OSCILLATION THEOREMS FOR SYSTEMS
OF LINEAR DIFFERENTIAL EQUATIONS

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Report 68-4

January, 1968

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1. The systems to be considered in this paper are of the

form

$$(1.1) \qquad\qquad y' = Ay \,,$$

where $A = A(x)$ is a continuous $n \times n$ matrix on an x-
interval R, and y is an n-dimensional column vector.
We shall assume that the elements of A are real, and we
shall consider only real solution vectors of (1.1). This is
not an essential restriction since, in the complex case, (1.1)
can be replaced by an equivalent real system with a $2n \times 2n$
coefficient matrix.

We shall say that a nontrivial solution vector $y = (y_1, \ldots, y_n)$
of (1.1) is <u>oscillatory</u> on R if each of its components takes
the value zero at some point of R, i.e., $y_k(x_k) = 0$, $x_k \in R$,
$k = 1, \ldots, n$. The system (1.1) itself will be said to be
oscillatory if it possesses at least one oscillatory solution
vector. If there is no such solution vector, i.e., if every

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expressed in terms of a fundamental (i.e., nonsingular) solution matrix Y of the matrix-matrix equation

(1.4) $$Y' = AY$$

corresponding to (1.1). Since the general solution of (1.1) is of the form $y = Y\alpha$, where α is an arbitrary constant vector, condition (1.2) is equivalent to $\alpha Y^*(s)Y(t)\alpha > 0$, i.e., to the condition that the symmetric part of the matrix $Y^*(s)Y(t)$ be positive-definite for all $s \in R$, $t \in R$. In particular, if we choose a fundamental solution Y_s of (1.3) which reduces to the unit matrix I for $x = s$, the condition becomes $\alpha Y_s(t)\alpha > 0$. With the help of this version of condition (1.2) it is easy to establish the following property of suborthogonal systems.

If the system (1.1) is suborthogonal, so is the adjoint system

(1.5) $$w' = -A^*w \qquad .$$

Indeed, if the matrix $W_s(x)$ is the fundamental solution (with $W_s(s) = I$) of the matrix-matrix equation corresponding to (1.5), we have

$$(W_s^*Y_s)' = -W^*AY + W^*AY = 0$$

and therefore $W_s^*Y_s = I$. Hence, if β is an arbitrary constant vector, and we set $\alpha = Y_s^{-1}(t)\beta = W_s^*(t)\beta$, we have $\beta W_s(t)\beta = \beta W_s^*(t)\beta = \alpha Y_s(t)\alpha > 0$. Since s,t and the

constant vector β were arbitrary, the assertion follows.

2. The principal aim of this paper is to obtain conditions -- expressed in terms of the coefficient matrix -- which guarantee the nonoscillation of the system (1.1) on a given interval. All these conditions will follow from two basic inequalities, which we state here in the form of a theorem.

Theorem 2.1. Let y and w be nontrivial solution vectors of the systems:

(2.1) $$y' = Ay \ ,$$

(2.2) $$w' = Bw \ ,$$

respectively, where the n × n matrices A,B are continuous on the interval [a,b] . If u,v are the unit vectors

(2.3) $$u = \frac{y}{\|y\|} \quad , \quad v = \frac{w}{\|w\|}$$

and C is an arbitrary constant orthogonal matrix, then

(2.4) $$\left| \text{arc sin}[u(b)Cv(b)] - \text{arc sin}[u(a)Cv(a)] \right| \leq \int_a^b (\|A\| + \|B\|) \, dx \ ,$$

where $\|A\|$ denotes the norm $\sup_{\|\alpha\|=1} \|A\alpha\|$.

If one of the systems (2.1), (2.2) is oscillatory on [a,b] , (2.4) can be replaced by the stronger inequality

(2.5) $$\left| \text{arc sin}[u(b)Cu(b)] \right| + \left| \text{arc sin}[u(a)Cu(a)] \right| \leq \int_a^b (\|A\| + \|B\|) \, dx \ .$$

Proof. Differentiating (2.3), we obtain

$$u' = \frac{y'}{\|y\|} - \frac{y(yy')}{\|y\|^3} \quad ,$$

and a similar expression for v' . In view of (2.1) and (2.2), this leads to

$$(2.6) \qquad u' = Au - u(uAu)$$

and

$$(2.7) \qquad v' = Bv - v(vBv) \ .$$

If C is a constant orthogonal matrix, we have

$$(uCv)' = u'Cv + v'C*u$$

and thus, by (2.6) and (2.7) ,

$$(2.8) \quad (uCv)' = [Cv - (uCv)u]Au + [C*u - (uCv)v]Bv \ .$$

Hence, since u and v are unit vectors,

$$|(uCv)'| \leq \|Cv - (uCv)u\| \ \|A\| + \|C*u - (uCv)v\| \ \|B\| \ .$$

Because of

$$\|Cv - (uCv)u\|^2 = \|Cv\|^2 - (uCv)^2 = 1 - (uCv)^2$$

and

$$\|C*u - (uCv)v\|^2 = \|C*u\|^2 - (uCv)^2 = 1 - (uCv)^2 \ ,$$

this implies

$$(2.9) \qquad \frac{\overline{(uCv)'}}{\sqrt{1 - (uCv)^2}} \leq \|A\| + \|B\| \; ,$$

and an integration establishes (2.4) .

We now turn to the proof of inequality (2.5). Since $uCv = vC*u$, we may assume without loss of generality that w (and thus also v) is oscillatory on $[a,b]$. If $v_1, \ldots, v_n$ are the components of v , there will thus exist a set of points $x_1, \ldots, x_n$ in $[a,b]$, containing at least two different points (since otherwise w would reduce to the trivial solution), such that $v_k(x_k) = 0$, $k = 1, \ldots, n$. Evidently, the vector $Cv(x_k)$ is not changed if the elements c_{ik} , $i = 1, \ldots, n$ in the k-th column of the matrix C are replaced by different numbers. We shall take advantage of this fact by substituting $-c_{ik}$ for $c_{ik} (i = 1, \ldots, n)$, and we note that this change does not affect the orthogonal character of the matrix. Proceeding from a to b , and making this change whenever a point x_k is crossed, we obtain a matrix function $C(x)$ which is constant and orthogonal in the intervals between adjacent points x_k . By the construction of $C(x)$, the vector function $C(x)v(x)$ is continuous on $[a,b]$, and we evidently have $C(b) = -C(a) = -C$.

In any interval between adjacent points x_k we may use (2.9) with $C(x)$ substituted for C . We integrate, and add up the contributions from all the intervals making up $[a,b]$.

Since $C(x)v(x)$ is continuous, and since $C(b) = -C(a) = -C$, we obtain

$$\left| \text{arc } \sin[u(b)Cv(b)] + \text{arc } \sin[u(a)Cv(a)] \right| \leq \int_a^b (\|A\| + \|B\|) \, dx \; .$$

Combining this with (2.4), we obtain (2.5). It is easy to see that this argument remains valid if some (or even all) of the points x_k coincide with either a or b.

In the special case in which C is a diagonal matrix whose elements c_{kk} are either 1 or -1, the continuity of $C(x)$ is not affected by changing c_{kk} into $-c_{kk}$ at a point at which the k-th component of either u or v is zero. In order to obtain inequality (2.5) it is therefore sufficient to assume that, for each k $(k = 1,\ldots,n)$, the k-th component of at least one of the vectors y,w vanishes on $[a,b]$. If, for $x = a$, we take C to be the unit matrix, this leads to the following result.

Theorem 2.2. Let y,w, A,B, u,v have the same meaning as in Theorem (2.1). If, for each k $(k = 1,\ldots,n)$, the k-th component of at least one of the vectors y,w vanishes at a point of $[a,b]$, then

$$(2.9) \quad \left| \text{arc } \sin[u(b)v(b)] \right| + \left| \text{arc } \sin[u(a)v(a)] \right| \leq \int_a^b (\|A\| + \|B\|) \, dx \; .$$

3. As a first application of Theorem 2.1 we derive the following sufficient condition for suborthogonality.

Theorem 3.1. If, for some continuous real function $\mu = \mu(x)$ on $[a,b]$, we have

$$(3.1) \qquad \int_a^b \|A + \mu E\| dx < \frac{\pi}{2} \quad ,$$

then the system (2.1) is suborthogonal on $[a,b]$. The constant $\pi/2$ in (3.1) is the best possible; in fact, the conclusion does not necessarily hold if the sign of equality is permitted in (3.1).

Proof: We use (2.4), with $B = 0$ and $C = I$. Since (2.2) is solved by an arbitrary constant vector, v may be taken to be an arbitrary constant unit vector. If $a \leq s < t \leq b$, and we set $v = u(t)$, (2.4) becomes

$$(3.2) \qquad |\frac{\pi}{2} - \text{arc} \sin[u(s)u(t)]| \leq \int_s^t \|A\| dx \leq \int_a^b \|A\| dx \ .$$

If (2.1) is not suborthogonal on $[a,b]$, there exist $s,t \in [a,b]$ such that $u(s)u(t) = 0$. In this case we thus must have

$$\frac{\pi}{2} \leq \int_a^b \|A\| dx \ .$$

If $\mu \equiv 0$, this conflicts with (3.1) and thus proves Theorem (3.1) in this particular case. The case of a general continuous function μ is easily reduced to the case $\mu = 0$, since the

general solution of the system $\sigma' = (A + E\mu)\sigma$ is of the form $\sigma = gy$, where $g(x)$ is the scalar function $\exp\{\int^x.\mu dx\}$ and y is the general solution of (2.1). We have $\sigma(s)\sigma(t) = g(s)g(t)[y(s)y(t)]$ and, since $g \neq 0$, the system $\sigma' = (A + \mu E)\sigma$ is suborthogonal if, and only if, the same is true of the system (2.1).

To show that the constant in Theorem 3.1 is the best possible, we set $n = 2m$, where m is a positive integer, and we consider the coefficient matrix A whose elements a_{ik} are defined as follows: $a_{k,k+1} = 1$, $k = 1,\ldots,n - 1,$ $a_{n,1} = (-1)^m$; all other elements of A are zero. It is easily confirmed that the system (2.1) associated with this matrix has a solution vector $y = (y_1,\ldots,y_{2m})$ with $y_{2k+1} = (-1)^k \sin x$, $k = 0,\ldots,m - 1$, and $y_{2k} = (-1)^{k+1} \cos x$, $k = 1,\ldots,m$. Accordingly, we have $y(0)y(\frac{\pi}{2}) = 0$, i.e., the system is not suborthogonal on $[0,\frac{\pi}{2}]$. On the other hand, it is easily confirmed that $\|A\| = 1$, and thus

$$\int_0^{\frac{\pi}{2}} \|A\| dx = \frac{\pi}{2} \quad .$$

This shows that (3.1) (with the particular choice $\mu = 0$) is the best possible condition of its kind and that suborthogonality does not necessarily obtain if the sign of equality holds in (3.1). We also note for further reference that the exhibited

solution vector is oscillatory on $[0, \pi/2]$.

Turning now to criteria for nonoscillation, we set $B = 0$, $C = I$ in (2.5) and, as before, we identify the arbitrary constant unit vector v with $u(a)$. An application of Theorem 2.1 then leads to the following result.

Theorem 3.2. _If the solution vector y of (2.1) is oscillatory on $[a,b]$, then_

$$(3.3) \qquad \frac{\pi}{2} + \text{arc sin } \frac{|y(a)y(b)|}{\|y(a)\|\|y(b)\|} \leq \int_a^b \|A\| dx \ .$$

As an immediate corollary of this result we find that the condition

$$\int_a^b \|A\| dx < \frac{\pi}{2}$$

is sufficient to guarantee the nonoscillation of the system (2.1) on $[a,b]$. However, this criterion can be given a more general form with the help of an arbitrary diagonal $n \times n$ matrix P , whose diagonal elements p_{kk} are continuously differentiable and do not vanish on $[a,b]$. If $w = Py$ and y is a solution of (2.1), the vector w is a solution of

$$(3.3) \qquad w' = (PAP^{-1} + P'P^{-1})w$$

and, as remarked by B. Schwarz [16], the system (3.3) is nonoscillatory on an interval if and only if the same is true

of the system 2.1; indeed, if y_k and w_k are the components
of y and w, respectively, then $w_k = p_{kk}y_k$, and $p_{kk} \neq 0$.
We thus have the following result.

Theorem 3.3. Let P be a diagonal $n \times n$ matrix whose
diagonal elements are continuously differentiable and do
not vanish on $[a,b]$, and let A be a continuous $n \times n$
matrix on this interval. If

$$(3.4) \qquad \int_a^b \|PAP^{-1} + P'P^{-1}\|dx < \frac{\pi}{2},$$

then the system (2.1) is nonoscillatory on $[a,b]$. The
constant $\pi/2$ in (3.4) is the best possible, and the con-
clusion does not necessarily hold if the sign of equality is
permitted in (3.4).

A weaker form of condition (3.4) (with the constant 1
instead of $\pi/2$) was recently obtained by W. J. Kim [7].
(For nonoscillation criteria of a different type see [11,12,16]).
The sharpness of (3.4) can be verified (for $P = I$) with the
help of the same example which was used to show that Theorem 3.1
is the best possible of its kind.

4. The presence of the n arbitrary functions in the main
diagonal of P lends a great deal of flexibility to condition
(3.4). For a given A, the best choice of P would be that
which minimizes the integral on the left-hand side (and thus
increases the interval to which the condition may be applied).

Since the resulting variational problem will in general
present great technical difficulties, it will often be more
rewarding to choose a matrix P of simple type which depends
on some arbitrary parameters, and then to find the best
criterion obtainable in this way.

We shall illustrate this method in the case of a system
(2.1) which is equivalent to the n-th order scalar differential
equation

$$(4.1) \qquad \sigma^{(n)} + r(x)\sigma = 0 \quad ,$$

where $v(x)$ is continuous on the interval $[a,b]$. If A
is the n × n matrix whose only non-zero elements a_{ik} are
$a_{k,k+1} = 1$, $k = 1,\ldots,n-1$ and $a_{nl} = -r$, the solution
vector y of (2.1) has the components $\sigma,\sigma',\ldots,\sigma^{(n-1)}$,
where σ is the solution of (4.1). The nonoscillation of
the system is thus equivalent to the condition that, for
any solution σ of (4.1), at least one of the functions
$\sigma,\sigma',\ldots,\sigma^{(n-1)}$ does not vanish in the interval in question.
An equation with this property is said to be <u>disfocal</u> on the
interval [13]. It may be noted that, by Rolle's theorem
a real disfocal equation is <u>a fortiori</u> disconjugate, i.e.,
none of its solutions can have more than n - 1 zeros on the
interval.

<u>Theorem 4.1.</u> <u>Let</u> R <u>be a</u> <u>closed</u> x-<u>interval and</u> <u>let</u> S
<u>be a</u> <u>measurable</u> <u>subset of</u> R <u>of</u> <u>Lebesgue</u> <u>measure</u> $\mu(S)$.
<u>If</u> $r(x)$ <u>is</u> <u>continuous on</u> R ,

if r(x) ≠ 0 except on a null set, and if

(4.2) $\sup_{S}[\mu(S)]^{n-1}\int_{R-S}|r|dx < (n-1)^{n-1}(\frac{\pi}{2n})^n$,

then equation (4.1) is disfocal on R . The constant in
(4.2) is the best possible.

If r is of constant sign and |r| is monotonic on R ,
the expression (4.2) can be simplified. For instance, if
|r| is nondecreasing and R is the interval [a,b] , (4.2)
may evidently be replaced by the condition

(4.3) $\sup_{c}(c-a)^{n-1}\int_{c}^{b}|r|dx < (n-1)^{n-1}(\frac{\pi}{2n})^n$, c∈[a,b] .

To prove Theorem 4.1, we use a constant diagonal matrix P .
If we set $p_{kk} = \beta^{n-k}$, k = 1,...,n , where β is a positive
constant, the matrix $PAP^{-1} = (b_{ik})$ has the elements
$b_{k,k+1} = \beta$, k = 1,...,n - 1 , $b_{n,1} = -r\beta^{-n+1}$, and all
the other elements b_{ik} are zero. It is easy to see that

$$\|PAP^{-1} + P'P^{-1}\| = \|PAP^{-1}\| = \max[\beta, |r|\beta^{-n+1}] ,$$

and we may therefore conclude from Theorem 3.3 that the
system associated with the coefficient matrix A is non-
oscillatory on R (and, therefore, the equation (4.1) is
disfocal on R) if

$$\int_{R}\{\max[\beta, |r|\beta^{-n+1}]\}dx < \frac{\pi}{2} .$$

If S denotes the subset of R on which $|r| \leq \beta^n$, this may be written in the form

$$(4.4) \qquad \beta\mu(S) + \beta^{-n+1} \int_{R-S} |r|\,dx < \frac{\pi}{2} \quad .$$

It is possible to choose β in such a way that

$$(4.5) \qquad \frac{\beta^n}{n-1} = \frac{\int_{R-S} |r|\,dx}{\mu(S)} \quad .$$

Indeed, the set S depends on β , and it is easy to see that the right-hand side of (4.5) varies continuously from 0 to ∞ if β^n decreases from $\max|r|$ to 0 . Hence, there must exist a positive β for which (4.5) holds. For this particular value of β , the left-hand side of (4.4) takes the form

$$(4.6) \qquad \frac{n}{n-1} \left\{ (n-1)[\mu(S)]^{n-1} \int_{R-S} |r|\,dx \right\}^{\frac{1}{n}} \quad ,$$

and condition (4.4) will thus certainly be satisfied if (4.2) holds. This completes the proof of Theorem 4.1.

To show that the constant in (4.2) is the best possible, we consider the equation

$$\sigma^{(2m)} - (-1)^m \sigma = 0$$

on the interval $[0, \pi/2]$. The solution $\sin x$ of this equation, as well as all its derivatives, vanish at either 0 or $\pi/2$, and the equation is thus not disfocal on $[0, \pi/2]$. On the other hand,

$$\max_c c^{n-1} \int_c^{\frac{\pi}{2}} |r| dx = \max_c c^{n-1}(\frac{\pi}{2} - c) = (n - 1)^{n-1}(\frac{\pi}{2n})^n , \quad (n=2m)$$

and this shows that the constant in condition (4.2) (which in this case is equivalent to condition (4.3)) cannot be improved upon.

5. In this section we explore some of the consequences of Theorem 2.2. In our first application, we use a matrix B whose only non-zero elements b_{ik} appear in the n-th column, and we set $b_{1,n} = b_{2,n} = \ldots = b_{m,n} = \epsilon$ $(1 \le m \le n - 1)$, where ϵ is a small positive number, and $b_{m+1,n} = \ldots = b_{n,n} = 0$. We have $\|B\| = \epsilon \sqrt{m}$, and it is easily confirmed that the system (2.2) associated with this matrix B has the solution vector $w = (\epsilon(x - x_1), \epsilon(x - x_2), \ldots, \epsilon(x - x_m), c_{m+1}, \ldots c_{n-1}, 1)$, where $x_1, \ldots, x_m$, $c_{m+1}, \ldots, c_{n-1}$ are arbitrary constants. If $x_r \epsilon [a,b]$, $r = 1, \ldots, m$, the first m components of w have zeros in $[a,b]$, and Theorem 2.2 can be applied if $y = (y_1, \ldots, y_n)$ is a solution vector of (2.1) whose component, $y_{m+1}, \ldots, y_n$ vanish at points of $[a,b]$. For $\epsilon \to 0$, we have $\|B\| \to 0$ and $v = w/\|w\|$ tends to a constant unit vector $v = (v_1, \ldots, v_u)$ with $v_1 = v_2 = \ldots = v_m = 0$, $v_k = \alpha_k$, $k = n + 1, \ldots, n$. This leads to the following result.

Theorem 5.1. Let $y = (y_1, \ldots, y_n)$ be a solution vector of the system (2.1). If each of the components $y_{m+1}, \ldots, y_n (1 \leq m \leq n-1)$ vanishes at some point of the interval $[a,b]$, then

$$(5.1) \quad \left| \arc \sin \sum_{k=m+1}^{n} u_k(b)\alpha_k \right| + \left| \arc \sin \sum_{k=m+1}^{n} u_k(a)\alpha_k \right| \leq \int_a^b \|A\| dx .$$

where $u = (u_1, \ldots, u_n) = y/\|y\|$, and the α_k are such that $\alpha_{m+1}^2 + \ldots + \alpha_n^2 = 1$ and are otherwise arbitrary.

Our next application refers to the case in which (2.2) is the system adjoint to A, i.e., $B = -A^*$. Since $\|B\| = \|B^*\|$, Theorem 2.2 yields the following result.

Theorem 5.2. Let y and w be, respectively, solution vectors of the system (2.1) and its adjoint (1.5), and set $u = y/\|y\|$, $v = w/\|w\|$. If, for each k $(1 \leq k \leq n)$, the k-th component of either y or w has a zero on $[a,b]$, then

$$(5.2) \quad \left| \arc \sin[u(b)v(b)] \right| + \left| \arc \sin[u(a)v(a)] \right| \leq \int_a^b \|A\| dx .$$

This result leads to interesting consequences in the case in which the system 2.1 is equivalent to an n-th order differential equation

$$(5.3) \qquad \sigma^{(n)} + r_{n-1}\sigma^{(n-1)} + \ldots + r_0\sigma = 0 ,$$

i.e., if the non-zero elements a_{ik} of A are given

by $A_{k,k+1} = 1$, $k = 1,\ldots,n - 1$, $a_{nk} = - r_{k-1}$, $k = 1,\ldots,n$.
If σ is a solution of (5.3), then the vector $y = (\sigma,\sigma',\ldots,\sigma^{(n-1)})$
is a solution of the system (2.1), and if τ is a solution of

the equation

$$(5.4) \qquad \tau^{(n)} - (r_{n-1}\tau)^{(n-1)} + - \ldots + (-1)^n r_o \tau = 0$$

adjoint to (5.3), the vector $w = (\tau^{(n-1)},\tau^{(n-2)},\ldots,\tau',\tau)$
solves the adjoint system (1.5) (see e.g., [2]). Let now σ be
a solution of (5.3) which has k zeros ($0 \leq k \leq n$) in $[a,b]$,
and denote by τ a solution of (5.4) with $n - k$ zeros in
this interval. By Rolle's theorem, the functions $\sigma,\sigma',\ldots,\sigma^{(k-1)}$,
$\tau,\tau',\ldots,\tau^{(n-k-1)}$ all have zeros in $[a,b]$, and it follows
that, for each m ($1 \leq m \leq n$) , the n-th component of either y
or w has a zero in $[a,b]$. In view of Theorem (5.2), this
establishes the following statement.

Theorem 5.3. Let σ and τ be, respectively, nontrivial
solutions of equation (5.3) and its adjoint (5.4), and set

$$(5.5) \qquad T(x) = \sum_{k=1}^{n} \frac{\sigma^{(k-1)}\tau^{(n-k)}}{\sqrt{\sum_{k=1}^{n} [\sigma^{(k-1)}]^2 \sum_{k=1}^{n} [\tau^{(k-1)}]^2}} \qquad ,$$

$$(\sigma = \sigma(x) \ , \ \tau = \tau(x)) \ .$$

If A is the companion matrix of equation (5.3) and

$$\int_a^b \|A\|dx < \text{arc } \sin|T(b)| + \text{arc } \sin|T(a)| \; ,$$

then the combined number of zeros of σ and τ in $[a,b]$ cannot exceed $n - 1$.

REFERENCES

[1] Barrett, J. H., 'Fourth order boundary value problems and comparison theorems', Canadian J. Math. 13(1961), 625-638.

[2] Coddington, E. A. and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

[3] Hartman, P., Ordinary Differential Equations, John Wiley and Sons, New York, 1964.

[4] Hartman, P. and A. Wintner, 'On disconjugate differential systems', Canadian J. Math 8(1956), 72-81.

[5] Hunt, R. W., 'The behavior of solutions of ordinary, self-adjoint differential equations of arbitrary even order', Pacific J. Math. 12(1962), 945-961.

[6] Hunt, R. W., 'Oscillation properties of even-order linear differential equations', Trans. Amer. Math. Soc. 115(1965), 54-61.

[7] Kim, W. J., 'Disconjugacy and disfocality of differential systems', to appear.

[8] Leighton, W. and Z. Nehari, 'On the oscillation of solutions of self-adjoint linear differential equations of the fourth order', Trans. Amer. Math. Soc. 89(1958), 325-377.

[9] Levin, A. Ju., 'Some questions on the oscillation of solutions of linear differential equations', Doklady Akad. Nauk 148(1963), 512-515, (Translation: Soviet Math. 4(1963), 121-124).

[10] Levin, A. Ju., 'Distribution of the zeros of solutions of a linear differential equation', Dokl. Akad. Nauk. 156(1964), 1281-1284. (Translation: Soviet Math. 5(1964), 818-821).

[11] London, D. and B. Schwarz, 'Disconjugacy of complex differential systems and equations', Trans. Amer. Math. Soc.

[12] Nehari, Z., 'On an inequality of Lyapunov', Studies in Mathematical Analysis and Related Topics, pp. 256-261, Stanford University Press, Stanford, California, 1962.

[13] Nehari, Z., 'Disconjugate linear differential operators', Trans. Amer. Math. Soc. 129(1967), 500-516.

[14] Reid, W. T., 'Principal solutions of non-oscillatory self-adjoint linear differential equations', Pacific J. Math. 8(1958), 147-169.

[15]   Reid, W. T., 'Oscillation criteria for self-adjoint differential systems', Trans. Amer. Math. Soc. 101(1961), 91-106.

[16]   Schwarz, B., 'Disconjugacy of complex differential systems', Trans. Amer. Math. Soc. 125(1966), 482-496.

[17]   Sherman, T. L., 'Properties of solutions of n-th order linear differential equations', Pacific J. Math. 15(1965), 1045-1060.

[18]   Sherman, T. L., 'Properties of solutions of quasi-differential equations', Duke J. Math. 32(1965), 297-304.

[19]   Sternberg, R. L., 'Variational methods and non-oscillation theorems for systems of differential equations', Duke Math. J. 19(1952), 311-322.

[20]   Sternberg, H. M. and R. L. Sternberg, 'A two-point boundary problem for ordinary self-adjoint differential equations of fourth order', Canadian J. Math. 6(1954), 416-419.

[21]   Wintner, A., 'On the non-existence of conjugate points', Amer. J. Math. 73(1951), 368-380.

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