ON A CLASS OF NON-LINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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1. Introduction. In [5], a theorem is proved which asserts the existence of a non-trivial solution to the problem

(1) $Ay + yF(y^2,x) = 0$ in O, $y|_{ao} = 0,$

where A is the Laplace operator, *Q* is a bounded region in R^N for which the Dirichlet problem is solvable, and F is a function locally Hölder continuous on R,x£} satisfying, for some € > 0 and all $x \in \Omega$,

(2) $0 < nl^{£}F(ri_{ls}x) \leq r?2^{€}F(77_2,x)$, for $0 < rj \wedge r?_2 < oo$,

and also, for all *xefl,*

(3) $F(?7,x) \leq cr)^7 + or$, $0 < 7] < oo$,

where c,a and *y* are positive constants, (N-2)*y* < 2. This result is the analogue of a result of [8] concerning a boundary value problem for a non-linear ordinary differential equation. The result of [5] concerning (1) was obtained by treating the integral equation equivalent to (1) by methods similar to those used in [9].

In this note we shall derive from the results of [5] an existence theorem for a boundary value problem of the form

(4) $TU = uF(u^2,x)$ in fi, $D\% | \wedge = 0$, $|a| \leq m-1$,

where r is an elliptic operator of order $2m$, $D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$, $|\alpha| = \alpha_1 + \dots + \alpha_N$; $m \geq 1$, $N \geq 1$. The result obtained here was suggested by the main theorem of Berger's study, [4], of a non-linear elliptic eigenvalue problem.

2. The differential operator, We shall assume throughout this section that *Q,* is a region of class C^{2m} (see def. 9.2, [3]) and that the differential operator r, given in the form

(5) $$\tau = \sum_{|\alpha|,|\beta|\le m} (-1)^{|\alpha|} D^{\alpha} a_{\alpha\beta}(x) D^{\beta},$$

has real coefficients satisfying

(6) $$a_{\alpha\beta}(x) \in C^{\max(|\alpha|,|\beta|)}(\overline{\Omega}), \qquad \text{all } \alpha,\beta.$$

In addition we assume that T is uniformly strongly elliptic in *Cl* and that there exists a positive constant c_o such that

(7) B[<p,<p]$\geq$ c_0||cp||^2 , all *cpeC*$^{™}_{o}$ (Q),

where

$$Bl<P\wedge] = \sum_{|o|,T|8|\le m} \int_{\wedge} a_{\alpha\beta}(x) \; D^{\alpha}\varphi(x) \; D^{\beta}\psi(x)\,dx.$$

We shall use standard notation. For r > 1, ||. . . $||_{m,r}$ is the Sobolev norm defined as follows,

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for <p having strong L^r-derivatives of order up to m in Q; $W^{m,r}$ *(Cl)* is the space of all such functions (because of the smoothness assumption concerning *Cl* this is equivalent to the more usual definition of W^3nclj>r(a)); W^m^r(O) is normed by ||. . . $||_{m,r}$. Finally, $I\!\!N^{m,r}_{o}$*(Cl)* is the closure of C^{∞}_{o} *(Cl)* in $W^{m,r}(\Omega)$. Without exception the function space considered here will be understood to consist of real valued functions.

By a standard result, see Theorem 8.2, [3], the generalized Dirichlet problem with zero boundary data

$$B[\varphi,u] = (\varphi,f), \qquad \text{all } \varphi\in C_0^\infty(\Omega),$$

has a unique solution $u\in W_0^{m,2}(\Omega)$ for each $f\in L^2(\Omega)$. Actually the same is true for $f\in L^q(\Omega)$ provided

(8) $$q > q_0 = \max(1,2N/(N+2m)), \qquad q < 2 .$$

This follows from the fact that, because of the Sobolev imbedding theorem, $W_0^{m,2}(\Omega)$ is stronger than $L^p(\Omega)$ when $\frac{1}{p}+\frac{1}{q} = 1$ and (8) holds. Thus for $f\in L^q(\Omega)$, $\varphi\in W_0^{m,2}(\Omega)$,

(9) $$|(\varphi,f)| \le \|\varphi\|_p \|f\|_q \le \text{const}.\ \|\varphi\|_{m,2} \|f\|_q .$$

The same proof as in the case where $f\in L^2$ then works for $f\in L^q$.

We define an operator A, whose domain is $\bigcup_{q>q_0} L^q(\Omega)$ and whose range is contained in $W_0^{m,2}(\Omega)$, by

(10) $$B[\varphi,Af] = (\varphi,f), \qquad \text{all } \varphi\in W_0^{m,2}(\Omega),$$

upon taking $\varphi = Af$ in (10) it follows from (7) and (9) that A acts as a bounded operator from $L^q(\Omega)$ to $W_0^{m,2}(\Omega)$ for each $q > q_0$.

We shall require the following results from [1] and [2].

(*) <u>If</u> $f\in L^r(\Omega)$, $r > q_0$ <u>then</u> $Af\in W^{2m,r}(\Omega)$, <u>and there exists a constant</u> k_r <u>such that</u>

(11) $$\|Af\|_{2m,r} \le k_r(\|f\|_r + \|Af\|_r) .$$

(**) II. $f \in L^{\infty}(\Omega)$ and $u = Af$, then (after modification on a set of measure zero) $u \in C^{2m-1}(\bar{\Omega})$ and u satisfies the boundary conditions

(12) $$D^{\alpha} u = 0, \quad \text{on } \partial\Omega, \ |\alpha| < m .$$

(***) if $f \in C^{0,\mu}(\bar{\Omega})$ and if

(13) ^Wed^W'W'tm , all a,fi ,

for some μ: $0 < \mu < 1$, then $u = Af \in C^{2m}(\bar{\Omega})$ and u satisfies (12).

$C^{k,\mu}(\bar{\Omega})$ is the space consisting of those functions in $C^{k}(\bar{\Omega})$ whose k^{th} order derivatives are uniformly Holder continuous of order μ in Ω.

The first assertion, (*), follows from Theorem 8.2, [2],[1] the proof of (**) is also in [2]; (***) follows from Theorem A5.1, Appendix 5, [1]. Although we do not use the fact here it is interesting to note that the assertion (**) is equivalent to the assertion that for $r > q_0$, a function u belongs to $W_0^{m,2}(\Omega) \cap W^{2m,r}(\Omega)$ if and only if it is the limit in $W^{2m,r}(\Omega)$ of a sequence (u_n), where, for each n, $u_n \in C^{2m-1}(\bar{\Omega}) \cap W^{2m,r}(\Omega)$ and $u = u_n$ satisfies (12).

If $r > 0$ then Tu can be defined for $u \in W^{2m,r}(\Omega)$, if $r > q_0$ then $W^{2m,r}$ can be imbedded in $W^{m,2}$, we shall denote by A_r, for $r > q_0$ the operator in $L^2(\Omega)$ whose domain is $M^r = W^{2m,r}(\Omega) \cap W_0^{m,2}(\Omega)$ and which sends $u \to Tu$. Since

$$B[\varphi, \psi] = (\varphi, T\psi) = (\varphi, A_r \psi), \qquad \text{for } \varphi \in C_0^{\infty}(\Omega), \ \psi \in M^r ,$$

[1] Actually this is not so unless (6) is strengthened. An adaptation of the arguments in [2] to the case where the operator is given in divergence form gives a result implying (*) under condition (6).

it follows that

(14) $A\&_r u = u,$ for $u \in M$.

On the other hand, by (*), A maps $L^r(Cl)$ into M^r so

$$B[<p,Af] = ((p,*>_r Af), \quad \text{for } <pe C_O^{™} \{d), \; f \in L^r(Q),$$

thus

(15) $*>_r Af = f$, for $f \in L^r(O)$.

We put $A_r = A|L^r(O)$.

Lemma 1. For each $r > q_O$,

$$A_r : L^r(O) - M^r$$

is a bijection. Moreover there are positive constants c_r, c_r' such that for $f \in L^r(O)$

(16) OrK'112-.r i «f»r £ «i»V»2.,r-

Proof, We have shown that A_r and $\&_r$ are inverses of one another. It readily follows from (6) that $\|\&_r u\| \leq$ const, $\|u\|_{«m^r}$ which implies the second inequality of (16). M is a subspace of $W^{2m,r}(a)$ so the first inequality of (16) follows from the open mapping theorem.

Lemma 2. Let $r > q$. If $2mr < N$ then A maps $L^r(0)$ compactly into $L^s(Q)$ for

(17) $1 < s < Nr/(N-2mr)$;

if $2mr > N$ then A can be regarded as a compact mapping of $L^r(0)$ into $C(JT)$.

Proof. Let $q_o < r$, then by Lemma 1 A maps $L^r(Cl)$ into $W^{2m'r}(0)$. If $2mr \leq N$ then, by SoboleVs theorem, $W^{2m'r}(Q)$ can be imbedded compactly in $L^s(Cl)$ for any s satisfying (17). If $2mr > N$ then $W^{2m'r}(f2)$ can be imbedded compactly in $C(\bar{\Omega})$.

The following lemma will simplify matters by making the results of [5] applicable,, as they stand, to the problem considered here.

Lemma 3. Let $a = N/(N-2m)$ or let $a\hat{} = OD$ according as $N > 2m$ O£ $N \leq 2m$. There exists a measurable function $G(x,t)$ on 0×0 such that the mapping

$$x - G(x, \bullet)$$

is uniformly continuous from O to $L^a(0)$ for $1 < a < a_o$, and

(18) $$\operatorname*{ess\,sup}_{xefl} \int |G(x,t)|^a dt < oo, \quad \operatorname*{ess\,sup}_{teO} \int |G(x\hat{}t)|^a dx < \infty,$$

for $1 < a < a_Q$. For $f \in L^r(O)$, $r > q_Q$,

(19) $$[Af](x) = \int_\Omega (G(K,t)f(t)dt, \qquad \text{a.e. in } 0.$$

Proof. Let $2mr > N$, so that A can be regarded as a map of $L^r(Q$ into $C(C\bar{2})$. By a well known representation theorem, (Theorem VI.7.1, [6]) there is a continuous map $x \to G(x, \bullet)$ of $\bar{SI}$ into $L^a(Q$, where $\frac{1}{r} + \frac{1}{a} = 1$, such that (19) holds everywhere in Q for $f€L^r(fi)$. It is easily seen that the function $G(x,t)$ is independent of the particular choice of r. Let

(20) $$r^* = \sum_{|,\,|\beta| \le m} (-1)^{|\beta|} D^{\beta} a_{\alpha\beta}(x) D^{\alpha},$$

(notice the symmetry of (6)), and let A^* be defined by

(21) $$B[A^*f,0] = (f,\psi), \qquad \psi \in W£^{,2}(\Omega), \; f \in L^q(\Omega), \; q > q_Q \,.$$

It follows that for $f^geL^q(fi)$, $q > q_o$, we have

(22) $$(A^*g,f) = fer,Af).$$

Because of the symmetry of (6), A^* has an integral representation analogous to (19) for $f€L^r(ft)$, $2mr > N$; let $G^*(x,t)$ denote the corresponding kernel. It follows then from (22) that

$$\int_\Omega\int_\Omega G^*(t,x)\,f(x)\,g(t)\,dxdt = \int_\Omega\int_\Omega G(x,t)\,f(x)\,g(t)\,dxdt,$$

for $f,geL^r(O)$, $2mr > N$. Thus we have

$$G^*(t,x) = G(x,t), \quad \text{a.e. in } QXO,$$

and from the uniform continuity of $x \to G(x,*)$^ and $t \to G^*(t,*)$ as mappings from Q, to $L^a(Q) >$ $1 < a < a_o$, follow the inequalities (18).

It remains to show that (19) is valid for $feL^-(£2)$ when $r > q_o$, and $2mr \le N$. In this case however it follows from (18) and Theorem 9.5.6, [7], that the right hand side of (19) defines a compact mapping from $L^r(£2)'$ to $L^s(Q>)$ for

$$1 < s < Nr/(N-2mr)\,.$$

Thus since (19) is valid for f in a dense subset of $L^r(\Omega)$, (namely for $f \in L^{r_1}(\Omega)$, $2mr_1 > N$), it follows from Lemma 2 that it is valid for $f \in L^r(\Omega)$.

3. *The non-linear problem*. Let Ω and Γ be as in Section 2. The main result of this note is the following.

Theorem. *Suppose that* (13) *holds, for some positive* μ *less than* 1 and that F is uniformly Hölder continuous on $\bar{R} \times \bar{\Omega}$. *Suppose also that* F *satisfies* (2) *for some* $\varepsilon > 0$ *and that* (3) *holds, for all* $x \in \Omega$, *with positive constants* c, σ *and* γ *where*

$$(23) \qquad 0 \le \gamma, \qquad \gamma(N-2m) < 2m.$$

Finally assume that Γ *is formally self-adjoint. Then there exists a function* $u \in C^{2m}(\Omega) \cap C^{m-1}(\bar{\Omega})$ *which is not identically zero and satisfies* (4) (in the ordinary sense).

Proof. We consider the operator equation

$$(24) \qquad u = AuF(u^2, x),$$

where A has the same meaning as in Section 2. By Lemma 3 this is equivalent to an integral equation

$$(25) \qquad u(x) = \int_{\Omega} G(x,t)u(t)F(u^2(t),t)\,dt,$$

where, since Γ is formally self-adjoint, $G(x,t)$ is symmetric; (18) holds for $1 < \alpha$, $\alpha(N-2m) < N$. It readily follows from (7) that A, regarded as an operator in $L^2(\Omega)$, is positive definite; the range of A contains $\overset{\circ}{C}{}^{\infty}(\Omega)$ and is therefore dense in $L^p(\Omega)$ for any $p > 1$. Now from Theorems 1 and 3 of [5] it follows that (25) has a non-trivial essentially bounded solution u; see also the remarks following the statement of Theorem 2 of [5]. From the equivalence of (24) and (25) it follows that u

satisfies (24). By (3), $u(x)F(u^2(x),x)$ is essentially bounded and thus, by (**), $ueC^{2*1''''1}(\bar{U})$ and u satisfies (12). Prom the differentiability of u and the hypothesis concerning F it follows that $u(x)F(u^2(x) \hat{} x)$ is uniformly Hölder continuous in Q. Finally by (***) we conclude that $ueC^{2}(O)$ and is an ordinary solution of (4). This completes the proof.

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