AN EXISTENCE THEOREM FOR A CLASS OF NON-LINEAR INTEGRAL EQUATIONS WITH APPLICATIONS TO A NON-LINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

Charles V. Coffman

Report 67-37

November, 1967

AN EXISTENCE THEOREM FOR A CLASS OF NON-LINEAR INTEGRAL EQUATIONS WITH APPLICATIONS TO A NON-LINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

Charles V. Coffman

In [5], Nehari proved that if *Q,* is a bounded real interval, K a symmetric positive definite kernel which is continuous on *£1* X *Q*$_9$ and F a non-negative continuous function on $\overline{R_+}$X Q satisfying, for some *e > 0* ,

(1) **o < T?~$^{€}_{1}$ F(r?$_1$,x) <_TJ$^{"€}_{2}$ F(r?$_{25}$x) , o < r?$_x$ < i?$_2$< OD ,**

for all *xetl* , then the integral equation

(2) $$y(x) = \int_\Omega K(x,t)y(t)F(y^2(t),t)dt ,$$

a©

has a non-trivial,solution which is continuous on £2 . The proof uses variational methods. In this note we shall prove, using arguments similar to those of [5]$_9$ that if *Q,* is a bounded region in R^n (n-dimensional Euclidean space), if K is a symmetric function on fl x fl such that for some pair of conjugate indices p, q, $1 < q \leq 2 \leq p < \infty$,

(3) $$[Au](x) = \int_\Omega K(x,t)u(t)dt ,$$

defines a completely continuous operator A from L^(fl) to *iP(ty* , which is positive definite in the sense that

$$\int_\Omega\int_\Omega K(x,t)u(x)u(t)dxdt > 0, \qquad U€L^q(0\"\{0],$$

and if in addition to (1)$_9$ F satisfies the appropriate polynomial growth inequality so that the mapping

$$y(x) \longrightarrow y(x)F(y^2(x),x)$$

is continuous from L^(fl) to L^(fiS) , then (2) has a non-trivial solution yelj^{P}(flD .

In the concluding section we apply the result described above to the boundary value problem

$$Au + uF(u^{-},x) = 0 \quad in \quad £2 \qquad u | ^{\wedge\wedge} = 0 ,$$

where A is the Laplace operator and Q is a bounded region in R^{n} for which the Diriclet problem is solvable.

Related to the problem (2) and the boundary value problem above are the eigenvalue problems,

(4) $$y(x) = A J K(x,t)y(t)F(y^{2}(t),t)dt ,$$

and

(5) $$\&u + AuF(u^{2},x) = 0 \quad in \quad a \qquad u | ^{\wedge} = 0 .$$

The first of these problems is treated in [3] and in [7]; problems of the form (5) are treated in [4] and in [6]. Also Berger, [1], has investigated a problem similar to (5) but with A replaced by a more general elliptic operator. All of these results contain the linear cases, the results obtained here are strictly non-linear in character.

2. Statement of results. Let *Cl* be a bounded region in Euclidean n-space and let K be a real valued symmetric function on £2 x *Q* . We assume that K(x, •) is measurable for almost all *xeCi* $_{3}$ and that : (H) the operator A defined by (3) is completely continuous from L^{q}(flj) to L^{p}(fiD for sbme pair of conjugate

indices p and q with

(6) $$1 < q \leq 2 \leq p \ .$$

Finally we assume that A is positive definite on L^(fi}, i.e.

(7) f [K(x,t)u(x)u(t)dtdx > 0 ,
Jfrn

.q
for uelj (flj , u (x) not almost everywhere equal to zero.

Let F be a real-valued function on "R^X *Q,9* (R+= {TJ: 77 real,77 >0)) which satisfies the following conditions i) the Caratheodory hypothesis (i.e. F(#,x) is continuous on R+ for almost all XGO and F(rj, •) is measurable for all 7?eR+) , ii) there is a positive constant *e* such that (1) holds for almost all (fixed) *xeCl ,* iii) there are positive constants *c,y* and there is a non-negative function *oeL (Q* such that for almost all (fixed) $x \in \Omega$,

(8) F(TJ,X) _< cV7 + cr(x) , 0 < T) < 00 .

Theorem 1. Let *Cl,* K and F be as^ above, in particular assume that (H) holds, and that (7) holds for all u€L^(flr) ,u ^ 0 . If

(9) *27* <_p - 2, cr€Lr($} , r = p/(p-2) ,

then the integral equation (2) has at least one non-trivial solution in *lP{Q* .

Theorem 2. Let *£1,* K and F satisfy the hypotheses of Theorem 1 • i£ p i^. fL non-negative real-valued function

(10) PeLr(q>, r = p/(p-2) ,

and if the least eigenvalue λ *of the symmetrizable linear integral equation*

(11) $$u(x) = \lambda \int_\Omega K(x,t)P(t)u(t)dt ,$$

is larger than 1 , *then the integral equation*

(12) $$y(x) = \int_\Omega K(x,t)y(t)(P(t) + F(y^2(t),t))dt ,$$

has a non-trivial solution in $L^p(\Omega)$.

We note here that the hypothesis (H) is implied by the condition

(13) $$\operatorname{ess\,sup}_{x\in\Omega} \int_\Omega |K(x,t)|^\alpha dt < \infty , \qquad \alpha > 1 ,$$

see [2], Theorem 9.5.6, p. 658. The positive definiteness of A in $L^q(\Omega)$ is equivalent, in the presence of hypothesis (H), to positive definiteness of A restricted to $L^2(\Omega)$ together with the density of the range of A in $L^p(\Omega)$. To see this we use the fact, [7], p. 189, that when (H) holds and A is positive definite in $L^2(\Omega)$ then for $u, v \in L^q(\Omega)$,

(13') $$\int_\Omega\int_\Omega K(x,t)u(x)v(t)dxdt = \sum_{k=1}^\infty \lambda_k^{-1} \int_\Omega u(t)\varphi_k(t)dt \int_\Omega v(t)\varphi_k(t)dt ,$$

where the λ_k are the eigenvalues of $A|L^2(\Omega)$ and, for each k, φ_k is the normalized (in $L^2(\Omega)$) eigenfunction of A corresponding to λ_k; the φ_k actually belong to $L^p(\Omega)$. If A has dense range in $L^p(\Omega)$ it follows, using (13'), that for non-zero $u \in L^q(\Omega)$, $\int u(t)\varphi_k(t)dt$ cannot vanish for all k. Putting $v = u$ in (13') we conclude, since u was arbitrary, that A is positive definite on $L^q(\Omega)$ when $A|L_2$ is positive definite and A has dense range in $L^p(\Omega)$. The other direction of the equivalence is obvious. One can construct a symmetric kernel satisfying (13) for some $a > 2$, which is positive definite on L^2 but is not positive definite on L^q for any $q < 2$.

The next theorem gives conditions under which an L^p-solution of (2) or of (12) will be essentially bounded.

Theorem 3. *Suppose that the symmetric kernel K satisfies (13) for some $p > 2$, and let F be a non-negative Caratheodory function on $R \times \Omega$ satisfying (8) with $\gamma < (p-2)/2 >$ and $V \in L^\infty$. Then every L^p solution of (2) is essentially bounded. If we assume the same hypothesis, and in addition assume that* $P \in L^\infty$ then every L^p solution of (12) is essentially bounded.

3. *Formulation of the variational problem*. We shall show that the existence of a non-trivial solution of (2) is implied by the existence of a solution to the variational problem formulated below.

We define a function G with the same domain as that of F

$$G(\eta,x) = \int_0^\eta F(s,x)ds \quad . \tag{14}$$

The variational problem is formulated in terms of functionals $J(u,v)$, $N(y)$, $H(y)$ which are defined, for $u, v, y \in L^p$, as follows,

$$J(u,v) = \int_\Omega \int_\Omega K(x,t)u(x)F(u^2(x),x)v(t)F(v^2(t),t)dxdt \,, \tag{15}$$

$$H(y) = \int_\Omega [y^2(x)F(y^2(x),x) - G(y^2(x),x)]dx \tag{16}$$

$$N(y) = \int_\Omega y^2(x)F(y^2(x),x)dx - J(y,y) \,. \tag{17}$$

Notice that H vanishes identically for the linear problem.

Lemma 1. The functionals H and N are continuous on L^p.

Proof. In view of (8), (9), the mapping f defined by

$$[fy](x) = y(x)F(y^2(x),x)$$

is a continuous mapping from L^p to L^q ; see [7], Theorem 19.1, p. 154. It then follows from hypothesis (H) that Af is a continuous mapping of L^p into itself ; the continuity of J follows from the continuity of the inner product on $L^p X L^q$. Appealing again to (8), (9) and [7], Theorem 19.1, we easily see that the mappings $y(x) \longrightarrow y^2(x)F(y^2(x),x)$, and $y(x) \longrightarrow G(y^2(x),x)$ are continuous from L^p to L^1 ; the continuity of H and N on L^p follows.

A function $y \in L^p$ will be called admissible if it is not almost everywhere equal to zero and can be represented in the form

$$y = Au , \qquad u \in L^q . \tag{18}$$

If $y \in L^p$ and y is not almost everywhere equal to zero (but not necessarily admissible), then it follows from (1) and the positive definiteness of A that v , defined by

$$v(x) = \int K(x,t)y(t)F(y^2(t),t)dt ,$$

is admissible.

(*) UL Y iJi *ML* admissible function then there is <a positive constant *a* such that

$$N(\alpha y) = 0 . \tag{19}$$

(**) *If* y *is admissible and satisfies*

(20) $$N(y) = 0 ,$$

and if

(21) $$v(x) = a \int_\Omega K(x,t)y(t)F(y^2(t),t)dt ,$$

where $a > 0$ *is chosen so that*

(22) $$N(v) = 0 ,$$

then (v is admissible and)

(23) $$a^2 \int_\Omega y^2(x)F(y^2(x),x)dx \leq \int_\Omega v^2(x)F(y^2(x),x)dx ,$$

and

(24) $$H(v) \leq H(y) .$$

Equality holds in (24) *if and only if* y *is a solution of* (2).

Finally, if $y \in L^p$, *then*,

(25) $$H(y) \geq e(1+\epsilon)^4 \int_\Omega y^2(x)F(y^2(x),x)dx .$$

Results analogous to the assertions of (*) and (**) are proved in [5]; the proofs are essentially the same for the case considered here.

The variational problem which we consider is that of minimizing the functional $H(y)$ within the class of admissible functions and subject to the side condition $N(y) = 0$. It is clear from (**) that a solution y of this variational problem must be a solution of the integral equation (2).

4. Solution of the variational problem. In this section we prove the existence of a solution to the variational problem posed above. We assume throughout that the hypotheses of Theorem 1 are satisified.

Lemma 2. *There is a positive constant* m *such that for any admissible function,* y , *satisfying* (20) ,

$$\int y^2(x)F(y^2(x),x)dx \geq m . \tag{26}$$

Moreover there are positive constants k_1, k_2 such that

$$\left(\int_\Omega \left|\int_\Omega K(x,t)y(t)F(y^2(t),t)dt\right|^p dx\right)^{1/p} \leq \left(k_1 + k_2 \int_\Omega y^2(t)F(y^2(t),t)dt\right)^{1/q}. \tag{27}$$

Proof, Because of hypothesis (H), (17) and (20) we have

$$\int y^2(x)F(y^2(x),x)dx \leq M\left(\int |y(x)F(y^2(x),x)|^q dx\right)^{2/q} , \tag{28}$$

where M is the norm of the operator A . By Holder's inequality

$$\int_\Omega |y(x)F(y^2(x),x)|^q dx \leq \left(\int_\Omega |F(y^2(x),x)|^r dx\right) \left(\int_\Omega y^2(x)F(y^2(x),x)dx\right)^{q/2} , \tag{29}$$

where, as before, $r = q/(2-q)$. Combining (28) and (29) we get, since the term on the left in (28) is positive,

$$1 \leq M\left(\int_\Omega |F(y^2(x),x)|^r dx\right)^{1/r} . \tag{30}$$

The first assertion of the lemma is now proved as follows. Suppose the assertion is false and that $\{y_n(x)\}$ is a sequence of admiss-ible functions satisfying (20) and

$$J\ y_n^2(x)F(y_n^2(x),x)dx \quad 0 \ , \qquad \textbf{as n oo .}$$

We can then conclude, using (1), that a subsequence of $\{y_n(x))$, which can be assumed to be the full sequence, converges almost everywhere to zero. Since *Cl* has finite measure we can assume that *y* = (p-2)/2 , and then $a^{1\wedge 2\ y} \in L^p$. Let E_n denote the subset of *Q* where $|y_n(x)| \geq$. $ff^{1,2y}$, by (8) and (9) we then have, on E_n,

$$|F(y_n^2(x),x)|^r = |F(y_n^2(x),x)||F(y_n^2(x),x)|^{r-1}$$

$$\leq c_1^{r-1}\ y_n^2(x)F(y_n^2(x),x),$$

where c_1 = (c+1) . Thus

$$\int_{E_n} |F(y_n^2(x),x)|^r dx \longrightarrow 0, \qquad \text{as } n \to co\ .$$

If $E^\wedge = fl \backslash E_n$, then $\backslash F(yfa),x)|^r \leq |p((a(x))^{1/y},x)|^r$ for XGE_n' . Thus since $y_n(x) \to 0$ a.e. in *Q*, , it follows from the dominated convergence theorem that

$$\int_X |F(y_n^2(x)_5 x)|^r dx \longrightarrow * 0, \qquad \text{as } n \to oo\ .$$

Thus our supposition has led to a contradiction of (30) and (26) is proved.

For an arbitrary yeL^p we have, for almost all $x \in \Omega$,

$$|F(y^2(x),x)|^r \leq c_{1Y}{}^2(x)F(y^2(x)_5 x) + F((a(x))^{17}T\ x)\ .$$

Upon integrating this inequality over *Q* we obtain

$$|\ |F(y^2(x),x)|^r dx < c, F\ y^2(x)F(y^2(x),x)dx + f\ F((a(x))^{X/}Tx)\,dx.$$

Using this inequality in (29) and using the continuity of A from L^q to L^p we obtain (27).

We now show that the problem

(31) $$H(y) = \min., \quad N(y) * 0,$$

has a solution within the class of admissible functions. First we observe that (25) and (26) imply

(32) $$/x = \inf\{H(y) : y \text{ is admissible and } N(y)' = 0) > 0 .$$

Let y,v and a be as in (**). The function *F(Tj,x)* is increasing in *rj* for almost all x , therefore

$$0 \leq f \; (v^2(x) - y^2(x))\,(F(v^2(x),x) - F(y^2(x),x))\,dx,$$

and this implies, in view of (24) and (25) ,

$$J_{S2}\, v^2(x)F(y^2(x),x)dx \leq J_{S2}\,(v^2(x)F(v^2(x),x) + y^2(x)\,F(y^2(x),x))\,dx$$

$$\leq 2\epsilon^{-1}(1+\epsilon)\,H(y) .$$

Using (25) this gives ,

$$a^2 \, J_{\Omega}\, y^2(x)F(y^2(x),x)dx \leq 2e^{-1}(1+e)\,H(y) .$$

From (26) follows

(33) $$a^2 \leq CH(y) , \quad C = 2(me)^{-1}(1+e) ,$$

and finally from (21), (27), (33) and (25) we have

(34) $$\|v\|_p < C_0(1+H(y))^{(2+q)/2q} .$$

For B c L^\{0] we denote by 0(B) the set of all (admissible) functions of the form (21), where *a* > 0 is chosen so that (22) holds ,

0(B) = {v : v = OAfy, yeB; «€R+, N(v) = 0} .

We choose $|X_1$ > #4 and take

B = {y : y admissible, N(y) = 0, H(y) $\leq$ j^} .

Then by (34) ,

(35) 0(B) c $\{y\epsilon L^p(\Omega) : \|y\|_p \leq c_o(1 + \mu_1{}^{(2+q)/2q})\}$,

and since f maps bounded sets in I^p into bounded sets in L^q it follows from (35) that

(36) $0^2(B)$ = 0(0(B)) c A{ueL^q : $||u||_q \leq$ const.}.

The ball {ueL^ : $||u||_q$ _< const.} is weakly compact and A is completely continuous so the set on the right in (35) is compact (in L^). Thus B^ , the closure of 0^2(B) in L^ , is compact in L^ and any non-zero function in B_1 is admissible. By Lemma 1,

(37) inf{H(y) : yeBj} = inf(H(y) : ye0^2(B)} ,

and since 0^2(B) consists entirely of admissible functions

(38) /i<_inf{H(y) : ye0^2(B)} <_inf{H(y) : yeB} = /x ;

the second inequality in (38) follows from (**) and the definition of 0 . From the first inequality of (38) and from (37) it follows, using (32), that O^B^ *3* therefore all functions in B_1 are admissible. Since B_1 is compact it follows from (37), (38), and Lemma 1 that there is a $y_o \epsilon B_1$ with $H(y_o) = \mu$. By Lemma 1, N vanishes identically on B_1, thus the admissible function y_o is a solution of the variational problem (31). As we have already observed, an admissible solution of (31) satisfies (2). Thus we have proved Theorem 1.

Remark.* Let C denote the cone of almost everywhere non-negative functions in L^p and assume that K is noh-negative, or let C denote some other closed convex cone in L^p and assume that K and F are such that $\int K(x,t)y(t)F(y^2(t),t)dt$ is in C whenever y is. Then one can add to the definition of admissibility the condition that $y \epsilon C$; with this definition of admissibility the argument given above implies that (2) has a non-trivial solution in C. In particular, if we add non-negativity to the definition of admissibility, then the condition that A be positive definite in L^g can be replaced by the condition that K be positive a.e. in $Q_1 \times Q_9$ and that AJL⁻ be non-negative definite.

5. Proof of Theorem 2. The only place in the proof of Theorem 1 where the argument can break down when F is replaced by $F_1 = P + F$ is in the demonstration (for which the reader was referred to [5]) that the normalization (19) is possible for any admissible function y_o. However if F is replaced by F_1 in (15) and (17) then the normalization (19) is still possible provided the least eigenvalue of (11) exceeds 1. The proof is the same as in [5]. All of the rest of the arguments above remain

valid as they stand when F is replaced by F_1. It should be noted that $H(y)$ remains unchanged when F is replaced by F_1.

6. Boundedness. Proof of Theorem 3. The following lemmas constitute the proof of Theorem 3.

Lemma 3. *Suppose that* (13) *holds for some* $p > 2$, *and that* F *satisfies* (8) *where*

(39) $$0 \le \gamma < (p-2)/2, \quad f \in L^s, \quad s > p/2\gamma .$$

Then Af maps L^{p_1} into L^{p_2} for

(40) $$p \le p_1 \le 2\gamma s, \quad p_2^{-1} > (2\gamma+1)/p_1 - (a-1)/a.$$

Proof. For $y \in L^{p_1}$, $p_1 \ge p$, we have by (8) and (39),

(41) $$fy \in L^{q_1},$$

for $q_1 \le p_1 \min((2\gamma+1)^{-1}, s/(s+p_1))$. If $p_1 \le 2\gamma s$ then $s/(s+p_1) \ge (2\gamma+1)^{-1}$ and thus (41) holds for $q_1 \le p_1/(2\gamma+1)$. Now by Theorem 9.5.6 [2], A maps L^{q_1} into L^{p_2} for

$$p_2^{-1} > q_1^{-1} - (a-1)/a, \quad 1 \le p_2 \le \infty,$$

and we conclude that the composite Af maps L^{p_1} into L^{p_2} for p_1, p_2 satisfying (40).

Lemma 4. *Assume the hypothesis of Lemma* 3. *Then any* L^p-*solution of* (2) *belongs to* L^{p_2} *for every* p_2 *satisfying*

$$p_2^{-1} > (2\gamma+1)/2\gamma s - (a-1)/a, \quad 1 \le p_2 \le \infty .$$

<u>Proof</u>. It is clear from Lemma 3 that there is a positive integer k such that $(Af)^k$ maps L^p into L^{p_1} where $p_1 = 2ys$. It then follows, again by Lemma 3, that $(Af)^{k+1}(L^p) \subset L^{p_2}$ for any p_2 satisfying (42).

It follows from Lemma 4 that any L^p solution of (2) will belong to L^∞ if $s > (2y+1)a/(2y(a-1))$, we shall prove the following stronger result which, however, is not needed for the proof of Theorem 3.

Lemma 5. <u>Assume again the hypothesis of Lemma</u> 3. <u>If</u>

(43) $$s > (4y+1)a/4y(a-1) ,$$

<u>then any</u> L^p <u>solution of</u> (2) <u>belongs to</u> L^∞.

<u>Proof</u>. By (8), (39) and Lemma 4, if y is an L^p solution of (2) then $fy \in L^{q_2}$ for every q_2 satisfying

(44) $$q_2^{-1} > 2s^{-1} + (2ys)^{-1} - (a-1)/a + 2y \max[0, s^{-1} - (a-1)/a] .$$

Using Hölder's inequality, it follows from (13) that A maps L^{q_2} into L^∞ if

(45) $$q_2^{-1} \le (a-1)/a .$$

In order that (44) and (45) be satisfied simultaneously by some q_2 it suffices that

$$0 > 2s^{-1} + (2ys)^{-1} - 2(a-1)/a + 2y \max[0, s^{-1} - (a-1)/a] ,$$

and this holds if and only if (43) holds. Clearly if there is a q_2 satisfying both (44) and (45) then the conclusion of the Lemma follows.

We can conclude from Lemma 4 that, under the hypothesis of Theorem 3, any L^p solution of (2) actually belongs to L^∞. The condition (1) is not used in this section so the assertion of Theorem 3 concerning solutions of (12) follows by applying Lemma 4 with F replaced by $F_1 = P + F$.

The argument used above to prove boundedness is an integral equations version of the 'bootstrap procedure* for proving regularity of a generalized solution of a boundary value problem,

7. Application to a non-linear elliptic boundary value problem. We consider the boundary value problem

$$Ay + yF(y^2,x) = 0, \text{ in } 0, \qquad y|\wedge_0 = 0 , \tag{46}$$

where Q is a bounded region in n-space for which the Diriclet problem is solvable and A is the Laplace operator. If $G(x,t)$ is the Green[1]s function for the Diriclet problem in Q then

$$\underset{xeft}{\text{ess sup}} \int_Q |G(X,t)|^a dt < oo, \quad \text{for } a < n/(n-2) ,$$

thus if we put $K = G$ the hypothesis H is satisfied for any pair p,q with

$$2 < p < 2n/(n-2) . \tag{47}$$

Let such a p be chosen and let A be the operator from $L^\wedge$ to L^P defined by (3), with $K = G$. Then, as is well known, $A|L^2$ is positive definite and the range of A contains all C^3 functions with compact support in f2 . Thus, in view of the remarks following the statement of Theorem 2, A is positive definite on L^q. If we assume that F satisfies (1) and (8), where

(48) $y < 2/n-2)$, $a = const.$,

then, since p can be chosen anywhere on the range (47) it follows from Theorems 1 and 3 that the integral equation

(49) $$y(x) = \int_{\Omega} G(x,t)y(t)F(y^2(t),t)dt ,$$

has a non-trivial solution in L^{∞}. From (8), (48) and the properties of the Green's function it follows that an L^{∞} solution y of (49) is continuously differentiable in ft , and continuous in $\bar{ft}$ and that yKrj = 0 . If we assume in addition that F is locally Httlder continuous on $\bar{R}_{+}$x ft , then y will be twice continuously differentiable in ft . Thus we have proved the following.

Theorem 4. Let F b£ Ideally Holder continuous on $\bar{R}_{+}$x 0 , and satisfy (8) where (48) holds. Assume also that for some e > 0, (1) holds for all xeft . Then the boundary value problem (46) has IL non-trivial solution y which is continuous in $\bar{ft}$ and of class C^2 in ft .

Similarly we can prove the following.

Theorem 5. Assume the hypothesis of Theorem 4, and assume that P jjs[i* bounded non-negative function which is locally Hö'lder continuous on ft . If the least eigenvalue of

$$Au + AP(x)u = 0 , \text{ in ft }, \qquad u|_{\partial\Omega} = 0 ,$$

exceeds 1 , then the boundary value problem

$$Ay + y(P(x) + F(y^2,x)) = 0 , \qquad \text{in ft }, \quad y|_{\partial\Omega} = 0 ,$$

has *a*. non-trivial solution y which is continuous in $\overline{ft}$ and of class C^2 in ft •

Remarks. 1. From Remark 1 at the end of section 4 it follows that the solutions whose existence is obtained in Theorems 4 and 5 can also be asserted to be positive in £i .

2. In [6] it is shown that if

$0= (xeR^n||x| < 1\}$, $n > 2$, and if $F(r?,x) = r\backslash^y$ where $y \geq 2/(n-2)$ then (46) does not have a solution which is positive and of class C^2 in *Q* and continuous in *Cl.*

References

1. M. S. Berger, 'An eigenvalue problem for nonlinear elliptic partial differential equationsJ Trans. Amer. Mat . Soc. 120 (1965).

2. R. E. Edwards, Functional Analysis, Holt Rinehart and Winston, Inc., New York , 1965.

3. M. A. Krasnosel'skii, Topological Methods in the Theory of Nonlinear Integral Equations, MacMillan, New York, 1964.

4. N. Levinson, 'Positive eigenfunctions for Au + Af(u) = 0,* Arch. Rational Mech. Anal., JL1 (1962), 258-272.

5. Z. Nehari, 'On a class of nonlinear integral equations,' Math. Zeit., 22 (1959), 175-183.

6. S. I. Pohozaev, 'Eigenfunctions of the equation Au + 7vf(u) = 0,' Dokl. Akad. Nauk SSSR, 165 (1965), 36-39. Soviet Math., 5, (1965), 1408-1411.

7. M. M. Vainberg, Variational Methods for the Study of Nonlinear Operators, Holden-Day, Inc., San Francisco, 1964.