcal{R}$:

$$x^i = x^i(X^\alpha, t), \qquad i,\alpha = 1,2,3. \tag{4.1}$$

For a rectilinear extension# there exists a fixed single

#The isochoric "steady extension" considered by Coleman and Noll (1962) for incompressible simple fluids is a special case of the motion (4.2). So also are the "homogeneous extension" and the "simple extension" analyzed for hypoelastic materials by Truesdell (1955a) and Green (1956).

Cartesian coordinate system in which each x^i is independent of X^k for $k \neq i$:

$$x^1 = x^1(X^1,t), \qquad x^2 = x^2(X^2,t), \qquad x^3 = x^3(X^3,t), \qquad -\infty < t < \infty. \tag{4.2}$$

Let us assume $\dfrac{\partial x^i}{\partial X^i} > 0$, for each i. Clearly, for such a motion

$$\underset{\sim}{F} = \underset{\sim}{U} = \underset{\sim}{V}, \qquad \underset{\sim}{R} = \underset{\sim}{1}, \tag{4.3}$$

and, in the Cartesian system in which (4.2) holds, (3.3) also holds with

$$\frac{\partial}{\partial X^i} x^i(X^i, t-s) = \alpha_i(t-s) = \alpha_i^t(s) \quad \text{(unsummed)}, \qquad 0 \leq s < \infty. \tag{4.4}$$

Hence, by (3.6)-(3.9) in this coordinate system we have

$$\tilde{S}_{ij} = S_{ij} = \sigma_i \delta_{ij} \quad \text{(unsummed)}, \tag{4.5}$$

with δ_{ij} the Kronecker delta. Thus the principal axes of stress, as well as the left and right principal directions of stretch, coincide with the axes of the coordinate system. Furthermore, by (3.20),

$$\sigma_i(t) = \mathcal{f}(\alpha_i^t, \alpha_j^t, \alpha_k^t; \rho_{\mathcal{R}}) \tag{4.6}$$

for each permutation (i,j,k) of the numbers 1,2,3.

It follows from (4.2) and (4.5) that whenever the motion of an isotropic body is a simple extension, Cauchy's dynamical equations,

$$\operatorname{div} \underset{\sim}{S} + \rho \underset{\sim}{b} = \rho \dot{\underset{\sim}{v}}, \tag{4.7}$$

with ρ the density at time t, $\underset{\sim}{b}$ the specific body force, and $\dot{\underset{\sim}{v}}$ the material time derivative of the velocity $\underset{\sim}{v}$, reduce to

$$\frac{\partial \sigma_i}{\partial x^i} + \rho b_i = \rho\left[\frac{\partial v_i}{\partial t} + v^i \frac{\partial v_i}{\partial x_i}\right], \quad \text{(unsummed)}. \tag{4.8}$$

By (4.4) and (4.6), σ_i, in general, depends on x^1, x^2, and x^3 as well as t. Thus, even if we assume a simple form for b_i, the three equations (4.8) are coupled and are difficult to discuss. There are, however, two exceptional cases.

(I) In those special motions called homogeneous extensions, it is assumed that $\underset{\sim}{F}$ is constant in space and hence that $\frac{\partial \sigma_i}{\partial x^i} = 0$ for each i. The general theory of homogeneous motion in compressible simple materials subject to constant and uniform body forces has been discussed by Truesdell and Noll.# The interested reader will have no difficulty in

#(1965, §28, pp. 61-63). The kinematics in the earlier papers of Truesdell (1955b) and Noll (1955) dealing with homogeneous motions in special materials can be usefully applied to general simple materials. See also Truesdell and Toupin (1960, §143, pp. 434-437).

working out details for isotropic bodies in pure extension, using the present formulae (4.5) and (4.6) for the stress.

(II) For a rectilinear longitudinal extension the equations (4.2) take the special form

$$x^1 = x^1(X^1, t), \qquad x^2 = X^2, \qquad x^3 = X^3, \qquad -\infty < t < \infty. \tag{4.9}$$

Hence,

$$\alpha_1^t(s) = \frac{\partial}{\partial X^1} x^1(X^1, t-s), \qquad \alpha_2^t(s) \equiv \alpha_3^t(s) \equiv 1, \qquad 0 \leq s < \infty. \tag{4.10}$$

Since, by Remark 4 after Theorem 5,

$$\sigma_1 = \sigma_1(X^1, t) = \mathcal{g}(\alpha_1^t, \rho_{\mathcal{R}}), \tag{4.11}$$

$$\sigma_3 = \sigma_2 = \sigma_2(X^1, t) = \mathcal{h}(\alpha_1^t, \rho_{\mathcal{R}}), \tag{4.12}$$

we here have

$$\frac{\partial\sigma_2}{\partial x^2} = \frac{\partial\sigma_3}{\partial x^3} = 0. \tag{4.13}$$

Of course (4.9) implies

$$v_2 = v_3 = 0. \tag{4.14}$$

If we now assume that body forces act only in the 1-direction, i.e. that

$$b_1 = b, \qquad b_2 = 0, \qquad b_3 = 0, \tag{4.15}$$

then, by (4.13) and (4.14), two of the three dynamical equations (4.8) reduce to 0 = 0, and the remaining one becomes

$$\frac{\partial\sigma_1}{\partial x^1} + \rho b = \rho\left[\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1}\right], \tag{4.16}$$

which is equivalent to

$$\frac{\partial}{\partial X^1} \mathcal{G}(\alpha_1^t;\rho_{\mathcal{R}}) + \rho_{\mathcal{R}} b = \rho_{\mathcal{R}} \ddot{x}^1. \tag{4.17}$$

With the functional $\mathcal{G}$ assigned, (4.17) is a functional-differential equation for $x^1 = x^1(X^1,t)$. If $\mathcal{G}$ has appropriate smoothness properties, the theory of this functional-differential equation can be expected to broadly generalize that of the quasi-linear hyperbolic partial differential equation describing plane longitudinal motions of elastic materials.#

#Cf. Coleman, Gurtin, and Herrera (1965), Coleman and Gurtin (1965), Coleman, Greenberg, and Gurtin (1966), and Greenberg (1967). The present proof that the equations (4.11), (4.12), and (4.17) follow from (4.9) is that referred to by Coleman, Gurtin, and Herrera (1965, pp. 15 and 19, ref. [15]) as "forthcoming".

One usually thinks of a rectilinear longitudinal extension as a motion taking place in a body which has in its reference configuration the form of a cylinder with its axis along the 1-axis of our Cartesian coordinate system. It follows from (4.9) that such a body remains a cylinder at all times. The forces which must be applied to the bounding cylindrical surfaces to maintain a purely longitudinal motion are, by (4.5), normal to the surface, and have, by (4.12), a value, in units of force per unit area, equal to $\mathscr{h}(\alpha_1^t;\rho_{\mathcal{R}})$. In the same units, $\mathscr{g}(\alpha_1^t;\rho_{\mathcal{R}})$ gives the cross-sectional tension in the cylinder. In general, both the internal tension and the normal forces vary with time and distance along the cylinder.

5. Inflation and Stretch of a Circular Tube

Let us now suppose that in its undistorted reference configuration the body under consideration has the form of a circular tube which may be "hollow", i.e. bounded by cylinders with radii R_I and R_O, or "full", i.e. with $R_O = 0$. Let us employ a single fixed cylindrical coordinate system with its z-axis along the axis of the cylinders and put

$$\left.\begin{array}{lll} z = x^1, & r = x^2, & \theta = x^3, \\ Z = X^1, & R = X^2, & \Theta = X^3. \end{array}\right\} \qquad (5.1)$$

If the equations (4.1) take the form

$$r = r(R,t), \qquad z = z(Z,t), \qquad \theta = \Theta, \qquad -\infty < t < \infty, \qquad (5.2)$$

then we say that the motion of the tube is a simultaneous inflation and stretch. In such a motion the body remains a circular tube. At time t the inner and outer radii of the tube are

$$r_I(t) = r(R_I,t), \qquad r_O = r(R_O,t). \qquad (5.3)$$

Let $\underset{\sim}{e}_z$, $\underset{\sim}{e}_r$, $\underset{\sim}{e}_\theta$ be the normalized natural basis at $\underset{\sim}{x}$ for the coordinate system z, r, θ. An elementary calculation, based on (1.2), (2.12), and (5.2), yields the following expression for the matrices

$[C\langle k\ell\rangle]$ and $[B\langle k\ell\rangle]$ of the components of the Cauchy-Green Tensors $\underset{\sim}{C}$ and $\underset{\sim}{B}$ relative to $\underset{\sim}{e}_z$, $\underset{\sim}{e}_r$, $\underset{\sim}{e}_\theta$:

$$[C\langle k\ell\rangle] = [B\langle k\ell\rangle] = \begin{bmatrix} \left(\frac{\partial z}{\partial Z}\right)^2 & 0 & 0 \\ 0 & \left(\frac{\partial r}{\partial R}\right)^2 & 0 \\ 0 & 0 & \left(\frac{r}{R}\right)^2 \end{bmatrix} . \tag{5.4}$$

Thus, in a simultaneous inflation and stretch the right and left principal directions of stretch lie along the coordinate lines; i.e.

$$\bar{\underset{\sim}{u}}_1 = \underset{\sim}{u}_1 = \underset{\sim}{e}_z, \qquad \bar{\underset{\sim}{u}}_2 = \underset{\sim}{u}_2 = \underset{\sim}{e}_r, \qquad \bar{\underset{\sim}{u}}_3 = \underset{\sim}{u}_3 = \underset{\sim}{e}_\theta. \tag{5.5}$$

Because the orthonormal basis $\underset{\sim}{e}_z$, $\underset{\sim}{e}_r$, $\underset{\sim}{e}_\theta$ changes only when the θ-coordinate changes, it follows from (5.2) and (5.5) that this basis is constant along the path lines of the motion. Hence, although the principal directions of stretch $\underset{\sim}{u}_i$ vary from particle to particle, they are constant in time at a given particle: <u>the local motion at each particle is an extension</u>. Since $\underset{\sim}{U}^2 = \underset{\sim}{C}$, it is clear from (3.3) and (5.4) that the principal stretch ratios for this extension are

$$\alpha_1(t-s) = \frac{\partial}{\partial Z} z(Z,t-s), \qquad \alpha_2(t-s) = \frac{\partial}{\partial R} r(R,t-s), \qquad \alpha_3(t-s) = \frac{1}{R} r(R,t-s). \tag{5.6}$$

These observations, when combined with Theorems 4 and 5, now imply that relative to the basis $\underset{\sim}{e}_r$, $\underset{\sim}{e}_z$, $\underset{\sim}{e}_\theta$, the components of the stress tensor $\underset{\sim}{S}$

are

$$
\left.
\begin{aligned}
&S\langle r\theta\rangle = S\langle rz\rangle = S\langle z\theta\rangle = 0,\\
&S\langle zz\rangle = \mathcal{f}(\alpha_z^t, \alpha_r^t, \alpha_\theta^t; \rho_{\mathcal{R}}),\\
&S\langle rr\rangle = \mathcal{f}(\alpha_r^t, \alpha_z^t, \alpha_\theta^t; \rho_{\mathcal{R}}),\\
&S\langle \theta\theta\rangle = \mathcal{f}(\alpha_\theta^t, \alpha_z^t, \alpha_\theta^t; \rho_{\mathcal{R}}),\\
\text{with}\quad &\alpha_r^t(s) = \frac{\partial}{\partial R}\, r(R,t-s), \quad \alpha_z^t(s) = \frac{\partial}{\partial Z}\, z(Z,t-s), \quad \alpha_\theta^t(s) = \frac{1}{R}\, r(R,t-s), \quad 0 \leq s < \infty.
\end{aligned}
\right\} \tag{5.7}
$$

Thus, the principal stresses may be regarded as functions of r, z, and t (or R, Z, and t). Since the principal axes of stress are along the coordinate axes, there are no shearing tractions on either the normal cross sections or the cylindrical boundaries of the tube.

If we denote by $v\langle r\rangle$, $v\langle z\rangle$, $v\langle\theta\rangle$ and $b\langle r\rangle$, $b\langle z\rangle$, $b\langle\theta\rangle$ the physical components, (i.e. the components relative to $\underset{\sim}{e}_r$, $\underset{\sim}{e}_z$, e_θ) of the velocity $\underset{\sim}{v}$ and the body force $\underset{\sim}{b}$, then, by (5.2),

$$
\begin{aligned}
v\langle z\rangle &= \frac{\partial}{\partial t}\, z(Z,t) = v\langle z\rangle(z,t),\\
v\langle r\rangle &= \frac{\partial}{\partial t}\, r(R,t) = v\langle r\rangle(r,t),\\
v\langle\theta\rangle &= 0.
\end{aligned} \tag{5.8}
$$

The dynamical equations (4.7) can be satisfied only under the assumption that $b\langle z\rangle$ and $b\langle r\rangle$ are independent of θ and $b\langle\theta\rangle \equiv 0$. Making this assumption and using (5.7), one finds, after a routine calculation, that the dynamical equations are equivalent to the two equations

$$\frac{\partial S\langle zz\rangle}{\partial z} + \rho b\langle z\rangle = \rho \frac{\partial v\langle z\rangle}{\partial t} + \rho v\langle z\rangle \frac{\partial v\langle z\rangle}{\partial z} . \tag{5.9}$$

$$\frac{\partial S\langle rr\rangle}{\partial r} + \frac{S\langle rr\rangle - S\langle\theta\theta\rangle}{r} + \rho b\langle r\rangle = \rho \frac{\partial v\langle r\rangle}{\partial t} + \rho v\langle r\rangle \frac{\partial v\langle r\rangle}{\partial r} , \tag{5.10}$$

There are two important special cases of (5.2).

(I) When (5.2) takes the form

$$z = z(Z,t), \qquad r = R, \qquad \theta = \Theta, \qquad -\infty < t < \infty. \tag{5.11}$$

the motion reduces to a <u>rectilinear longitudinal extension</u> (4.9) with $X^1 = Z$ and $x^1 = z$. In this special case we must assume $b\langle r\rangle = 0$; equation (5.10) then reduces to $0 = 0$, while (5.9) is identical to the equation (4.16). According to (4.17) the functional-differential equation governing the evolution of a rectilinear longitudinal motion is determined by the material functional $\mathcal{G}$ of (3.28).

(II) In the <u>inflation of a circular tube</u> (5.2) has the special form

$$z = Z, \qquad r = r(R,t), \qquad \theta = \Theta, \qquad -\infty < t < \infty. \tag{5.12}$$

and (5.6) yields

$$\alpha_z^t = 1^\dagger(s) \stackrel{\text{def}}{=} 1, \qquad 0 \le s < \infty. \tag{5.13}$$

Therefore, it follows from (3.25) that in such a motion (5.7) reduces to

$$\left.\begin{aligned}
S\langle r\theta\rangle &= S\langle rz\rangle = S\langle z\theta\rangle = 0,\\
S\langle zz\rangle &= \mathfrak{n}(\alpha_r^t,\alpha_\theta^t;\rho_{\mathcal{R}}),\\
S\langle rr\rangle &= \mathfrak{m}(\alpha_r^t,\alpha_\theta^t;\rho_{\mathcal{R}}),\\
S\langle \theta\theta\rangle &= \mathfrak{m}(\alpha_r^t,\alpha_\theta^t;\rho_{\mathcal{R}}),\\
\text{with}\quad \alpha_r^t(s) = \frac{\partial}{\partial R}\, r(R,t-s),\quad & \alpha_\theta^t(s) = \frac{1}{R}\, r(R,t-s),\quad 0 \le s < \infty.
\end{aligned}\right\} \tag{5.14}$$

The tractions on the bounding cylindrical surfaces are normal thrusts given, in units of force per unit area, by

$$\left.\begin{aligned}
-S\langle rr\rangle(R_I,t) &= -\mathfrak{m}(\alpha_r^t,\alpha_\theta^t;\rho_{\mathcal{R}})\Big|_{R=R_I},\\
-S\langle rr\rangle(R_0,t) &= -\mathfrak{m}(\alpha_r^t,\alpha_\theta^t;\rho_{\mathcal{R}})\Big|_{R=R_0}.
\end{aligned}\right\} \tag{5.15}$$

The cross-sectional tension $S\langle zz\rangle$ in the cylinder is given by $(5.14)_1$ and need not be zero; although $S\langle zz\rangle$ is independent of z it may vary with r and t. Let us assume that body forces act only in the r-direction. The equation (5.9) then becomes 0 = 0 and the only equation to be satisfied is (5.10). Since it follows from (5.14) that $S\langle rr\rangle$ and $S\langle\theta\theta\rangle$ in (5.10) are functions of r and t (or R and t) alone, we must here further assume that $b\langle r\rangle$ is independent of both θ and z. In the material description, the functional-differential equation (5.10) governing the inflation of the tube takes the form#

#It appears that apparatus recently described by Ensminger and Fyfe (1966) should be capable of generating in isotropic solids non-isochoric motions obeying (5.16).

$$\frac{r}{R}\frac{\partial}{\partial R}\mathfrak{m}(\alpha_r^t,\alpha_\theta^t;\rho_{\mathcal{R}}) + \frac{1}{R}\left[\mathfrak{m}(\alpha_r^t,\alpha_\theta^t;\rho_{\mathcal{R}}) - \mathfrak{m}(\alpha_\theta^t,\alpha_r^t;\rho_{\mathcal{R}})\right]\frac{\partial r}{\partial R} + \rho_{\mathcal{R}} b\langle r\rangle = \rho_{\mathcal{R}}\ddot{r}. \quad (5.16)$$

6. Inflation of a Spherical Shell

If the equations (4.1) take the form

$$r = r(R,t), \qquad \theta = \Theta, \qquad \phi = \Phi, \qquad -\infty < t < \infty, \tag{6.1}$$

in a single, fixed, spherical coordinate system with

$$\left.\begin{array}{lll} r = x^1, & \theta = x^2, & \phi = x^3, \\ R = X^1, & \Theta = X^2, & \Phi = X^3, \end{array}\right\} \tag{6.2}$$

then the motion is called a spherical expansion. If in its reference configuration the body has the shape of a spherical shell with inner radius R_I, outer radius R_O, and center of curvature at $R = 0$, then according to (6.1) it is a spherical shell also at time t with radii $r_I(t) = r(R_I,t)$ and $r_O(t) = r(R_O,t)$.

It follows from (1.2), (2.12), and (6.1) that the matrices of the components of $\underset{\sim}{C}$ and $\underset{\sim}{B}$, relative to the normalized natural basis $\underset{\sim}{e}_r$, $\underset{\sim}{e}_\theta$, $\underset{\sim}{e}_\phi$, at r, θ, ϕ, are

$$[C\langle k\ell\rangle] = [B\langle k\ell\rangle] = \begin{bmatrix} \left(\frac{\partial r}{\partial R}\right)^2 & 0 & 0 \\ 0 & \frac{r^2}{R^2} & 0 \\ 0 & 0 & \frac{r^2}{R^2} \end{bmatrix} . \tag{6.3}$$

Hence the orthonormal basis $\underset{\sim}{e}_r$, $\underset{\sim}{e}_\theta$, $\underset{\sim}{e}_\phi$ gives both the right and left principal directions of stretch. Since, by (6.1), this basis is constant along each path line, the local motion of each particle is an extension. Furthermore, since we here have

$$\alpha_1(t-s) = \frac{\partial}{\partial R} r(R,t-s), \qquad \alpha_2(t-s) = \alpha_3(t-s) = \frac{r}{R}, \qquad 0 \leq s < \infty, \tag{6.4}$$

the local extensions are symmetric. It follows from Theorem 4 and Remark 2 after Theorem 5 that here

$$\left.\begin{aligned} S\langle r\theta\rangle &= S\langle r\phi\rangle = S\langle \theta\phi\rangle = 0, \\ S\langle rr\rangle &= \mathcal{l}(\alpha_r^t, \alpha_\theta^t; \rho_{\mathcal{R}}) \\ S\langle\theta\theta\rangle &= S\langle\phi\phi\rangle = \mathcal{k}(\alpha_r^t, \alpha_\theta^t; \rho_{\mathcal{R}}) \\ \text{with} \quad \alpha_r^t(s) &= \frac{\partial}{\partial R} r(R,t-s), \qquad \alpha_\theta^t(s) = \frac{1}{R} r(R,t-s), \qquad 0 \leq s < \infty. \end{aligned}\right\} \tag{6.5}$$

The functionals $\mathcal{l}$ and $\mathcal{k}$ are related to $\mathcal{f}$ as shown in (3.23). It is clear from (6.5) that the physical components of $\underset{\sim}{S}$ may be regarded as functions of either r and t or R and t. Of course, the velocity has for its only non-zero component

$$v\langle r\rangle = \frac{\partial}{\partial t} r(R,t) = v\langle r\rangle(r,t). \tag{6.6}$$

If we assume that the body forces depend on r alone and act only in the r-direction, then it follows from (6.5) and (6.6) that for a spherical expansion the dynamical equations are equivalent to the single equation

$$\frac{\partial S\langle rr\rangle}{\partial r} + \frac{2}{r}\left[S\langle rr\rangle - S\langle\theta\theta\rangle\right] + \rho b\langle r\rangle = \rho \frac{\partial v\langle r\rangle}{\partial t} + \rho v\langle r\rangle \frac{\partial v\langle r\rangle}{\partial r}. \tag{6.7}$$

This functional-differential equation is determined when the material functionals $\mathcal{l}$ and $\mathcal{k}$ are specified. In the material description, (6.7) takes the form

$$\left(\frac{r}{R}\right)^2 \frac{\partial}{\partial R}\mathcal{l}(\alpha_r^t,\alpha_\theta^t;\rho_{\mathcal{R}}) + \frac{2r}{R^2}\,\mathcal{h}(\alpha_r^t,\alpha_\theta^t;\rho_{\mathcal{R}})\frac{\partial r}{\partial R} + \rho_{\mathcal{R}} b\langle r\rangle \;=\; \rho_{\mathcal{R}}\ddot{r}, \qquad (6.8)$$

with $\mathcal{h}$ the material functional defined by the relation

$$\mathcal{h}(\alpha^*,\beta^*;\rho_{\mathcal{R}}) \;=\; \mathcal{l}(\alpha^*,\beta^*;\rho_{\mathcal{R}}) - \mathcal{k}(\alpha^*,\beta^*;\rho_{\mathcal{R}}) \;=\; \mathcal{f}(\alpha^*,\beta^*,\beta^*;\rho_{\mathcal{R}}) - \mathcal{f}(\beta^*,\alpha^*,\beta^*;\rho_{\mathcal{R}}), \qquad (6.9)$$

which holds for all pairs of positive functions α^*,β^* on $[0,\infty)$.

7. A Bending Motion

In the special global motions we have considered so far the coordinates x^1, x^2, x^3 and X^1, X^2, X^3 shown in (4.1) have been referred to the same orthogonal coordinate system. For certain motions, however, it is convenient to use different coordinate systems for the present and reference configurations. If g_{km} and $G_{\alpha\beta}$ are the covariant components of the unit tensor in the respective coordinate systems x^1, x^2, x^3 and X^1, X^2, X^3, and if we put

$$F^i{}_\alpha = \frac{\partial}{\partial X^\alpha} x^i(X^\beta, t), \tag{7.1}$$

then the covariant components of $\underset{\sim}{C}$ in the system X^1, X^2, X^3 are

$$C_{\alpha\beta} = F^k{}_\alpha F^m{}_\beta g_{km}, \tag{7.2}$$

while the contravariant components of $\underset{\sim}{B}$ in the system x^1, x^2, x^3 are

$$B^{km} = F^k{}_\alpha F^m{}_\beta G^{\alpha\beta}. \tag{7.3}$$

For the physical components of $\underset{\sim}{C}$ in the orthogonal system X^1, X^2, X^3 we have

$$C\langle\alpha\beta\rangle = \frac{C_{\alpha\beta}}{\sqrt{G_{\alpha\alpha}G_{\beta\beta}}}, \quad \text{(unsummed)}; \tag{7.4}$$

the physical components of $\underset{\sim}{B}$ in the orthogonal system x^1, x^2, x^3 are

$$B\langle km\rangle = \frac{B^{km}}{\sqrt{g^{kk}g^{mm}}}, \quad \text{(unsummed)}. \tag{7.5}$$

The general formulae (7.2)-(7.5) are well known. From them we may read off the following observation.

Theorem 6: Suppose the motion is such that there exist two orthogonal, possibly distinct, coordinate systems x^1, x^2, x^3 and X^1, X^2, X^3 relative to which the equations $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X}, t)$ take the special forms

$$x^1 = x^1(X^1, t), \qquad x^2 = x^2(X^2, t), \qquad x^3 = x^3(X^3, t), \qquad -\infty < t < \infty. \tag{7.6}$$

Let $\underset{\sim}{e}_i$, $i = 1,2,3$, and $\underset{\sim}{e}_\alpha$, $\alpha = 1,2,3$, be the orthonormal bases pointing along the coordinate directions x^i and X^α, respectively. The matrices of the components of $\underset{\sim}{B}$ relative to $\underset{\sim}{e}_i(\underset{\sim}{x})$ and $\underset{\sim}{C}$ relative to $\underset{\sim}{e}_\alpha(\underset{\sim}{X})$ are then given by

$$[B\langle ij\rangle] = [C\langle\alpha\beta\rangle] = \begin{bmatrix} \frac{g_{11}}{G_{11}}\left(\frac{\partial x^1}{\partial X^1}\right)^2 & 0 & 0 \\ 0 & \frac{g_{22}}{G_{22}}\left(\frac{\partial x^2}{\partial X^2}\right)^2 & 0 \\ 0 & 0 & \frac{g_{33}}{G_{33}}\left(\frac{\partial x^3}{\partial X^3}\right)^2 \end{bmatrix}. \tag{7.7}$$

Thus the right principal directions of stretch coincide with the vectors $\underset{\sim}{e}_\alpha(\underset{\sim}{X})$ which are constant along path lines, i.e. fixed at each particle, and the motion (7.6) is a motion of extension. The principal stretch ratios are

$$\alpha_i(t-s) = \sqrt{\frac{g_{ii}}{G_{\alpha\alpha}}}\; \frac{\partial}{\partial X^\alpha} x^i(X^\alpha, t-s)\Big|_{\alpha=i}, \quad \text{(unsummed)}. \tag{7.8}$$

The left principal directions of stretch coincide with the vectors $\underset{\sim}{e}_i(\underset{\sim}{x})$ and can vary along path lines. The rotation tensor $\underset{\sim}{R}$ is determined by the relation $\underset{\sim}{e}_i(\underset{\sim}{x}) = \underset{\sim}{R}(t)\underset{\sim}{e}_\alpha(\underset{\sim}{X})$.

When this almost trivial theorem is placed alongside Theorems 4 and 5 it attains a practical significance. For an example of its use we may consider motions describing the bending of a block into a cylindrical wedge.# Let

#The advantage of using separate coordinate systems at $\underset{\sim}{x}$ and $\underset{\sim}{X}$ for describing the bending of a block was pointed out by Truesdell (1952, §42I), who derived, in essence, the formula (7.12). See also Truesdell and Toupin (1960, §50, pp. 300-301).

$$r = r(X,t), \qquad \theta = \theta(Y,t), \qquad z = z(Z,t), \qquad -\infty < t < \infty, \tag{7.9}$$

where the cylindrical coordinate system,

$$x^1 = r, \qquad x^2 = \theta, \qquad x^3 = z, \tag{7.10}$$

and the Cartesian system,

$$X^1 = X, \qquad X^2 = Y, \qquad X^3 = Z, \tag{7.11}$$

have a common origin and a common z-Z axis. Since we here have $G_{\alpha\beta} = \delta_{\alpha\beta}$, $g_{22} = r^2$, $g_{11} = g_{33} = 1$, and $g_{km} = 0$ if $k \neq m$, Theorem 6 tells us that the local motion of each particle is an extension. For this extension

$$\alpha_1(t-s) = \frac{\partial}{\partial X} r(X,t-s), \qquad \alpha_2(t-s) = r\frac{\partial}{\partial Y}\theta(Y,t-s), \qquad \alpha_3(t-s) = \frac{\partial}{\partial Z} z(Z,t-s), \tag{7.12}$$

and the basis $\underset{\sim}{e}_\theta(\underset{\sim}{x})$, $\underset{\sim}{e}_r(\underset{\sim}{x})$, $\underset{\sim}{e}_z(\underset{\sim}{x})$ gives the left principal directions of stretch at time t. This motion differs from those we considered in Sections 4, 5, and 6 in that here the left and right principal directions of stretch do not coincide, and thus, by (3.1), $\underset{\sim}{R} \neq \underset{\sim}{1}$. In fact, an analysis given by Truesdell and Toupin shows that $\underset{\sim}{R}$ at r, θ, z is the tensor describing a rotation of θ radians about the z-Z axis.#

#[1960, eqs. (50.3)-(50.6).]

Using now Theorems 4 and 5, we find that for the motion (7.9) the components of the stress tensor S relative to the basis $\underset{\sim}{e}_r(\underset{\sim}{x})$, $\underset{\sim}{e}_\theta(\underset{\sim}{x})$, $\underset{\sim}{e}_z(\underset{\sim}{x})$ are

$$\left.\begin{array}{l} S\langle r\theta\rangle = S\langle rz\rangle = S\langle \theta z\rangle = 0, \\ S\langle rr\rangle = \mathscr{f}(\alpha_r^t, \alpha_z^t, \alpha_\theta^t; \rho_{\mathcal{R}}), \\ S\langle \theta\theta\rangle = \mathscr{f}(\alpha_\theta^t, \alpha_z^t, \alpha_r^t; \rho_{\mathcal{R}}), \\ S\langle zz\rangle = \mathscr{f}(\alpha_z^t, \alpha_\theta^t, \alpha_r^t; \rho_{\mathcal{R}}), \\ \text{with } \alpha_r^t(s) = \frac{\partial}{\partial X} r(X,t-s), \quad \alpha_\theta^t(s) = r\frac{\partial}{\partial Y}\theta(Y,t-s), \quad \alpha_z^t(s) = \frac{\partial}{\partial Z} z(Z,t-s), \quad 0 < s \leq \infty. \end{array}\right\} \tag{7.13}$$

In the special case

$$r = r(X,t), \qquad \theta = \theta(Y,t), \qquad z = Z, \qquad -\infty < t < \infty, \tag{7.14}$$

we have

$$\alpha_z^t = 1^\dagger, \tag{7.15}$$

and employing (3.25) we find that (7.13) reduces to

$$\left.\begin{aligned}
S\langle r\theta\rangle &= S\langle rz\rangle = S\langle \theta z\rangle = 0, \\
S\langle rr\rangle &= \mathfrak{m}(\alpha_r^t, \alpha_\theta^t; \rho_{\mathcal{R}}), \\
S\langle \theta\theta\rangle &= \mathfrak{m}(\alpha_\theta^t, \alpha_r^t; \rho_{\mathcal{R}}), \\
S\langle zz\rangle &= \mathfrak{n}(\alpha_r^t, \alpha_\theta^t; \rho_{\mathcal{R}}), \\
\text{with} \quad \alpha_r^t(s) = \frac{\partial}{\partial X} r(X, t-s), &\qquad \alpha_\theta^t(s) = r \frac{\partial}{\partial Y} \theta(Y, t-s), \qquad 0 \leq s < \infty.
\end{aligned}\right\} \tag{7.16}$$

Under the assumption that the body forces are zero, the dynamical equations reduce to two equations which in the spatial description have the form

$$\left.\begin{aligned}
\frac{\partial S\langle rr\rangle}{\partial r} + \frac{S\langle rr\rangle - S\langle \theta\theta\rangle}{r} &= \rho \frac{\partial v\langle r\rangle}{\partial t} + \rho v\langle r\rangle \frac{\partial v\langle r\rangle}{\partial r} - r\rho (v\langle \theta\rangle)^2, \\
\frac{1}{r}\frac{\partial S\langle \theta\theta\rangle}{\partial \theta} &= \rho \frac{\partial v\langle \theta\rangle}{\partial t} + \rho v\langle \theta\rangle \frac{\partial v\langle \theta\rangle}{\partial \theta} + 2\rho v\langle r\rangle v\langle \theta\rangle.
\end{aligned}\right. \tag{7.17}$$

If the material description is used, the functional-differential equations (7.17) become

$$\left.\begin{aligned}
r\left(\frac{\partial \theta}{\partial Y}\right) \frac{\partial}{\partial X} \mathfrak{m}(\alpha_r^t, \alpha_\theta^t; \rho_{\mathcal{R}}) + [\mathfrak{m}(\alpha_r^t, \alpha_\theta^t; \rho\ ) - \mathfrak{m}(\alpha_\theta^t, \alpha_r^t; \rho_{\mathcal{R}})]\left(\frac{\partial \theta}{\partial Y}\right)\left(\frac{\partial r}{\partial X}\right) &= \rho_{\mathcal{R}} \ddot{r} - \rho_{\mathcal{R}} r \dot{\theta}^2, \\
\left(\frac{\partial r}{\partial X}\right) \frac{\partial}{\partial Y} \mathfrak{m}(\alpha_\theta^t, \alpha_r^t; \rho_{\mathcal{R}}) &= \rho_{\mathcal{R}} \ddot{\theta} + 2\rho_{\mathcal{R}} \dot{\theta} \dot{r},
\end{aligned}\right. \tag{7.18}$$

with $\mathfrak{m}$ the material functional defined in $(3.25)_1$.

III. Sheared Extensions

8. Reduced Forms Valid in Sheared Extensions

There are motions other than extensions for which the constitutive equations of an isotropic material can be reduced to simpler forms. Among these are the shearing motions to be discussed in Section 9. Both shears and extensions are contained in a class of motions called sheared extensions which we now define.

Let $\underset{\sim}{F}(t-s)$ be the deformation gradient at a particle X at time t-s relative to an undistorted reference configuration $\mathcal{R}$. We say that the history of X up to time t has been a *sheared extension* if

$$\underset{\sim}{F}(t-s) = \underset{\sim}{P}(t-s)\underset{\sim}{N}(t-s), \qquad 0 \leq s < \infty, \tag{8.1}$$

where $\underset{\sim}{P}(t-s)$ is orthogonal for each s and $\underset{\sim}{N}(t-s)$ is such that there exists an orthonormal basis $\underset{\sim}{h}_i$, independent of s, relative to which the components of $\underset{\sim}{N}(t-s)$ have the form

$$[\underset{\sim}{N}(t-s)] = \begin{bmatrix} \beta_1(t-s) & 0 & 0 \\ \zeta(t-s) & \beta_2(t-s) & 0 \\ 0 & 0 & \beta_3(t-s) \end{bmatrix}, \qquad \beta_i(t-s) > 0. \tag{8.2}$$

We call $\underset{\sim}{h}_i$ the canonical basis of the sheared extension.

It follows from (2.12) and the orthogonality of $\underset{\sim}{P}(t-s)$ in (8.1) that

$$\underset{\sim}{C}(t-s) = \{\underset{\sim}{P}(t-s)\underset{\sim}{N}(t-s)\}^T\underset{\sim}{P}(t-s)\underset{\sim}{N}(t-s) = \underset{\sim}{N}(t-s)^T\underset{\sim}{N}(t-s). \tag{8.3}$$

Therefore, if (8.2) holds then relative to $\mathbf{h}_i$ we have

$$[\mathbf{C}] = \begin{bmatrix} \beta_1^2+\zeta^2 & \zeta\beta_2 & 0 \\ \zeta\beta_2 & \beta_2^2 & 0 \\ 0 & 0 & \beta_3^2 \end{bmatrix}, \tag{8.4}$$

where the dependence of the matrix elements of t-s is understood. It is easy to see that (8.4) is not only a necessary, but also a sufficient condition for (8.1) and (8.2) to hold. Hence one can make the following

Remark: The history of X up to time t is sheared extension if and only if there exists a basis $\mathbf{h}_i$, independent of s, relative to which the matrix of the components of the right Cauchy-Green tensor has the form

$$[\mathbf{C}(t-s)] = \begin{bmatrix} \gamma_1(t-s) & \xi(t-s) & 0 \\ \xi(t-s) & \gamma_2(t-s) & 0 \\ 0 & 0 & \gamma_3(t-s) \end{bmatrix}, \tag{8.5}$$

$$\gamma_1 > \frac{\xi(t-s)^2}{\gamma_2(t-s)}, \qquad \gamma_i(t-s) > 0$$

for all s, $0 \leq s < \infty$. This basis $\mathbf{h}_i$ is the canonical basis of the sheared extension. Furthermore

$$\xi = \zeta\beta_2, \qquad \gamma_1 = \beta_1^2 + \zeta^2, \qquad \gamma_2 = \beta_2^2, \qquad \gamma_3 = \beta_3^2, \tag{8.6}$$

or, equivalently,

$$\zeta = \frac{\xi}{\sqrt{\gamma_2}}, \qquad \beta_1 = \sqrt{\gamma_1 - \xi^2/\gamma_2}, \qquad \beta_2 = \sqrt{\gamma_2}, \qquad \beta_3 = \sqrt{\gamma_3}. \tag{8.7}$$

The following theorem gives the reduced form taken by the constitutive equation of an isotropic material when the local motion is known to be a sheared extension.

Theorem 7: If, relative to an undistorted reference configuration $\mathcal{R}$, the motion of an isotropic material up to time t is a sheared extension with canonical basis $\underset{\sim}{h}_i$, then, the matrix of the components, relative to $\underset{\sim}{h}_i$, of the material stress tensor (2.19) has the form

$$[\underset{\sim}{\bar{S}}(t)] = \begin{bmatrix} \bar{S}[11](t) & \bar{S}[12](t) & 0 \\ \bar{S}[21](t) & \bar{S}[22](t) & 0 \\ 0 & 0 & \bar{S}[33](t) \end{bmatrix}, \qquad \bar{S}[ij](t) \overset{\text{def}}{=} \underset{\sim}{h}_i \cdot \underset{\sim}{\bar{S}}(t)\underset{\sim}{h}_j. \tag{8.8}$$

Moreover, there are three scalar material functionals $\mathfrak{z}$, $\mathfrak{s}$, $\mathfrak{t}$, of four function variables, which then determine the non-zero components in (8.8) as follows:

$$\left.\begin{aligned} \bar{S}[21](t) = \bar{S}[12](t) &= \mathfrak{z}(\xi^t, \gamma_1^t, \gamma_2^t, \gamma_3^t; \rho_{\mathcal{R}}), \\ \bar{S}[11](t) &= \mathfrak{s}(\xi^t, \gamma_1^t, \gamma_2^t, \gamma_3^t; \rho_{\mathcal{R}}), \\ \bar{S}[22](t) &= \mathfrak{s}(\xi^t, \gamma_2^t, \gamma_1^t, \gamma_3^t; \rho_{\mathcal{R}}), \\ \bar{S}[33](t) &= \mathfrak{t}(\xi^t, \gamma_1^t, \gamma_2^t, \gamma_3^t; \rho_{\mathcal{R}}), \\ \text{with} \quad \xi^t(s) = \xi(t-s), \quad \gamma_i^t(s) &= \gamma_i(t-s), \quad i = 1,2,3, \quad 0 \le s < \infty. \end{aligned}\right\} \tag{8.9}$$

<u>These functionals obey the identities,</u>

$$\left.\begin{aligned}
\mathcal{Z}(\xi^*,\gamma_2^*,\gamma_1^*,\gamma_3^*;\rho_{\mathcal{R}}) &= \mathcal{Z}(\xi^*,\gamma_1^*,\gamma_2^*,\gamma_3^*;\rho_{\mathcal{R}}),\\
\mathcal{Z}(-\xi^*,\gamma_1^*,\gamma_2^*,\gamma_3^*;\rho_{\mathcal{R}}) &= -\mathcal{Z}(\xi^*,\gamma_1^*,\gamma_2^*,\gamma_3^*;\rho_{\mathcal{R}}),\\
\mathcal{S}(-\xi^*,\gamma_1^*,\gamma_2^*,\gamma_3^*;\rho_{\mathcal{R}}) &= \mathcal{S}(\xi^*,\gamma_1^*,\gamma_2^*,\gamma_3^*;\rho_{\mathcal{R}}),\\
\mathcal{T}(\xi^*,\gamma_2^*,\gamma_1^*,\gamma_3^*;\rho_{\mathcal{R}}) &= \mathcal{T}(\xi^*,\gamma_1^*,\gamma_2^*,\gamma_3^*;\rho_{\mathcal{R}}),\\
\mathcal{T}(-\xi^*,\gamma_1^*,\gamma_2^*,\gamma_3^*;\rho_{\mathcal{R}}) &= \mathcal{T}(\xi^*,\gamma_1^*,\gamma_2^*,\gamma_3^*;\rho_{\mathcal{R}}),
\end{aligned}\right\} \tag{8.10}$$

<u>for all functions</u> ξ^*, γ_1^*, γ_2^*, γ_3^* <u>in their domain of definition</u>.

<u>Proof</u>: The proof rests on the Corollary to Theorem 3 and its consequence (2.25). It follows from (2.25) that if a particular orthogonal tensor $\underset{\sim}{Q}$ commutes with $\underset{\sim}{C}(t-s)$ for all s then $\underset{\sim}{Q}$ must commute with the material stress tensor $\bar{\underset{\sim}{S}}(t)$:

$$\underset{\sim}{Q}\underset{\sim}{C}^t\underset{\sim}{Q}^{-1} = \underset{\sim}{C}^t \implies \underset{\sim}{Q}\bar{\underset{\sim}{S}}(t)\underset{\sim}{Q}^{-1} = \bar{\underset{\sim}{S}}(t). \tag{8.11}$$

When $\underset{\sim}{C}(t-s)$ is given by (8.5), $\underset{\sim}{C}(t-s)$ clearly commutes with the orthogonal tensor $\underset{\sim}{Q}$ whose components relative to the basis $\underset{\sim}{h}_i$ are

$$[\underset{\sim}{Q}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{8.12}$$

and an elementary argument shows that the symmetric tensor $\bar{\underset{\sim}{S}}(t)$ commutes with this tensor if and only if the components $\bar{S}[ij](t)$ of $\tilde{\underset{\sim}{S}}(t)$ relative to $\underset{\sim}{h}_i$ obey#

#For the details of the argument see Coleman and Noll [1959a, p. 294, eqs. (3.7), (3.9)-(3.11)] or [1961, p. 696, eqs. (5.15), (5.17)-(5.19)].

$$\bar{S}[13](t) = \bar{S}[31](t) = 0, \qquad \bar{S}[23] = \bar{S}[32](t) = 0, \tag{8.13}$$

which gives us (8.8).

It follows from (8.8) that once the canonical basis $\underset{\sim}{h}_i$ is specified, $\underset{\sim}{C}^t$ is determined when ξ^t and γ_i^t, $i = 1,2,3$, are prescribed. Since $\bar{\underset{\sim}{S}}(t)$ is given by a function $\mathcal{H}(\cdot;\rho_{\mathcal{R}})$ of $\underset{\sim}{C}^t$, each non-zero component of $\bar{\underset{\sim}{S}}(t)$ is given by a real-valued function of ξ^t and the γ_i^t. Hence we have $(8.9)_1$ and three equations of the form

$$\bar{S}[ii](t) = \mathcal{S}_i(\xi^t, \gamma_1^t, \gamma_2^t, \gamma_3^t; \rho_{\mathcal{R}}), \qquad i = 1,2,3. \tag{8.14}$$

An argument analogous to that given in Section 3 after (3.10) shows that (2.25) implies that the functionals $\mathcal{Z}$, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ are unaltered if the basis $\underset{\sim}{h}_i$ be changed. Hence, these functionals $\mathcal{Z}$, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, depending only on $\rho_{\mathcal{R}}$ and determined by $\mathcal{H}$, may be called <u>material functionals</u>.

Let us now take for $\underset{\sim}{Q}$ in (2.25) the orthogonal tensor whose components with respect to $\underset{\sim}{h}_i$ are given by (3.11). A direct calculation using (8.14) and (3.11) gives

$$[\underset{\sim}{Q}\underset{\sim}{C}(t-s)\underset{\sim}{Q}^{-1}] = \begin{bmatrix} \gamma_2(t-s) & \xi(t-s) & 0 \\ \xi(t-s) & \gamma_1(t-s) & 0 \\ 0 & 0 & \gamma_3(t-s) \end{bmatrix}. \tag{8.15}$$

Of course, (2.25) asserts that the material stress tensor corresponding to $\underset{\sim}{Q}\underset{\sim}{C}^t\underset{\sim}{Q}^{-1}$ is $\underset{\sim}{Q}\underset{\sim}{S}(t)\underset{\sim}{Q}^{-1}$, but (8.8) and (3.11) yield

$$[\underset{\sim}{Q}\bar{\underset{\sim}{S}}(t)\underset{\sim}{Q}^{-1}] = \begin{bmatrix} \bar{S}[22](t) & \bar{S}[21](t) & 0 \\ \bar{S}[12](t) & \bar{S}[11](t) & 0 \\ 0 & 0 & \bar{S}[33](t) \end{bmatrix}. \tag{8.16}$$

Hence, interchanging the functions γ_1^t and γ_2^t, with ξ^t and γ_3^t held fixed, results in an interchange of $S[11](t)$ and $S[22](t)$, with $S[12](t)$ and $S[33](t)$ unchanged. This implies that $\mathcal{A}_1$ and $\mathcal{A}_2$ must be related as follows:

$$\mathcal{A}_2(\xi^t,\gamma_1^t,\gamma_2^t,\gamma_3^t;\rho_{\mathcal{R}}) = \mathcal{A}_1(\xi^t,\gamma_2^t,\gamma_1^t,\gamma_3^t;\rho_{\mathcal{R}}). \tag{8.17}$$

Since (8.17) holds throughout the domain of $\mathcal{A}_1$ and $\mathcal{A}_2$, (8.17) is equivalent to asserting that if we put

$$\mathcal{A} = \mathcal{A}_1 \tag{8.18}$$

then

$$\mathscr{s}_2(\xi^t,\gamma_1^t,\gamma_2^t,\gamma_3^t;\rho_{\mathcal{R}}) = \mathscr{s}(\xi^t,\gamma_2^t,\gamma_1^t,\gamma_3^t;\rho_{\mathcal{R}}). \tag{8.19}$$

Putting

$$\mathscr{t} = \mathscr{s}_3, \tag{8.20}$$

we observe that (8.18) and (8.19), when combined with (8.14), imply $(8.9)_{2,3\&4}$. Furthermore, since $S[12]$ and $S[33]$ remain fixed when γ_1^t and γ_2^t are interchanged, the material functionals $\mathscr{z}$ and $\mathscr{t}$ must obey the identities $(8.10)_1$ and $(8.10)_4$. If we now take for $\underset{\sim}{Q}$ in (2.25) the orthogonal tensor whose matrix with respect to $\underset{\sim}{h}_i$ is

$$[\underset{\sim}{Q}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{8.21}$$

we find that

$$[\underset{\sim}{Q}\underset{\sim}{C}(t-s)\underset{\sim}{Q}^{-1}] = \begin{bmatrix} \gamma_1(t-s) & -\xi(t-s) & 0 \\ -\xi(t-s) & \gamma_2(t-s) & 0 \\ 0 & 0 & \gamma_3(t-s) \end{bmatrix} \tag{8.22}$$

and

$$[\underset{\sim}{Q}\underset{\sim}{\bar{S}}(t)\underset{\sim}{Q}^{-1}] = \begin{bmatrix} \bar{S}[11](t) & -\bar{S}[12](t) & 0 \\ -\bar{S}[12](t) & \bar{S}[22](t) & 0 \\ 0 & 0 & \bar{S}[33](t) \end{bmatrix} \tag{8.23}$$

Hence, changing ξ^t to $-\xi^t$ with the functions γ_i^t held fixed results in a change of $\bar{S}[12](t)$ to $-\bar{S}[12](t)$ with the diagonal elements $\bar{S}[ii](t)$ held fixed. Clearly, by (8.9), this is true throughout the domain of the functionals $\mathfrak{z}$, $\mathfrak{s}$, $\mathfrak{t}$ if and only if these functionals obey the identities $(8.10)_{2,3,\&5}$; q.e.d.

Of course, an <u>extension</u> is a sheared extension with $\xi(t-s) \equiv 0$. In fact, when $\xi(t-s) \equiv 0$, we have $\gamma_i(t-s) = \alpha_i^2(t-s)$ and $\underset{\sim}{h}_i(t) = \underset{\sim}{u}_i(t)$, where the α_i are principal stretches and the $\underset{\sim}{u}_i$ right principal directions of stretch. Clearly Theorem 4 is a corollary of the present Theorem 7,
pt but Theorem 5 is not quite a corollary. To prove Theorem 5 one must consider an orthogonal tensor, such as (3.16), which interchanges γ_1^t and γ_3^t. The reader will easily verify that for a sheared extension with ξ^t not identically zero, use of (2.25) with an orthogonal tensor of the form (3.16) does not yield any further reductions of our material functionals.

An example of a global motion involving sheared extensions is furnished by considering one for which there exists a single fixed <u>Cartesian</u> coordinate system in which

$$x^1 = x^1(X^1,t), \qquad x^2 = x^2(X^1,X^2,t), \qquad x^3 = x^3(X^3,t), \qquad -\infty < t < \infty, \tag{8.24}$$

with $\dfrac{\partial x^i}{\partial X^i} > 0$ for each i. It follows from (8.24) that the Cartesian components of the deformation gradient $\underset{\sim}{F}$ have the matrix

$$[\underset{\sim}{F}] = \begin{bmatrix} \dfrac{\partial x^1}{\partial X^1} & 0 & 0 \\ \dfrac{\partial x^2}{\partial X^1} & \dfrac{\partial x^2}{\partial X^2} & 0 \\ 0 & 0 & \dfrac{\partial x^3}{\partial X^3} \end{bmatrix}, \tag{8.25}$$

and the motion is, therefore, a sheared extension with its canonical basis equal to the natural basis of the Cartesian system, with $\underset{\sim}{P}$ in (8.1) identically equal to $\underset{\sim}{1}$, and with

$$\zeta = \frac{\partial x^2}{\partial X^1}, \qquad \beta_1 = \frac{\partial x^1}{\partial X^1}, \qquad \beta_2 = \frac{\partial x^2}{\partial X^2}, \qquad \beta_3 = \frac{\partial x^3}{\partial X^3}, \tag{8.26}$$

or, by (8.6),

$$\left.\begin{aligned} \xi &= \left(\frac{\partial x^2}{\partial X^1}\right)\left(\frac{\partial x^2}{\partial X^2}\right), \qquad & \gamma_1 &= \left(\frac{\partial x^1}{\partial X^1}\right)^2 + \left(\frac{\partial x^2}{\partial X^1}\right)^2, \\ \gamma_2 &= \left(\frac{\partial x^2}{\partial X^2}\right)^2, & \gamma_3 &= \left(\frac{\partial x^3}{\partial X^3}\right)^2 . \end{aligned}\right\} \tag{8.27}$$

Theorem 7 gives us the material stress tensor $\bar{\underset{\sim}{S}}(t)$ in the motion (8.24). When written in terms of $\bar{\underset{\sim}{S}}$, (4.7) takes the form

$$\mathrm{div}_{\underset{\sim}{X}}(\underset{\sim}{F}\bar{\underset{\sim}{S}}) + \rho_{\mathcal{R}}\underset{\sim}{b} = \rho_{\mathcal{R}}\ddot{\underset{\sim}{x}} , \tag{8.28}$$

and in Cartesian coordinates this equation becomes

$$\frac{\partial}{\partial X^\beta}\left(\frac{\partial x^i}{\partial X^\alpha}\,\bar{S}^{\alpha\beta}\right) + \rho_{\mathcal{R}} b^i = \rho_{\mathcal{R}}\ddot{x}^i . \tag{8.29}$$

It follows from (8.8) that for the motion we are considering the dynamical equations (8.29) simplify considerably, and we have, at each time t,

$$\left.\begin{aligned}
\frac{\partial}{\partial x^1}\left(\frac{\partial x^1}{\partial X^1}\,\bar{S}[11]\right) + \frac{\partial}{\partial X^2}\left(\frac{\partial x^1}{\partial X^1}\,\bar{S}[12]\right) + \rho_{\mathcal{R}} b^1 &= \rho_{\mathcal{R}} \ddot{x}^1, \\
\frac{\partial}{\partial X^1}\left(\frac{\partial x^2}{\partial X^1}\,\bar{S}[11] + \frac{\partial x^2}{\partial X^2}\,\bar{S}[21]\right) + \frac{\partial}{\partial X^2}\left(\frac{\partial x^2}{\partial X^1}\,\bar{S}[21] + \frac{\partial x^2}{\partial X^2}\,\bar{S}[22]\right) + \rho_{\mathcal{R}} b^2 &= \rho_{\mathcal{R}} \ddot{x}^2, \\
\frac{\partial}{\partial X^3}\left(\frac{\partial x^3}{\partial X^3}\,\bar{S}[33]\right) + \rho_{\mathcal{R}} b^3 &= \rho_{\mathcal{R}} \ddot{x}^3.
\end{aligned}\right\} \quad (8.30)$$

Of course, the equations (8.9) and (8.27) are to be used to evaluate the quantities $\bar{S}[ij] = \bar{S}[ij](t)$ appearing here. Thus, (8.30) is a set of three functional-differential equations for the three functions x^i in (8.24).

9. Shearing Motions

A shearing motion, or, for short, a shear is a sheared extension for which

$$\beta_1(t-s) \equiv \beta_2(t-s) \equiv \beta_3(t-s) \equiv 1 \qquad (9.1)$$

in (8.2).# Clearly, for a shear the equations (8.6) reduce to

#Shearing motions are here defined relative to an unspecified undistorted reference configuration and the derived expressions for the stress are, of course, valid for both compressible solids and compressible fluids. For incompressible fluids, or whenever the present configuration can be regarded as undistorted and the density is not a parameter, the present analysis of shears includes as special cases Coleman and Noll's analyses of steady viscometric flows (1959a & b) and unsteady lineal shearing flows (1961, §5, pp. 694-699).

$$\xi = \zeta, \qquad \gamma_1 = 1 + \zeta^2, \qquad \gamma_2 = \gamma_3 = 1, \qquad (9.2)$$

and we can make the following

Remark: The history of X up to time t is a shear if and only if there exists a basis $\underset{\sim}{h}_i$, independent of s, relative to which the components of

$\underset{\sim}{C}(t-s)$ have the form

$$[\underset{\sim}{C}(t-s)] = [\underset{\sim}{h}_i \cdot \underset{\sim}{C}(t-s)\underset{\sim}{h}_j] = \begin{bmatrix} 1+\xi(t-s)^2 & \xi(t-s) & 0 \\ \xi(t-s) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (9.3)$$

Let us call $\xi(t-s)$ the <u>amount of shear</u> at X at time t-s, and let us continue to call $\underset{\sim}{h}_i$ the <u>canonical basis</u> at X. Of course, (9.3) implies that $\rho(t-s) \equiv \rho_R$. When the history is known to be a shearing motion, specification of the density, the real-valued function ξ^t, and the canonical basis $\underset{\sim}{h}_i$ determines $\underset{\sim}{C}^t$ and hence, by (2.22), the material stress tensor $\bar{\underset{\sim}{S}}(t)$.

We now use the three material functionals $\mathcal{z}$, $\mathcal{s}$, $\mathcal{t}$ to define four new material functionals $\mathcal{y}$, $\mathcal{w}_1$, $\mathcal{w}_2$, $\mathcal{w}_3$ of a single real-valued function ξ^* on $[0,\infty)$:

$$\begin{aligned} \mathcal{y}(\xi^*;\rho) &= \mathcal{z}(\xi^*, 1^\dagger+\xi^{*2}, 1^\dagger, 1^\dagger;\rho), \\ \mathcal{w}_1(\xi^*;\rho) &= \mathcal{s}(\xi^*, 1^\dagger+\xi^{*2}, 1^\dagger, 1^\dagger;\rho), \\ \mathcal{w}_2(\xi^*;\rho) &= \mathcal{s}(\xi^*, 1^\dagger, 1^\dagger+\xi^{*2}, 1^\dagger;\rho), \\ \mathcal{w}_3(\xi^*;\rho) &= \mathcal{t}(\xi^*, 1^\dagger+\xi^{*2}, 1^\dagger, 1^\dagger;\rho), \end{aligned} \qquad (9.4)$$

$$1^\dagger(s) \equiv 1, \qquad \xi^{*2}(s) = \left(\xi^*(s)\right)^2, \qquad 0 \le s < \infty.$$

The restrictions $(8.10)_{2,3\&5}$ on $\mathcal{Z}$, $\mathcal{S}$, $\mathcal{t}$ require that $\mathcal{Y}$ and ω_i obey the identities

$$\left.\begin{aligned} \mathcal{Y}(-\xi^*;\rho) &= -\mathcal{Y}(\xi^*;\rho) \\ \omega_i(-\xi^*;\rho) &= \omega_i(\xi^*;\rho), \qquad i = 1,2,3; \end{aligned}\right\} \tag{9.5}$$

i.e., $\mathcal{Y}$ is an odd functional and ω_i, $i = 1,2,3$ an even functional. The following theorem is now an obvious corollary to Theorem 7.

Theorem 8: If, relative to an undistorted reference configuration with density ρ, the motion of an isotropic material up to time t is a shear with canonical basis $\underset{\sim}{h}_i$ and amount of shear $\xi(t-s)$, then the components with respect to $\underset{\sim}{h}_i$ of the material stress tensor $\bar{\underset{\sim}{S}}(t)$ are

$$\left.\begin{aligned} \bar{S}[13](t) = \bar{S}[31](t) = \bar{S}[32](t) = \bar{S}[23](t) &= 0, \\ \bar{S}[12](t) = S[21](t) &= \mathcal{Y}(\xi^t;\rho), \\ \bar{S}[ii](t) &= \omega_i(\xi^t;\rho), \qquad i = 1,2,3, \\ \bar{S}[ij](t) = \underset{\sim}{h}_i \cdot \underset{\sim}{S}(t)\underset{\sim}{h}_j \quad \text{and} \quad \xi^t(s) &= \xi(t-s), \qquad 0 \le s < \infty; \end{aligned}\right\} \tag{9.6}$$

here $\mathcal{Y}$ and ω_i are material functionals obeying the identities (9.5).

10. Curvilineal Shears

Letting the spatial coordinates x^i and material coordinates X^α in (4.1) be referred to the same coordinate system, we say that the motion is a curvilineal shear# if it satisfies the following three

#The curvilineal shears defined here are closely related to the curvilineal flows discussed by Noll and Truesdell (1965, §107, pp. 432-434). Here I assume that the path-line equations take the simple form (10.1) when a fixed undistorted configuration is taken as reference, while Noll and Truesdell assume, at bottom, that an equation of the form (10.1) holds whenever the present configuration is taken as reference. For fluids, the present analysis may be replaced by theirs. Their treatment is not directly applicable to isotropic solids, whereas the present is, although it follows theirs in several details.

conditions:

(1) There exists an orthogonal curvilinear coordinate system which gives (4.1) the special form

$$\left.\begin{aligned} x^1 &= X^1 \\ x^2 &= X^2 + \eta(X^1, t) \\ x^3 &= X^3 + \lambda(X^1, t), \qquad -\infty < t < \infty. \end{aligned}\right\} \qquad (10.1)$$

(2) The ratio of the X^1 derivatives, η' and λ', of η and λ do not depend on t; i.e. these derivatives have the form

$$\eta'(X',t) = f(X^1)q(X^1,t), \qquad \lambda'(X,t) = h(X')q(X',t). \qquad (10.2)$$

(3) The covariant components $g_{kk}(\underset{\sim}{x})$ of the unit tensor are constant along path lines and are equal to $g_{kk}(\underset{\sim}{X})$.

If, in particular, the g_{kk} depend on only x^1, then it follows from $(10.1)_1$ that $g_{kk}(\underset{\sim}{x}) = g_{kk}(x^1) = g_{kk}(X^1)$ and hence that the g_{kk} are constant along the curves defined by (10.1). In a similar way we see that the g_{kk} are constant along such curves if $\lambda(X^1,t) \equiv 0$ and if the g_{kk} depend on only x^1 and x^3. L. E Bragg has shown that the former situation holds if and only if the coordinate system is either Cartesian or cylindrical with x^1 the radial coordinate.#

#As yet unpublished. See also Truesdell and Noll (1965, p. 433).

Let $\underset{\sim}{e}_\alpha(\underset{\sim}{X})$, $\alpha = 1,2,3$, be the unit vectors along the coordinate directions X^1, X^2, X^3 at the point $\underset{\sim}{X}$ occupied by the particle X in the reference configuration. Clearly the numbers $\underset{\sim}{e}_\alpha \cdot \underset{\sim}{C}(t-s)\underset{\sim}{e}_\beta$ equal the physical components $C\langle\alpha\beta\rangle$ shown in the general formula (7.4) and evaluated at time t-s. For a curvilinear shear, an elementary calculation yields

$$\left.\begin{aligned} [\underset{\sim}{e}_{\underset{\sim}{\alpha}} \cdot \underset{\sim}{C}(t-s)\underset{\sim}{e}_{\underset{\sim}{\beta}}(\underset{\sim}{X})] &= \begin{bmatrix} 1+\psi^2+\omega^2 & \psi & 0 \\ \psi & 1 & 0 \\ \omega & 0 & 1 \end{bmatrix}, \\ \text{with} \quad \psi = \eta'(X^1, t-s)\sqrt{\frac{g_{22}(\underset{\sim}{X})}{g_{11}(\underset{\sim}{X})}}, &\qquad \omega = \lambda'(X^1, t-s)\sqrt{\frac{g_{33}(\underset{\sim}{X})}{g_{11}(\underset{\sim}{X})}}. \end{aligned}\right\} \quad (10.3)$$

Let us now put

$$\xi(t-s) = \sqrt{\frac{g_{22}(\underset{\sim}{X})\lambda'(X^1,t-s)^2 + g_{33}(\underset{\sim}{X})\eta'(X^1,t-s)^2}{g_{11}(\underset{\sim}{X})}} = \sqrt{\psi^2+\omega^2}, \quad (10.4)$$

$$\mu = \frac{\eta'}{\xi}\sqrt{\frac{g_{22}}{g_{11}}} = \frac{\psi}{\sqrt{\psi^2+\omega^2}}, \qquad \nu = \frac{\lambda'}{\xi}\sqrt{\frac{g_{33}}{g_{11}}} = \frac{\omega}{\sqrt{\psi^2+\omega^2}}, \qquad \nu^2+\mu^2 = 1. \quad (10.5)$$

Although ξ depends on t-s, it follows from (10.2) that μ and ν are independent of t-s. The basis $\underset{\sim}{h}_i$, defined by

$$\left.\begin{aligned} \underset{\sim}{h}_1 &= \underset{\sim}{e}_1(\underset{\sim}{X}), \\ \underset{\sim}{h}_2 &= \mu\underset{\sim}{e}_2(\underset{\sim}{X}) + \nu\underset{\sim}{e}_3(\underset{\sim}{X}), \\ \underset{\sim}{h}_3 &= -\nu\underset{\sim}{e}_2(\underset{\sim}{X}) + \mu\underset{\sim}{e}_3(\underset{\sim}{X}), \end{aligned}\right\} \quad (10.6)$$

is orthonormal, and, for each particle, independent of t-s. An easy calculation, employing (10.3), (10.4), and (10.6), shows that the components of $\underset{\sim}{C}(t-s)$ relative to the basis $\underset{\sim}{h}_i$ have the matrix (9.3). Thus, we can assert the following

<u>Remark</u>: In a curvilinear shear the motion of each particle is a shearing motion with the canonical basis given by (10.6) and the amount of shear by (10.4).

Theorem 8 tells us that the components $\bar{\underset{\sim}{S}}_{ij}[ij](t)$ of $\bar{\underset{\sim}{S}}(t)$ relative to $\underset{\sim}{h}_i$ are given by (9.6). The physical components $\bar{\underset{\sim}{S}}\langle\alpha\beta\rangle$ of $\bar{\underset{\sim}{S}}(t)$ in the material coordinates X^1, X^2, X^3, i.e. the components of $\bar{\underset{\sim}{S}}(t)$ relative to the basis $\underset{\sim}{e}_\alpha(X)$, are related to the components $\bar{S}[ij](t)$ through the formula

$$\bar{\underset{\sim}{S}}\langle\alpha\beta\rangle(t) \stackrel{\text{def}}{=\!=} \underset{\sim}{e}_\alpha(\underset{\sim}{X})\cdot\bar{\underset{\sim}{S}}(t)\underset{\sim}{e}_\beta(\underset{\sim}{X}) = \left(\underset{\sim}{e}_\alpha(\underset{\sim}{X})\cdot\underset{\sim}{h}_k\right)\bar{S}[k\ell](t)\left(\underset{\sim}{h}_\ell\cdot\underset{\sim}{e}_\alpha(\underset{\sim}{X})\right), \tag{10.7}$$

with summation over k and ℓ understood. On substituting (9.6) and (10.6) into (10.7) we find that

$$\left.\begin{aligned}
\bar{S}\langle 11\rangle(t) &= \mathscr{W}_1(\xi^t;\rho),\\
\bar{S}\langle 22\rangle(t) &= \mu^2\mathscr{W}_2(\xi^t;\rho) + \nu^2\mathscr{W}_3(\xi^t;\rho),\\
\bar{S}\langle 33\rangle(t) &= \nu^2\mathscr{W}_2(\xi^t;\rho) + \mu^2\mathscr{W}_3(\xi^t;\rho),\\
\bar{S}\langle 12\rangle(t) &= \bar{S}\langle 21\rangle(t) = \mu\mathscr{Y}(\xi^t;\rho),\\
\bar{S}\langle 13\rangle(t) &= \bar{S}\langle 31\rangle(t) = \nu\mathscr{Y}(\xi^t;\rho),\\
\bar{S}\langle 23\rangle(t) = \bar{S}\langle 32\rangle(t) &= \mu\nu[\mathscr{W}_2(\xi^t;\rho) - \mathscr{W}_3(\xi^t;\rho)],\\
\xi^t(s) &= \xi(t-s), \qquad 0 \le s < \infty,
\end{aligned}\right\} \tag{10.8}$$

and we have

Theorem 9: If the motion of an isotropic body, relative to an undistorted configuration with density ρ, is a curvilinear shear (10.1), the physical components of the material stress tensor $\bar{\underset{\sim}{S}}$ are given by (10.8) with ξ, μ, and ν defined in (10.4) and (10.5).

Substitution of (10.8) into the dynamical equations (8.28) will, in general, yield an over-determined system of field equations. Exceptions occur when the material is incompressible and the coordinate system x^1, x^2, x^3 is either Cartesian or cylindrical. A detailed treatment of such motions will be given in a forthcoming article.

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References

Brauer, R. 1965 Arch. Rational Mech. Anal. **18**, 97-99.

Coleman, B. D., Greenberg, J. M., & Gurtin, M. E. 1966 Arch. Rational Mech. Anal. **22**, 333-354.

Coleman, B. D. & Gurtin, M. E. 1965 Arch. Rational Mech. Anal. **19**, 239-298.

Coleman, B. D., Gurtin, M. E., & Herrera-R., I. 1965 Arch. Rational Mech. Anal. **19**, 1-19.

Coleman, B. D. & Noll, W. 1959a Arch. Rational Mech. Anal. **3**, 289-303.

Coleman, B. D. & Noll, W. 1959b J. Appl. Phys. **30**, 1508-1512.

Coleman, B. D. & Noll, W. 1961 Ann. N. Y. Acad. Sci. **89**, 672-714.

Coleman, B. D. & Noll, W. 1962 Phys. Fluids **5**, 840-843.

Coleman, B. D. & Noll, W. 1964 Arch. Rational Mech. Anal. **15**, 87-111.

Ensminger, R. R. & Fyfe, M. 1966 J. Mech. Phys. Solids, **14**, 231-238.

Green, A. E. 1956 J. Rational Mech. Anal. **5**, 637-642.

Green, A. E. & Rivlin, R. S. 1957 Arch. Rational Mech. Anal. **1**, 1-21.

Greenberg, J. M. 1967 Arch. Rational Mech. Anal. **24**, 1-21.

Noll, W. 1955 J. Rational Mech. Anal. **4**, 3-81.

Noll, W. 1958 Arch. Rational Mech. Anal. **2**, 197-226.

Noll, W. 1965 Arch. Rational Mech. Anal. 18, 100-102.

Serrin, J. 1959 Mathematical principles of classical fluid mechanics. In the Encyclopedia of Physics, vol. VIII/1, pp. 125-263, edited by S. Flügge. Berlin-Göttingen-Heidelberg: Springer.

Truesdell, C. 1952 J. Rational Mech. Anal. 1, 125-300.

Truesdell, C. 1955a J. Rational Mech. Anal. 4, 83-133, 1019-1020.

Truesdell, C. 1955b Comm. Pure Appl. Math. 8, 123-132.

Truesdell, C. & Noll, W. 1965 The Non-linear Field Theories of Mechanics. In the Encyclopedia of Physics, vol. III/3, edited by S. Flügge. Berlin-Heidelberg-New York: Springer.

Truesdell, C. & Toupin, R. 1960 The Classical Field Theories of Mechanics. In the Encyclopedia of Physics, vol. III/1, pp. 226-902, edited by S. Flügge. Berlin-Göttingen-Heidelberg: Springer.