ON UNITARY PROPERTIES OF GRUNSKY'S MATRIX[1]

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1. Introduction. The Bieberbach conjecture asserts that the n-th coefficient of the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n ,$$

analytic and univalent in the unit disk, satisfies

$$|a_n| \leq n$$

with equality holding only for the Koebe function

$$K(z) = z/(1 - z)^2$$

or one of its rotations. This was proved for the second and third coefficients by Bieberbach [1] and Loewner [2] respectively. The proof for the fourth coefficient was first obtained by Garabedian and Schiffer [3] and was later simplified by Charzynski and Schiffer [4]. Their proof was based on the Grunsky inequality which gives necessary and sufficient conditions that an analytic function be univalent in the unit disk. Recently, a streamlined proof for $n = 4$ was given by Garabedian, Ross and Schiffer [5] in which even more striking use was made of Grunsky's inequality. In the latter paper the authors proved the local theorem in the sense that there exists an $\epsilon_n > 0$

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such that if

$$(1.1) \qquad \sum_{k=2}^{2n} |a_k - k|^2 < \epsilon_n$$

then Re $a_2 \leq 2n$. Garabedian and Schiffer [6] complemented this result by showing for odd n, Re $a_{2n+1} \leq 2n + 1$ if only

$$(1.2) \qquad 2 - \text{Re } a_2 < \epsilon_n$$

for some small $\epsilon_n > 0$. They also indicated that their methods imply a similar result for the even coefficients. The author [7] showed that the equivalence of the conditions (1.1) and (1.2) is a simple consequence of Loewner's formulas.

Grunsky's inequality is based on the fact that an analytic function $f(z)$ is univalent in the unit disk if and only if the series

$$(1.3) \qquad \log \frac{f(z) - f(\xi)}{z - \xi} = \sum_{m,n=0}^{\infty} d_{mn} z^m \xi^n$$

converges for $|z| < 1$, $|\xi| < 1$. Grunsky [8] showed that this is the case if and only if the symmetric infinite matrix

$$C = (C_{m,n}) \; , \quad C_{mn} = \sqrt{mn} \; d_{mn} \; ,$$

satisfies

$$(1.4) \qquad \left| \sum_{m,n=1}^{\infty} C_{mn} x_m x_n \right| \leq \sum_{n=1}^{\infty} |x_n|^2$$

for every complex vector $x = (x_1, x_2, \ldots, x_n, \ldots)$.

Grunsky's work prompted Schur [9] to show that every quadratic form satisfying (1.4) has a matrix of the form

$$C = U'EU \tag{1.5}$$

where U is unitary, and U' its transpose and E is a real diagonal matrix with elements $0 \leq e_i \leq 1$. This study suggests a close relationship between unitary matrices and Grunsky matrices and raises the question of characterizing those univalent functions which correspond to unitary matrices.

2. Unitary Grunsky Matrices. The Faber polynomials P_n associated with f are uniquely defined by the relations

$$P_n\left(\frac{1}{f(z)}\right) = \frac{1}{z^n} - \sum_{m=1}^{\infty} b_{nm} z^m ,$$

the coefficients b_{nm} being related to the coefficients C_{mn} by

$$(2.1) \qquad b_{nm} = \sqrt{\frac{m}{n}}\, C_{nm} ,$$

see, for example, Schiffer [10].

The following definition of slit mapping, which appears to include those in common use, is the most convenient one for our purpose.

Definition. Let $f(z)$ be analytic and univalent. We say that f defines a slit mapping if the complement of the range of f has measure zero (with respect to the ordinary Lebesgue measure in the plane).

The classical area theorem is proved by using Green's identity together with the fact that the area of a region bounded by a positively oriented Jordan curve is positive, see, for example, Nehari [11]. Golusin [12] obtained a generalization of the area theorem by using the fact that the integral of a non-negative function over such a region is non-negative.

Theorem 2.1. Suppose that $f(z)$ is normalized and analytic in the unit disk. A necessary and sufficient condition that f be univalent is that

$$(2.2) \qquad \sum_{m=1}^{\infty} \left| \sum_{m=1}^{\infty} x_n C_{nm} \right|^2 \leq \sum_{n=1}^{\infty} |x_n|^2$$

for every square summable complex vector $x = (x_1, x_2, \ldots, x_n, \ldots)$. Equality holds for all sequences if and only if f defines a slit mapping.

Proof: As was observed by Jenkin [13], it is a consequence of the Schwartz inequality that (1.4) is implied by the apparently stronger inequality (2.2). However, as was pointed out by Schiffer [14], it is only apparently stronger, for it is implied by (1.4) together with Schur's observations. Since it is illuminating to distinguish those properties which belong to matrix theory from those which require attributes of univalent functions we give a proof of this fact. Indeed, the left hand side of (2.2) is equal to $\|Cx\|^2$. It then follows from (1.5) that

$$\|Cx\|^2 = (U'EUx\ ,\ U'EUx) = (EU'^*U'EUx\ ,\ Ux)$$

$$= (E^2Ux, Ux) \leq \|E^2\| \|Ux\|^2 \leq \|x\|^2$$

since $\|E^2\| \leq 1$ and U is unitary.

It remains to prove the assertion regarding equality for slit mappings. This follows by noting that in the proof of Jenkins [14] there is equality in this case. Actually an examination of the proof of Grunsky's inequality shows that it yields both results. In the final expression there is one non-negative term which is dropped; this expression also yields Theorem 2.1.

Our application to the coefficient problem is based on

Theorem 2.2 If $f(z)$ is analytic in the unit disk, then f defines a univalent slit mapping if and only if the matrix C is unitary.

Proof: To prove the necessity, suppose that $f(z)$ is univalent. Let $x_n = 1$ and $x_m = 0$ if $m \neq n$. Since (2.2) is an equality for finite sequences, it follows that

$$\sum_{m=1}^{\infty} |C_{nm}|^2 = 1 , \tag{2.3}$$

that is, the rows of C have norm 1. Now let $x_n = 1$, and $x_m = 0$ if $m \neq n$ or k , $k \neq n$. Then

$$\sum_{m=1}^{\infty} |C_{nm} + x_k C_{km}|^2 = 1 + |x_k|^2 .$$

It follows by expanding the left side of the above equality and using (2.3) that

$$\operatorname{Re} \sum_{m=L}^{\infty} C_{nm} \overline{C}_{km} \overline{x}_k = 0 ,$$

or, since x_k is an arbitrary complex number

$$\sum_{m=1}^{\infty} C_{nm} \overline{C}_{km} = 0 ,$$

that is, the rows of C are pairwise orthogonal. Hence C is unitary.

The proof of the sufficiency follows from the fact that if C is unitary then (2.2) is an equality.

In connection with the method of Garabedian, Ross and Schiffer, it is interesting to ask for which univalent functions a truncated Grunsky matrix is unitary. The following theorem shows that this can happen only if the matrix is diagonal. Thus, if one could prove that within the class of matrices arising from univalent functions the extremal function corresponds to a truncated unitary matrix, one would have an easy proof of the Bieberbach conjecture.

Theorem 2.3. Let f be an analytic function in $|z| < 1$ with Grunsky matrix $C = (C_{jk})$. If there exists a finite set of integers $1 = \alpha_1 < \alpha_2 < \ldots < \alpha_n$ such that the matrix $(C_{\alpha_j \alpha_k})$ is unitary, then C is diagonal. Moreover,

$$f(z) = \frac{z}{e^{i\theta} z^2 + az + 1}$$

where θ is a real constant and a is constant.

Proof: Let $\hat{C} = (C_{\alpha_j \alpha_k})$ $j,k = 1,2,\ldots,n$. Since $\hat{C}$ and C are both unitary, it follows that $C_{\alpha_j \beta} = 0$ unless $\beta = \alpha_k$ for some k . In particular, since the first Faber polynomial is given by

$$P_1(w) = w - a \quad , \quad a = \text{const.} \, ,$$

we have

$$\frac{1}{f(z)} = a + \frac{1}{z} + \sum_{j=1}^{m} b_{1\alpha_j} z^{\alpha_j} \, , \quad b_{1\alpha_m} \neq 0 \, , \quad m \leq n \, .$$

The α_n-th Faber polynomial has the form

$$P_{\alpha_n}(w) = w^{\alpha_n} + \gamma_1 w^{\alpha_{n-1}} + \ldots + \gamma_{\alpha_{n-1}} w + \gamma_{\alpha_n} \quad ;$$

hence

$$b_{\alpha_n, \alpha_m \alpha_n} = b_{1\alpha_m}^{\alpha_n} \, .$$

But since $b_{\alpha_n, k} = 0$ if $k > \alpha_n$, it follows that $\alpha_m = 1$. The unitary property of C then shows that $b_{11} = e^{i\theta}$; hence

$$\frac{1}{f(z)} = a + \frac{1}{z} + e^{i\theta} z \, .$$

It is easy to see from (1.3) that if $1/f(z)$ differs from $1/g(z)$ by a constant then f and g have the same Grunsky matrix. In particular if $a = 0$ one sees that C is diagonal.

This completes the proof.

If C is a matrix let C_m denote the m-th row vector and $\delta C = C - I$ where I is the identity. The following theorem puts Theorem 2.3 into more useable form.

Theorem 2.4. If $C = (C_{mn})$ is a symmetric unitary matrix and $C_{mn} = r_{mn} + i\, s_{mn}$ where r_{mn} and s_{mn} are real, then

$$r_{mn} - \Delta_{mn} = -\frac{1}{2}(\delta C_m, \delta C_n) \ .$$

Here Δ_{mn} denotes the Kronecker delta.

Proof: Since C and I are unitary, we have

$$\begin{aligned}\Delta_{mn} &= (C_m, C_n) = (\delta C_m + I_m, \delta C_n + I_n) \\ &= (\delta C_m, \delta C_n) + (\delta C_m, I_n) + (I_m, \delta C_n) + (I_m, I_n) \\ &= (\delta C_m, \delta C_n) + r_{mn} + r_{nm} - \Delta_{mn} - i(s_{nm} - s_{mn}) \ .\end{aligned}$$

The result now follows immediately from the symmetry of C .

3. A Generalization of the Fourth Coefficient Problem. In this section we illustrate the previous results by proving the

Theorem 3.1. If $f(z)$ is normalized, analytic and univalent in the unit disk, then

$$|a_4 + \alpha a_2(a_3 - \tfrac{3}{4}a_2^2)| \leq 4$$

for all real α satisfying $|\alpha + 2| \leq \sqrt{75/17}$. Equality holds only for the Koebe function.

Proof: The polynomial

$$P_\alpha(a) = a_4 + \alpha a_2(a_3 - \frac{3}{4} a_2^2)$$

is homogeneous of degree three in the sense that replacing a_k by $e^{i(k-1)\theta} a_k$ brings out a factor of $e^{3i\theta}$ in front of $P_\alpha(a)$. We therefore may assume that if f is the extremal function then

$$(3.1) \qquad P_\alpha(a) \geq 0 \quad , \quad 0 \leq \operatorname{Re} a_2 \leq 2 .$$

It is easily shown that f satisfies a Schiffer differential equation and that hence f defines a slit mapping as does $\sqrt{f(z^2)}$. By direct computation one shows that

$$C_{11} = a_2/2 ,$$

$$C_{13} = \frac{\sqrt{3}}{2}(a_3 - \frac{3}{4} a_2^2)$$

and

$$a_4 = \frac{2}{3} C_{33} + \frac{10}{3} C_{11}^3 + \frac{8}{\sqrt{3}} C_{11} C_{13}$$

where (C_{jk}) is the Grunsky matrix of $\sqrt{f(z^2)}$. Setting $C_{jk} = r_{jk} + i\, s_{jk}$, $t = r_{11}$, $\delta P_\alpha = P_\alpha(a_2, a_3, a_4) - P_\alpha(2,3,4)$ and using (3.1), we have

$$\delta P_\alpha = \tfrac{2}{3}(r_{33} - 1) + \tfrac{10}{3}(t^3 - 1) - 10ts_{11}^2 + \frac{4\lambda t}{\sqrt{3}} r_{13} - \frac{4\lambda}{\sqrt{3}} s_{11}\, s_{13}$$

where $\lambda = \alpha + 2$ and $0 \le t \le 1$. It is a consequence of Theorem 2.4 that

$$(3.2)\quad \tfrac{2}{3}(r_{33} - 1) + \frac{4\lambda t}{\sqrt{3}} r_{13} = -\tfrac{1}{3}\|\delta C_3\|^2 - \frac{2\lambda t}{\sqrt{3}} (\delta C_3, \delta C_1)$$

$$= -\left\| \frac{1}{\sqrt{3}} \delta C_3 + \lambda t \delta C_1\right\|^2 + \lambda^2 t^2 \|\delta C_1\|^2 \le 2\lambda^2 t^2 (1 - t) .$$

It is clear that

$$(3.3)\quad -10t\, s_{11}^2 - \frac{4\lambda}{\sqrt{3}} s_{11}s_{13} = -\, t(10s_{11}^2 + \frac{4\lambda}{\sqrt{3}} s_{11}s_{13})$$

$$- \frac{4\lambda(1-t)}{\sqrt{3}}\, s_{11}s_{13} \le \frac{2}{15}\, t\lambda^2 s_{13}^2 + \frac{2}{\sqrt{3}}|\lambda|(1-t)(s_{11}^2 + s_{13}^2).$$

Now the area theorem (Theorem 2.1 with $x_1 = 1, x_n = 0$ otherwise) asserts that

$$r_{11}^2 + s_{11}^2 + s_{13}^2 \le 1 .$$

Substitution of the above into (3.3) yields

$$(3.4)\quad -10t\, s_{11}^2 - \frac{4\lambda}{\sqrt{3}} s_{11}s_{13} \le \left(\frac{2}{15} t\lambda^2 + \frac{2}{3} |\lambda|(1-t)\right)(1 - t^2) .$$

By substituting (3.3) and (3.4) into (3.2) we obtain

$$(3.5)\quad \delta P_\alpha \le \frac{10}{3}(t^3 - 1) + 2\lambda^2 t^2(1 - t) + \left(\frac{2}{15} t\lambda^2 + \frac{2}{\sqrt{3}}|\lambda|(1 - t)\right)(1 - t^2) .$$

It is clear that the above estimate is monotone in $|\lambda|$ when $0 \le t \le 1$. Choosing $|\lambda| = \sqrt{75/17}$, the largest value for which the right side of (3.5) is negative near $t = 1$, and using the crude estimate $\sqrt{25/17} < 3/2$, one obtains the inequality

$$\delta P_\alpha \le - (1 - t)^2\left(\frac{1}{3} + \frac{157}{51} t\right) \le 0$$

with equality only if $t = 1$. This completes the proof.

While the above method is adequate for the global theorem it does not give the best local estimate. Bombieri [15] proved that

$$\liminf_{t \to 1^-} \frac{4 - \mathrm{Re}\, a_4}{1 - t} > 1.6 \quad .$$

while the estimate (3.5) gives

$$\liminf_{t \to 1-} \frac{4 - \mathrm{Re}\, a_4}{1 - t} \ge \frac{14}{15} \quad .$$

This can be improved slightly by considering the contrabution of of the imaginary parts of the first two components of $\delta C_3/\sqrt{3} + 2t\delta C_1$.

Jenkins and Ozawa [16],[17] used Theorem 2.1 to derive the local result for the sixth and eighth coefficients. However, they picked special values of the parameters rather than using the unitary property.

4. Some Remarks on the Sixth Coefficient Problem. By applying Theorem 2.4 to the formulas for the sixth coefficient, see Garabedian, Ross and Schiffer [5], we have obtained partial results toward the solution of the sixth coefficient problem. If the coefficients are real one readily obtains the known result $|a_6| \leq 6$. If a_2 and a_3 are real the method gives the same estimates as for real coefficients. Schiffer [18] has announced a similar result. When a_2 is real and positive, our method can be used to prove $\mathrm{Re}\ \delta a_6 \leq 0$. The estimate differs from the estimate for real coefficients by $\lambda(t)(1 - t^2)$ where $\lambda(t)$ is the largest eigen value of a two by two matrix. Ozawa [19] proved the result for a_2 real by exploiting the classical form of Grunsky's inequality. In the general case we have shown that if f is normalized so that $|\mathrm{Arg}\ a_2| \leq \pi/5$, then

$$\delta\ \mathrm{Re}\ a_6 \leq Q(t) \quad , \quad t = 1/2\ \mathrm{Re}\ a_2 \ ,$$

where $Q(t)$ depends on t and the largest eigen values of three four by four matrices. We are now conducting computing machine experiments to see if $Q(t)$ is negative on $[0,1)$.

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