A NUMERICAL APPROACH TO THE SIXTH COEFFICIENT PROBLEM

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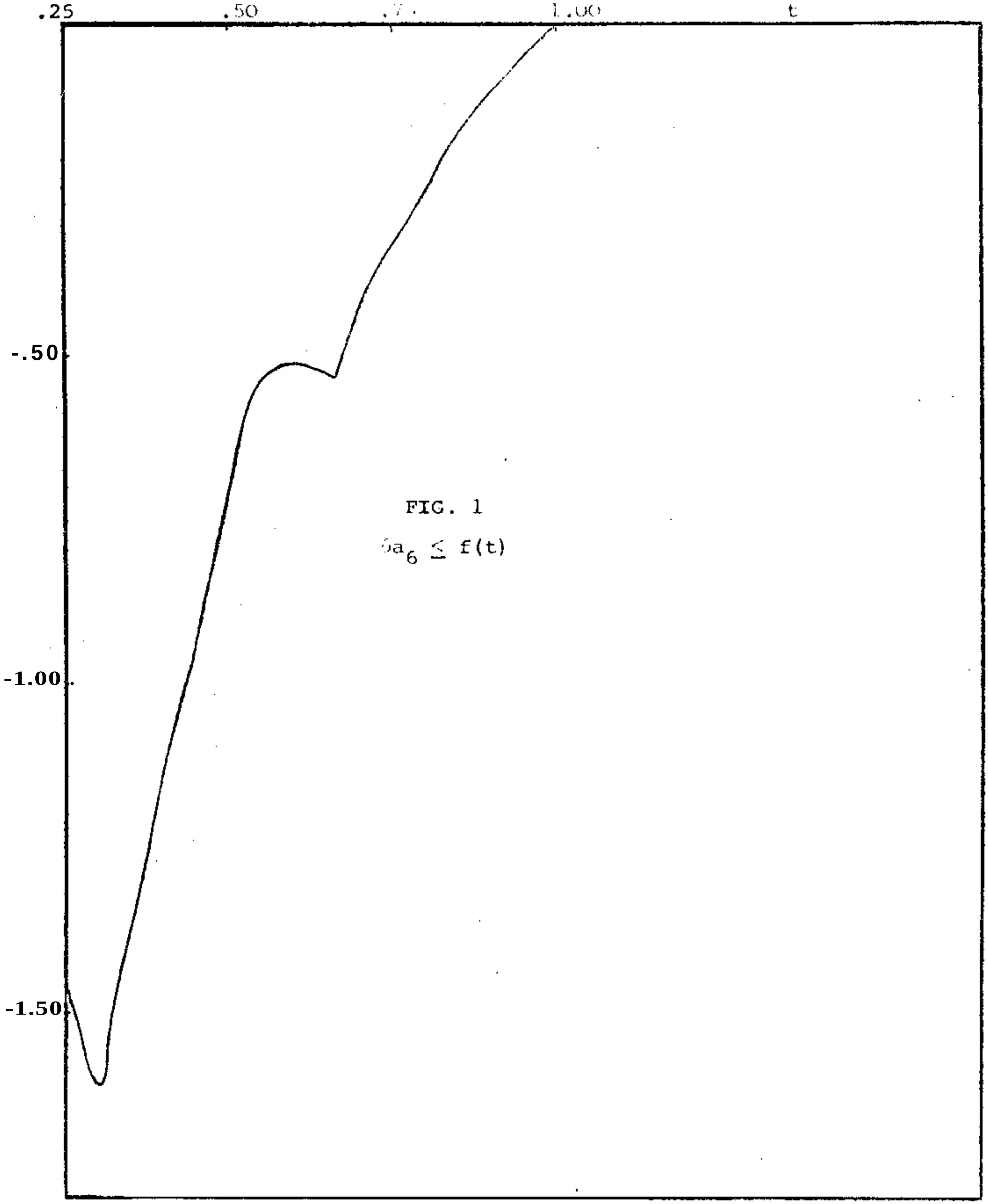

FIG. 1

$\delta a_6 \leq f(t)$

defined on the unit disk- C_m denotes the m-th row vector of C and 6C = C - I where I is the identity. $P_k(C)$ represents a polynomial which is homogeneous of degree k in the senSe that if C = $(e^{i(l/\wedge)\theta}C\wedge)$, then P_k<C) - $e^{ik\theta}P_k(C)$.

Garabedian, Ross and Schiffer noted that $a_6 = P\wedge Q\check{C}0$ where $\check{C}$ = ^C?_ν+1,9y +1^ ' v '^ ^ °'^'2 ' C beinS the Grunsky matrix of f(z $\overline{2}$) . They also noticed a polynomial relation $P_g(\check{C})$: 0 ;thus

(o) $a_6 = P_{10}(\& + (i| c\wedge - 1M c_{13}) R_6(\check{C}) = \tilde{P}_{10}(\lambda,\mu,\check{C})$

where A and M are Lagrange multipliers. The authors then motivated the choice A = /i = 1 and considered the more general problem of maximizing the resulting polynomial over the class of all symmetric three by three matrices which satisfy Grunsky's inequality. An application of Schur's diagonalization theorem for symmetric matrices then showed that in the more general setting the maximum occurs when $\check{C}$ is unitary•

The essential tool in our investigation is

Theorem 1. The following conditions are necessary and sufficient in order that the normalized analytic function f(z) map the unit disk 1 - 1 onto the complement of a set of measure zero:

(1) The Grunsky matrix C associated with f is unitary.

(2) $r_{mn} - I_{mn} = -i(6C_2, 6C_m)_n$.

The above theorem is due to the author [5] and was used to show that for real a in a neighborhood of the origin,

$|a_4 + a_2 a_3 - \frac{3}{2} a_2^3| \le 4$ with equality holding only for the Koebe function. The method seems to indicate that, for the higher even order coefficients, the unitary property of the infinite matrix gives one all of the advantages of a truncated unitary matrix while allowing one to retain the polynomial relations.

Assume that $f(z)$ defines a slit univalent mapping. By the homogeneity of $\tilde{P}_{10}$, it may be assumed that $a_6 > 0$ and that $|\operatorname{Arg} c_{11}| < \pi/5$; hence

(3) $0 \le t = \operatorname{Re} c_{11} \le 1$ and $|s_{11}| \le \theta t$, $\theta = \tan \pi/5$,

(4) $\delta a_6 = \operatorname{Re}\{\tilde{P}_{10}(\mu,\lambda,C) - \tilde{P}_{10}(\mu,\lambda,I)\}$.

Theorem *2.* If $0 \le t \le 1/4$, then $a_6 \le 6$.
This is proved by using (4) with $\mu = 1$, $\lambda = 1$ together with $|c_{11}| \le 5t/4$ (a consequence of (3)) and the unitary property of C.

When $t > 1/4$ a more delicate analysis is required. Define

$$U = \frac{1}{\sqrt{5}}\delta c_5 + \frac{2t}{3}\delta c_3 + \sqrt{\frac{3}{3}}(1-\mu)c_{13} + 2\lambda(t^2 + \beta)\delta c_1,$$

and

$$V = \frac{1}{\sqrt{3}}\delta c_3 + 2t\delta c_1 .$$

Here β is a quadratic form in c_i to be chosen later. It is an easy consequence of (2) that there exist polynomials $Q(\mu,\lambda,C,\overline{C})$ and $R(\mu,C,\overline{C})$ such that

(5) $-||U|1^2 + Q(M,A,C,\bar{C}) - 0$

and

(6) $-(t^2 + \frac{6\mu}{3} r_{1J}) \|v\|^2 + R(\mu,C,\bar{C}) = 0$.

Our approach is to combine (4), (5) and (6) and choose M and A so that, in some sense, one obtains very good estimates for functions with real coefficients. If $r_{13} \leq 0$ we choose A - 1/2 , [I - 0 and add (4) to (6) omitting the term $-||v||^2$. When $r_{13} > 0$ and .8 < t <_1 , we choose A = 1 , /i -= 0 and add (4)^ (5) and (6), dropping the contributions of $-1|u||^2$ and $-||v||^2$ other than $(\text{Im } U_k)^2$, k = 1,3 , $(\text{Im } V_k)^2$, k - 1,3,5 and $(\text{Re } V_1)^2$. If $r_{13} > 0$ and .25 < t < .9 we choose A - 1 , leave M undertermined, and add (4), (5) and (6) dropping the contributions of $-||uj|^2$ and $\sim||v||^2$ other than $-|v_1|^2$. The next step is to express all Grunsky coefficients in terms of the first row CL_1 . In the resulting expressions, there are perfect squares of the form $\sim(L(C_1) + M(CL_1))$ where L is linear and M contains no linear terms. These are estimated by $-(L(C_1))^2 - 2L(C_1)M(C_1)$. The quadratic form j8 is now chosen so that the coefficient of r_1g is zero. What remains is an estimate of the form

(7) $6a_6 \leq P(t) - A_Q(/i,t) | r_{31}|^3 - A_1(\mu,t)r_{13}^2 + A_2(\mu,t)|r_{13}|$

$+ Q_1(t,s) + s^{\wedge} Q_2(t,s) + Q_3(t,s)(L(t,s'))^2$

$+ Q_4(M,t,s)|r_{13}|$.

Here $P(t)$ and A, are polynomials, the Q^js are quadratic forms in $s = (s_{11}, s_{13}, s_{15}, s_{17})$, L is linear in $s' = (0, s_{13}, s_{15}, s_{17})$, the coefficients of the Q^js and L being polynomials in the indicated variables. The problem of proving that $6a_6 \le 0$ is thereby reduced to showing that the right side of (7) is negative subject to the condition

(8) $\quad r^*_{13} + ||s||^2 \le 1 - t^2 ,$

the area theorem, and (3).

Let $Q(t,x,y)$ represent a quadratic form in the variables $x = s_{11}$, $y = (s_{13}, s_{15}, s_{17})$ whose coefficients are polynomials in t. We write

$$Q(t,x,y) = Q_{11}(t)x^2 + 2xq(t,y) + \tilde{Q}(t,y)$$

where q is linear in y and set

$$\check{Q}(t,x,y) = Q(t,x,y) - Q_{11}(t)x^2 .$$

For each t the maximum of zero and the largest eigen value of Q is denoted by $\mu(Q)$. With θ defined by (4) we let

$$\nu_1(Q) = \mu(Q)(1 - t^2) ,$$

$$\nu_2(Q) = Q_{11}\theta^2 t^2 + \mu(\check{Q})(1 - t^2) ,$$

$$\nu_3(Q) = Q_{11}\theta^2 t^2 + 2\theta||q|| \, t\sqrt{1 - t^2} + \mu(\tilde{Q})(1 - t^2) ,$$

$$\nu_4(Q) = \theta\sqrt{Q_{11}^2 + 4||q||^2} \, t\sqrt{1 - t^2} + \mu(\tilde{Q})(1 - t^2)$$

and $\nu(Q) = \min[\nu_1, \nu_2, \nu_3, \nu_4]$. Set

$$P(Q) = \mu(Q) \quad \text{if} \quad \nu = \nu_1 ,$$

$$= \mu(\check{Q}) \quad \text{if} \quad \nu = \nu_2 ,$$

$$= 8\|q\| \; t \sqrt{1 - t^2} + \mu(\tilde{Q}) \quad \text{if} \quad \nu = \nu_3 ,$$

$$= \Theta \quad Q + 4\|q\|^2 \; t/2 \quad 1 - t^2 + \mu(\tilde{Q})$$

$$\text{if} \quad \nu = \nu_4 .$$

We are now in a position to estimate those terms in (7) which depend only on s and t .

Theorem 3. Let $\Delta = M(Q_1) - \mu(\check{Q}_1) - Q_{1,11} - \nu(\check{Q}_2)$, $\eta = \mu(\check{Q}_1) - \mu(Q_1) + \mu(Q_3)\|L\|^2$, $C = Q_{2,11}$ (1 + t2) and define ψ to be the root of the quadratic equation

$$Cz^2 - (\Delta + \eta)z + \eta = 0$$

which lies in the interval [0,1] if $\Delta > 0$; otherwise set $\psi = 1$. Define $a = \min\{\mu(Q_2), C\psi + \nu(\check{Q}_2)\}$. Then

$$Q_1(s) + s^2 Q_2(s) + Q_3(s)(L(s))^2 \leq M(Q_5)(1 - t^2 - r_{13}^2)$$

where

$$Q_5 = Q_1(s) + a s^2 + \mu(Q_3)(L(st))^2 .$$

In the one case where μ was not determined f_x is chosen to minimize $(A_2)^2/(A_1 + \mu(Q_3))$ over $\mu \geq 0$.

Theorem 4. If f(z) is a normalized univalent slit mapping in the unit disk which satisfies (4), then

$$\delta a_6 \leq P(t) - (A_0 + \mu(Q_3))|r_{13}|^3 - (A_1 + \mu(Q_5))r_{13}^2$$

$$+ (A_2 + \nu(Q_3))|r_{13}| + \mu(Q_5)(1 - t^2) .$$

By substituting for $|r_{13}|$ the value for which the cubic obtains its maximum or $1 - t^2$, whichever is smaller, we obtain the desired bound which depends only on t. Computing experiments indicate that this bound has a graph given by Figure 1, at least if $t = .25 + k(.01)$, $k = 0,1,\ldots,75$.

We are now engaged in a program designed to convert the above numerical procedure into a rigorous proof. It is proposed to use the data obtained from the computer to obtain tentative polynomial bounds for the various parameters. Proving that these are actual bounds is equivalent to showing that each of a finite number of polynomials is positive on an appropriate interval, a task which can be accomplished in a finite number of steps. If this procedure is successful, the problem will have been reduced to proving that a polynomial with rational coefficients is non-positive. This again requires only a finite number of steps.

For a more complete bibliography, the reader is referred to the papers listed below.

BIBLIOGRAPHY

[1] Bieberbach, L., "Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln", S. B. preuss. Akad. Wiss., 138(1916) 940-955.

[2] Loewner, K., "Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I", Math. Ann., 89(1923) 103-121.

[3] Garabedian, P. R. and M. Schiffer, "A proof of the Bieberbach conjecture for the fourth coefficient", J. Ratl. Mech. Anal., 4(1955) 427-465.

[4] Garabedian, P. R., Ross, G. G. and M. Schiffer, "On the Bieberbach conjecture for even n", J. Math. Mech., 2^(1^65) 975-930.

[5] Pederson, R., "On unitary properties of Grunsky's matrix", to appear in Arch. Ratl. Mech. and Anal.

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