A NONLINEAR OSCILLATION PROBLEM

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1. Consider the differential equation

$$y'' + yF(y^2,x) = 0, \tag{1}$$

where the function $F(t,x)$ is defined for $t \in [0,\infty)$, $x \in (0,\infty)$ and has there the following properties: (a) $F(t,x) \geq 0$; (b) $F(t,x)$ is continuous in x for fixed t; (c) in a neighborhood of every x in $(0,\infty)$, $F(t,x)$ satisfies a uniform Lipschitz condition.

A solution of (1) is said to be nonoscillatory if, for $a > 0$, the number of its zeros in (a,∞) is finite. The equation itself is said to be nonoscillatory if all its solutions have this property. We note here that the conditions imposed on $F(t,x)$ are not quite sufficient to guarantee that any local solution of (1) can be extended to the entire interval $(0,\infty)$ [3,2], and it may therefore seem to be advisable to use a different definition of nonoscillation. However, this is not necessary. An elementary argument [2] shows that, under our assumptions on $F(t,x)$, a solution of (1) which cannot be continued to the right of a point b must necessarily have an infinite of zeros in a left neighborhood of b. A nonoscillatory solution can thus ipso facto be continued throughout the interval $(0,\infty)$.

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Simple examples show that some solutions of an equation of type (1) may oscillate, while others do not [5,6]. Accordingly, one is led to the consideration of two different types of non-oscillation conditions: those which insure the existence of at least one nonoscillatory solution, and those which guarantee that the equation is non-oscillatory. While a number of conditions of the first type are known [1,4,5,6], the only criterion of the second type found in the literature seems to be the following condition: If, for $\alpha > 0$,

$$\int^{\infty} xF(\alpha x^2, x)\,dx < \infty$$

and if, for fixed t, $F(t,x)$ is a nonincreasing function of x, then (1) is nonoscillatory [1,6]. However, this condition guarantees, in addition, that all solutions of (1) are either $\sim cx$ or $\sim c$ (c constant) for large x, and it is clear that much less is required in order to make the equation merely non-oscillatory. The following statement describes a sufficient condition for nonoscillation which, in a sense to be specified, is the best of its kind.

<u>Theorem I</u>. <u>Let</u> $F(t,x)$ <u>be defined for</u> $t \in [0,\infty)$, $x \in (0,\infty)$ <u>and satisfy there the following conditions</u>: (a) $F(t,x) \geq 0$; (b) $F(t,x)$ <u>is continuous in</u> x <u>for fixed</u> t; (c) <u>in a neighborhood of every</u> x <u>in</u> $(0,\infty)$, $F(t,x)$ <u>satisfies a uniform Lipschitz condition</u>; (d) <u>for fixed</u> x, $F(t,x)$ <u>is a nondecreasing function of</u> t. <u>If</u> $G(t,x)$ <u>is defined by</u>

(2) $$G(t,x) = \int_0^t F(s,x)\,ds$$

and if, for some positive ϵ and all positive α, $xG(\alpha x^{1+\epsilon},x)$ is nonincreasing for $x \in (a,\infty)$ $(a > 0)$, then equation (1) is nonoscillatory. This condition is the best possible in the sense that the conclusion does not hold for $\epsilon = 0$.

In the case of the special equation

(3) $$y'' + p(x)y^{2n-1} = 0, \; p(x) > 0, \; n > 1,$$

we shall obtain the following stronger result.

Theorem II. If $p(x)$ is continuous and $p(x)(x \log x)^{n+1}$ is nonincreasing, equation (3) is nonoscillatory.

2. If a_1 and a_2 are two consecutive zeros of a solution y of (1), an elementary manipulation shows that

(4) $$\int_{a_1}^{a_2} y'^2\,dx = \int_{a_1}^{a_2} y^2 F(y^2,x)\,dx.$$

Since, for $x \in (a_1,a_2)$,

$$y^2(x) = \left(\int_{a_1}^{x} y'\,dx\right)^2 \leq (x - a_1)\int_{a_1}^{x} y'^2\,dx < x\int_{a_1}^{a_2} y'^2\,dx,$$

it follows from (4) that

(5) $$1 \leq \int_{a_1}^{a_2} xF(y^2,x)\,dx.$$

On the other hand, $F(t,x)$ is (for fixed x) a nondecreasing function of t. By (2), $G(t,x)$ is thus convex in t, and we have

$$G(\alpha x,x) \geq G(y^2,x) + (\alpha x - y^2)F(y^2,x).$$

Hence,

$$\alpha xF(y^2,x) \leq \alpha xF(y^2,x) + G(y^2,x) \leq y^2F(y^2,x) + G(\alpha x,x),$$

where α is an arbitrary positive number. Integrating this from a_1 to a_2, and using (5), are obtain

$$(6) \qquad \alpha \leq \int_{a_1}^{a_2} y^2F(y^2,x)\,dx + \int_{a_1}^{a_2} G(\alpha x,x)\,dx.$$

We note that our assumptions imply that $G(t,x)$ is nonincreasing in x for fixed t and that, as a result, the partial derivative $G_x(t,x)$ exists for almost all x. Indeed, since $xG(\alpha x^{1+\epsilon},x)$ is nonincreasing, so is $G(\alpha x^{1+\epsilon},x)$, and if α is chosen so that $\alpha x^{1+\epsilon} = t$ and $x_1 > x$, we have

$$G(t,x_1) \leq G(\alpha x_1^{1+\epsilon},x_1) \leq G(\alpha x^{1+\epsilon},x) = G(t,x).$$

It is understood that the following identities and inequalities involving $G_x(t,x)$ are to be used only at points at which this derivative exists.

We now introduce the function

$$(7) \qquad Q(t,x) = G(t,x) + tF(t,x) + xG_x(t,x)$$

and we use the two easily verified identities

$$(8) \qquad \begin{aligned} &\frac{d}{dx}\{x[y'^2 + G(y^2,x)] - yy'\} \\ &\qquad = (2xy' - y)[y'' + yF(y^2,x)] + Q(y^2,x), \end{aligned}$$

$$(9) \quad \frac{d}{dx}\{xG(\alpha x^{1+\epsilon},x)\} = Q(\alpha x^{1+\epsilon},x) + \epsilon\alpha x^{1+\epsilon}F(\alpha x^{1+\epsilon},x).$$

Since $xG(\alpha x^{1+\epsilon},x)$ is assumed to be nonincreasing for $x > a$, it follows from (9) that

$$Q(\alpha x^{1+\epsilon},x) + \epsilon\alpha x^{1+\epsilon}F(\alpha x^{1+\epsilon},x) \leq 0$$

for all positive α and almost all $x > a$. If y^2 is a positive number, and we set $\alpha = y^2x^{-1-\epsilon}$, we obtain

$$(10) \qquad Q(y^2,x) + \epsilon y^2F(y^2,x) \leq 0.$$

We now apply the identity (8) to a solution y of (1). Since the right-hand side of (8) reduces in this case to $Q(y^2,x)$, it follows from (8) and (10) that

$$(11) \qquad \frac{d}{dx}\{x[y'^2 + G(y^2,x)] - yy'\} + \epsilon y^2F(y^2,x) \leq 0.$$

Integrating this inequality between two consecutive zeros a_1, a_2 of y, and observing that, by (2), $G(0,x) = 0$, we have

$$a_2y'^2(a_2) - a_1y'^2(a_1) + \epsilon \int_{a_1}^{a_2} y^2F(y^2,x)\,dx \leq 0.$$

Combining this with (6), we arrive at the inequality

$$(12) \qquad \alpha\epsilon + a_2y'^2(a_2) - a_1y'^2(a_1) \leq \epsilon \int_{a_1}^{a_2} G(\alpha x,x)\,dx.$$

If $a, a_1, \ldots, a_m$ are consecutive zeros of y, addition of the corresponding inequalities (12) yields

$$m\alpha\epsilon \le ay'^2(a) - a_m y'^2(a_m) + \epsilon \int_a^{a_m} G(\alpha x,x)\,dx,$$

and thus,

$$(13) \qquad m \le \frac{a}{\alpha\epsilon} y'^2(a) + \frac{1}{\alpha} \int_a^{a_m} G(\alpha x,x)\,dx.$$

To obtain a bound for the integral, we observe that, because of the convexity of $G(t,x)$ (as a function of t),

$$(14) \qquad G(\beta x^{1+\epsilon},x) \ge G(\alpha x,x) + (\beta x^{\epsilon} - \alpha)xF(\alpha x,x),$$

where β may be any positive number. If we set $\beta = \alpha a^{-\epsilon}$, we will have $\beta x^{\epsilon} - \alpha \ge 0$. Since $F(t,x)$ is a nondecreasing function of t, it follows from (2) that $G(t,x) \le tF(t,x)$. Hence,

$$(\beta x^{\epsilon} - \alpha)xF(\alpha x,x) \ge (\beta x^{\epsilon} - \alpha)\alpha^{-1}G(\alpha x,x) = [(x/a)^{\epsilon} - 1]G(\alpha x,x),$$

and (14) shows that

$$(15) \qquad a^{\epsilon}G(\alpha a^{-\epsilon}x^{1+\epsilon},x) \ge x^{\epsilon}G(\alpha x,x).$$

By assumption, $xG(\alpha a^{-\epsilon}x^{1+\epsilon},x)$ is nonincreasing for $x > a$. Hence, $xG(\alpha a^{-\epsilon}x^{1+\epsilon},x) \le aG(\alpha a,a)$, and (15) leads to the inequality

$$G(\alpha x,x) \le \left(\frac{a}{x}\right)^{1+\epsilon} G(\alpha a,a).$$

Thus,

$$\int_a^{a_m} G(\alpha x,x)\,dx \le \int_a^{\infty} G(\alpha x,x)\,dx \le \frac{a}{\epsilon} G(\alpha a,a),$$

and inequality (13) yields the bound

$$(16) \qquad m \le \frac{a}{\alpha\epsilon}[y'^2(a) + G(\alpha a,a)]$$

for the number of zeros which a solution of (1), which vanishes at $x = a$, can have in any interval (a,b). Hence, all solutions are nonoscillatory, and the main assertion of Theorem I is proved. It may be noted that, for any particular $G(t,x)$, we may take advantage of the arbitrariness of the positive constant α to obtain the best possible bound (16).

It remains to be shown that the assumption $\epsilon > 0$ is essential, i.e., that an equation of the form (1) may have oscillatory solutions if it is only assumed that $xG(\alpha x,x)$ is nonincreasing for any positive constant α. To obtain an example of an equation (1) which exhibits this type of behavior, consider the equation

$$y'' + \frac{y}{x^2} H'\left(\frac{y^2}{x}\right) = 0,$$

where $H(0) = 0$ and $H(s)$ is an increasing, differentiable convex function for $s \in [0,\infty)$. We have

$$F(t,x) = \frac{1}{x^2} H'\left(\frac{t}{x}\right), \quad G(t,x) = \frac{1}{x} H\left(\frac{t}{x}\right),$$

and $xG(\alpha x,x) = H(\alpha)$. The general solution of the equation is

$$y(x) = x^{1/2} u(\log x),$$

where $u(t)$ is the general solution of

$$\ddot{u} - \frac{u}{4} + uH'(u^2) = 0.$$

It is easy to see that, unless $H'(s)$ reduces to a constant, the latter equation always has both oscillatory and nonoscillatory solutions. Any solution which vanishes at one point is necessarily periodic, and thus oscillatory [6].

3. We now turn to the proof of Theorem II. If Ψ denotes the quantity

$$\Psi = x[ny'^2 + py^{2n}] - nyy', \tag{17}$$

a computation shows that

$$\Psi' = n(2xy' - y)(y'' + py^{2n-1}) + \frac{y^{2n}(x^{n+1}p)'}{x^n}.$$

If y is a solution of (3), this reduces to

$$\Psi' = \frac{y^{2n}(x^{n+1}p)'}{x^n},$$

and it is easily confirmed that

$$[\Psi(\log x)^{n+1}]' = (n+1)(\log x)^n\left[\frac{\Psi}{x} - py^{2n}\right] + \frac{[(x\log x)^{n+1}p]'y^{2n}}{x^n}.$$

Since $(x \log x)^{n+1}p(x)$ is nonincreasing, the derivative of this function exists for almost all x, and is either negative or zero. Hence, for almost all x,

$$\frac{[\Psi(\log x)^{n+1}]'}{(\log x)^n} \le (n+1)\left[\frac{\Psi}{x} - py^{2n}\right]$$

and thus, for any $1 < a < a_1 < \infty$,

$$\int_a^{a_1} \frac{[\Psi(\log x)^{n+1}]'}{(\log x)^n}\,dx \le (n+1)\left[\int_a^{a_1} \frac{\Psi}{x}\,dx - \int_a^{a_1} py^{2n}dx\right]$$

An integration by parts transforms the left-hand side into

$$[\Psi \log x]_a^{a_1} + n \int_a^{a_1} \frac{\Psi}{x}\, dx,$$

and (17) shows that the last inequality can be brought into the form

$$[\Psi \log x]_a^{a_1} \leq n \int_a^{a_1} (y'^2 - py^{2n})\, dx - \int_a^{a_1} \frac{yy'}{x}\, dx.$$

If $y(a) = 0$ and $y(a_1) = 0$ or $y'(a_1) = 0$, it follows from (3) that

$$\int_a^{a_1} (y'^2 - py^{2n})\, dx = 0,$$

and the inequality simplifies to

$$\int_a^{a_1} \frac{yy'}{x} dx \leq \Psi(a) \log a - \Psi(a_1) \log a_1. \tag{18}$$

If a and a_1 are consecutive zeros of y, an integration by parts shows that

$$\int_a^{a_1} \frac{yy'}{x} dx = \frac{1}{2} \int_a^{a_1} \frac{y^2}{x^2} dx.$$

Since this is positive, it follows from (18) that $\Psi(a_1) \log a_1 \leq \Psi(a) \log a$. Hence, if $a, a_1, a_2, \ldots$ are consecutive zeros of y, the sequence $\{\Psi(a_k) \log a_k\}$ is nonincreasing. Since, by (17), $\Psi(a_k) = na_k y'^2(a_k)$, this shows that

$$a_k \log a_k y'^2(a_k) \leq A, \quad (k = 1, 2, \ldots), \tag{19}$$

where A is a positive constant which does not depend on k.

Let now a be a zero of y, and assume that $y'(a) > 0$ (replacing, if necessary, y by $-y$). If b is the smallest zero of y' in (a,∞), both y and y' will then be positive in (a,b). Furthermore, since, by (3),

$$[(x - a)y' - y]' = (x - a)y'' = -(x - a)py^{2n-1} < 0,$$

and since $y(a) = 0$, we have $(x - a)y' \leq y$ in this interval. Hence,

$$\int_a^b y'^2 dx = \int_a^b \frac{(x-a+a)}{x} y'^2 dx \leq \int_a^b \frac{yy'}{x} dx + a \int_a^b \frac{y'^2}{x} dx \tag{20}$$

and, similarly,

$$\int_a^b \frac{y'^2}{x} dx = \int_a^b \frac{(x-a+a)}{x^2} y'^2 dx \leq \int_a^b \frac{yy'}{x^2} dx + a \int_a^b \frac{y'^2}{x^2} dx$$

$$\leq \frac{1}{a} \int_a^b \frac{yy'}{x} dx + a \int_a^b \frac{y'^2}{x^2} dx.$$

Combining this inequality with (20), we obtain

$$\int_a^b y'^2 dx \leq 2 \int_a^b \frac{yy'}{x} dx + a^2 \int_a^b \frac{y'^2}{x^2} dx. \tag{21}$$

To estimate the last term, we observe that $y'' = -py^{2n-1} < 0$ and that, therefore, $y'(x) \leq y'(a)$ in (a,b). Hence

$$a^2 \int_a^b \frac{y'^2}{x^2} dx \leq a^2 y'^2(a) \int_a^b \frac{dx}{x^2} \leq ay'^2(a).$$

In view of (19), (21) thus leads to the inequality

$$\int_a^b y'^2 dx \leq 2 \int_a^b \frac{yy'}{x} dx + \frac{A}{\log a}. \tag{22}$$

Inequality (18) was shown to hold for solutions y of (3) for which $y(a) = y'(a_1) = 0$, and we may therefore set $a_1 = b$ in (18). By (17), $\Psi(b) > 0$, and we may assume that $b > 1$. Accordingly, (18) implies that

$$\int_a^b \frac{yy'}{x}\, dx \leq \Psi(a) \log a = nay'^2(a) \log a.$$

In view of (19) we may thus conclude from (22) that

$$\int_{a_k}^{b_k} y'^2 dx \leq 2nA + \frac{A}{\log a_k} < B \tag{23}$$

$(k = 1,2,\ldots)$, where B is a positive constant which does not depend on k, and b_k is the zero of y' between a_k and a_{k+1}.

On the other hand, it follows from (3) that

$$\int_{a_k}^{b_k} y'^2 dx = \int_{a_k}^{b_k} py^{2n} dx.$$

Since, for $x \in [a_k, b_k]$,

$$y^2(x) \leq (x - a_k) \int_{a_k}^{x} y'^2 dx \leq x \int_{a_k}^{x} y'^2 dx \leq x \int_{a_k}^{b_k} y'^2 dx,$$

this implies

$$1 \leq \left(\int_{a_k}^{b_k} y'^2 dx \right)^{n-1} \int_{a_k}^{b_k} px^n dx.$$

and thus, by (23),

$$1 \leq B^{n-1} \int_{a_k}^{b_k} px^n dx. \tag{24}$$

Since $(x \log x)^{n+1} p(x)$ is nonincreasing in (a,∞), we have $x^n p(x) \leq nCx^{-1}(\log x)^{-n-1}$, where C is a positive constant. Hence,

$$\int_{a_k}^{b_k} x^n p(x)\,dx < \frac{C}{(\log a_k)^n},$$

and it follows from (24) that

$$\log a_k < (B^{n-1}C)^{\frac{1}{n}} = \log D.$$

This shows that the zeros of y are confined to the interval $[a,D)$. The number of these zeros is necessarily finite. Indeed, suppose $y(a_k) = 0$, $a < a_1 < a_2 < \ldots < D$, and set $a_0 = \lim a_k$ for $k \to \infty$. Since $b_k < a_{k+1} < a_0$, it follows from (24) that

$$1 \leq B^{n-1} \int_{a_k}^{a_0} px^n dx,$$

and this leads to a contradiction if k becomes sufficiently large. This concludes the proof of Theorem II.

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