CAUCHY SPACES I
UNIFORMIZATION THEOREMS

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we present the definitions of the various structures to be discussed here (limit structures, uniform limit structures, and Cauchy structures) along with a general theory of such structures. The various functors connecting these structures are given in section 4, and there we also present the adjointness relations holding between these functors. Section 5 contains a list of axioms describing separation, uniformity, and regularity properties in a limit structure. The various connections among these axioms are also given there. After the uniformization theorems of section 6 we tie together these results with those of Kowalsky [9], who first studied Cauchy filters axiomatically.

This paper is based on results obtained in the author's thesis [12], prepared under the direction of Professor Oswald Wyler. The author also gratefully acknowledges Professor Wyler[f]s suggestions in the reorganization of this material.

2. Preliminaries

In a category C we use the notation C[A,B] for the set of all maps from the object A to the object B. If C_i and C_2 are two categories with functors T : C^ •* C_2 and S : C_2 + C^ we say that T is an <u>adjoint</u>[1] for S, denoted T -f S, iff there is a natural equivalence between C_2[TA,B] and C_1[A,SB] for each A e C_x and each B e C_2. Note that we omit parenthesis wherever possible so that T(A) becomes TA.

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1. The terminology "adjoint[11] is not standard and we have defined an adjoint in the sense of Freyd [8], but a coadjoint in the sense of Mitchell [10]. Our definition is the standard definition of "left adjoint.[11]

As a special case of the preceding definition, suppose C_2 is a subcategory of C_1 and S is the imbedding functor• If T is an adjoint for S, then T is called a <u>reflection</u> and C_2 is a <u>reflective subcategory</u> of C_1 (cf. [11, section 4]). Examples of reflective subcategories are the compact Hausdorff topological spaces in the category of topological spaces (by the Stone-Cech compactification functor) and the complete, regular T_1 Cauchy spaces in the category of Cauchy spaces (see [11, section 6]).

Turning now to filters, let (S,$) be a pre-ordered set; < is a reflexive and transitive relation on S. If B is a subset of S, we shall consider B to be pre-ordered, its order being inherited from S, and we shall suppress specific mention of the order on B. Given a subset B of S we define the

<u>initial closure</u> of B « B~ - {s e S : 3b(b e B & s ^ b)}

and <u>terminal closure</u> of B « B^+ - { s e S : 3b(b e B & b ^ s)}.

The set B is called <u>initial</u> iff B « B~, and B is <u>terminal</u> iff B • B^+. B is called a <u>filter base</u> in S iff every finite subset of B has a lower bound in B. A <u>filter</u> is a filter base B such that B « B^+. We may obtain the notions dual to these by replacing £ by its dual order in the definitions. Thus, for example, a <u>dual filter</u> is a subset D of S such that finite subsets of D are bounded above in D, and D « D~. We note that if B is a filter base (resp. dual filter base), then B* (resp. B~) is the filter (resp. dual filter) generated by B.

The set of all filters in S may be partially ordered by saying that £ is <u>finer than</u> £, denoted £ $ £, for two filters JF and £ iff £ O £ as sets. Dually, we may partially order the dual filters in S by saying that D^ is <u>finer than</u> $D_{2>}$ again denoted D_1 ^ D_2, iff

$D_1 \subset D_2$ as **sets.** We prefer to consistently use <c as the "finer than[11] symbol since it leads to a quite manageable formalism (cf. Kowalsky [9] **and** also the formulas for meet and join below).

We now suppose S to be a complete lattice with least element 0. The filter $\{0\}^+$ is called the null filter and all other filters in S will be called proper filters. The unmodified word "filter[11] will then always refer to a proper filter. The set of all filters of S, including the null filter, is a complete lattice with the join and meet of the filters JF and £ given by

j? v £ - (F V G : F e F and G *t G)*

and £ ^ £ " (F A G : F E F and G e £}.

We pote that F, A G, may well be the null filter even if jF and <S are proper filters.

More specifically, we shall often be dealing with filters on a fixed set E. In this case, S is the power set of E, ordered by inclusion, and filters Jji S are said to be filters on E. The set of all (proper) filters on E will be denoted FE. Given a point x e E we shall denote the filter Ux}}$^+$ by x', and we let DE • {i : x e E), Given two filters £ and (J on E we let][^{x} £ denote the filter on E * E generated by {F x G : F e £ and G e *G).* If $ and Y are filters on E x E, we let t""^{1} - {M""^{1} : M e •} and $of - {M»N : M e 4> and N e ¥}$^+$ where M"^{1} » {(y,x) : (x,y) e M} and

M©N - {(x,z) e E x E : 3y(y *e* E & (x,y) e M & (y,z) e N}.

If f is a mapping from the set E to some other set E^f, we let fE • {f(F) : F e £}"*" for a filter £ on E. The proofs of the following lemmas are omitted.

Lemma 2.1 If £ and, G are filters on E and f : E + E^f, then f(1 V G) « fF V f£, and for any x e E, if y «• f(x), then y « fx.

Lemma 2.2 Xj[$, •', and *" are filters on E * E$ then

Given a filter F on E, we shall frequently be referring to the filter *F - (£ * F) V *A*+» whe« A ls the diagonal of E * E. We have the following result involving filters of this sort.

Lemma 2,3 For filters F and G on E, 4> V 4> $ 4> o $ $ * # r G F £ £ y G

Moreover, if F A G is the null filter, then F • ^G • *p V 0Q •

3. Limit spaces^ uni^ortnJLjinitjpaces, and Cauchy spaces

Definition 3.1 Let E be a set and q a subset of FE x E satisfying:

Lim₁ : (x,x) z q for each x e E

Lim₂ : (F,x) e q and G i F •> (£,x) e q

Lim₃ : (£,x) e q and (G,x) e q •> (F vG, x) E q,

Then q is called a limit[2] structure on E and (E,q) is a limit[2] space. We shall frequently use the usual notation for such a binary relation, writing £ q x instead of (£,x) e q, and saying "£ q-converges to x," or simply, "P converges to x" if there is no confusion.

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2. Limit spaces are called Limesr'dume in [7,9] and convergence spaces in [1,3,A]. The term "convergence space" in [12] and elsewhere is preserved for structures q which need not satisfy condition Lim₁.

Definition 3,2 Let E be a set and J a dual filter of filters on E x E satisfying:

Uni_1 : {A}+ e J

Uni_2 : * e J »> S"1 e J

Uni_3 : $,¥ e J and *°f not the null filter ⇒ $ o Y e J.

Then J is called a uniform limit[3] structure on E and (E,J) is a uniform limit[3] space.

Peflnition 3,3 Let E be a set and C a collection of filters on E satisfying:

Cau_1 : _DE C C

Cau_2 : C+ A C" - C

Cau_3 : £, £, H e C and JF ^ £ and .F £ H «> G V H e C,

Then C is called a Cauchy structure on E and (E,C) is a Cauchy space.

While any topological space defines a limit space, convergence of Mikusifiski operators, and convergence a.e. in the space of measurable real valued functions on [0,1] give rise to limit spaces which are not topological (see [13] and [9]). Those uniform limit structures which are principal (i.e., for which there exists a filter ^ in J such that • e J iff $ £ ¥) have been characterized by Cook and Fischer [A, Theorem 6] as being equivalent to the uniform structures of Bourbaki [2]. Examples of uniform limit structures which are not uniform structures will be plentiful after the uniformization theorem (6.12). The Cauchy filters of any uniform limit space form a Cauchy structure, and again examples will become apparent in section 6.

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3. The terminology is chosen to agree with that used in Definition 3.1 and it differs from that of [4].

In order to provide a unified theory for the three types of structures defined above we note that in each case we have the following **situation:**

3.4 There is a functor E from sets to sets.

3.5 To each set E we associate a collection oE of subsets of EE (called "structures" on E), with 00 • {E0} .

3.6 aE is closed under set intersection. Therefore oE is a complete lattice when ordered by inclusion, and EE e aE.

3.7 If f : E_1 -• E_2 is a mapping and S_2 e *oE^9* then $f^*(S_2)$ £^{f} {s e EE_X : (Ef)(s) e S_2> e *oE±*.

In the special three structures defined above we have:

For limit spaces,

EE - FE x E

(Ef)(F,x) - (fF,fx)

oE is the set of limit structures on E.

For uniform limit spaces,

EE - F(E x E)

(Ef)($) - {fM : M e * }$^+$ where fM * {(fx,fy) : (x,y) e M}

aE is the set of uniform limit structures on E.

For Cauchy spaces,

EE - FE

(Ef)(F) - fF

aE is the set of Cauchy structures on E.

The proofs of 3.6 and 3.7 are routine for each of the three types of structures we are considering, and are omitted.

Given f : E_1 -> E_2 we see that we may define of = f* : aE_2 -• aE_1

by 3#7. It is easily verified that a defines a contravariant functor **from the category** of sets to the category of complete ordered sets (not **to complete** lattices, however, since f* need not preserve joins). **We may define** a **dual** to f* by letting

$$f*<^{s}l> {}^{m} \; D t S \; e \; aE_2 \; : \; Vs(s \; e \; S_{\pm} \; -> \; (Ef)(s) \; c \; S)\}.$$

The following properties are almost immediate:

3.8 f * ^) « S_2 <-> Sj^ $ f*(S_2).

3.9 (gf)* •- g* f* if gf is defined, and $(1_E)^*$ » l_{oE}.

3.10 f* preserves intersections, and f^ preserves unions.

3.11 f*(f*(S_2)) * S_2 and $S_{\pm}$ * f ^ M S ^) .

3.12 f*(f^(f*(S_2))) - f*(S_2) and f*(f*(f *(Sj»)« f^(S^ .

Property 3.8 may be proven directly from the definitions. 3.9 follows from 3.8 and the contravariant character of f*. The remaining properties again follow directly from 3.8.

Two other properties to be used in Cauchy Spaces II are:

3.13 f surjective -> f^ surjective and f* injective

f injective •> f* surjective and f^ injective.

3.14 f*(f*(S_2)) - S_2 if f is surjective, and f ^ M S ^) - Sj if f is injective.

To show the first half of 3.13, we recall that if f is surjective, then fg • $l_{p_{h_2}}$ for some mapping g. But then g* f* » f^ g^ « 1_{aE_2} , which gives the desired conclusion, and the second implication is proven dually, except in the case E^ » 0, and there we use the convention noted in 3.5. Property 3.14 then follows from 3.13 and 3.12.

DgfinjLtJLonL 3^15 Given a function f : E^ •> E_9 between two sets and

given S_1 e oE_1, S_2 *e oE₂*, we say that f is S^Sp-continuous (just **"continuous"** if there is no confusion) iff S^ £ $f^*(S_2)$ and we usually **write f** : $(E_1,S_1) \to (E_2,S_2)$. In the case *E^* « E_2 - E, the identity **map** 1_E : (E,Sj) -* (E,S_2) is continuous iff *S^* C S_2, and in this case we say that S_1 is finer than S_2 (equivalently, S_2 is coarser than S^, denoted S_1 « S_2.

From the properties above we see that we may consider the category of "spaces" where the objects are ordered pairs (E,S), with E a set and S e oE, and the mappings are continuous functions. We can define a subspace of (E,S) to be a pair (E',S') with E' C E and S' * j*(S) where j is the inclusion mapping from E' into E. Dually, a quotient space of (E,S) is an ordered pair (E'',S'') with E" a quotient set of E and S" • q*(S) where q : E •+ E'' is the quotient map.

Given a subset A of E we let [A] denote the structure generated by A, i.e., [A] * *(\{S* : S e *oE* and AC S}. Since we frequently deal with the generator of a structure we will need the following lemma.

Lemma 3^16 If f : E -• E' is a mapping and A C EE, then f^[A]) • [(Zf)(A)]. Thus, f : (E,[A]) + (E',S') is continuous iff (If)(A)CS' .

Proof. [(Ef)(A)] ^ S' <»> (Ef)(s) e S' for all s e A <»> A C$f^*(S')$ <=> [A] ^ $f^*(S')$.

Proposition 3.17 For mappings f : E -• E'' and g : E" -> E', Se oE and S' e *oE'*, the following are equivalent:

(a) gf : (E,S) -• (E',S') is continuous;

(b) f : (E,S) + $(E'',g^*(S'))$ is continuous;

(c) g : $(E'',f^*(S))$ -v (E',S') is continuous;

(d) f*(S) ^ $g^*(S')$.

Proof. Use Definition 3.15 and properties 3.8 and 3.9.

A« The Functors, U arid. T

Limit spaces, uniform limit spaces, and Cauchy spaces form the objects of three categories as we saw in section 3. We denote these categories by .L, 1J, and K^ respectively. The maps of these categories are in each case the continuous functions where continuity is given in Definition 3.15.

Definition 4.1 Let (E,q) be a limit space, (E,J) be a uniform limit space, and (E,C) be a Cauchy space. We make the following definitions:

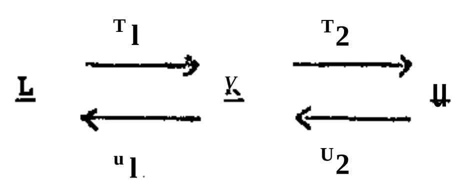
$U_2 J$ - {F e FE : F * F e J}

$U_1 C$ « {(F,x) e F E * E : F V x e C }

T2C * the uniform limit structure on E generated by {*_F : £ t C}

$T_1 q$ « the Cauchy structure on E generated by (F Vx: (F,x) e q},

It is easily verified *that with these definitions we may define functors lh, Uj, T|, and T2 between categories in accordance with the following diagram:



T_3 - T_{2T1}

U_3 = U_{1U2}

We have ^(E.S) - (E.U^S) and ^(E.S) - (E.TjS) for i - 1,2 and $U_\pm$(f) - $T_\pm$(f) - f for f : &X, S^) -* (E_2,S_2>.

Theorem A«2 As defined above. T_i -\ U_i for i - 1,2,3.

Proof. For 1 = 1, suppose f : T_1(E,q) •*• (F,C) is a function. Then f is continuous iff whenever F q x and y - f(x) we have f(£ V x) « f F v y e C and this happens iff f : (E,q) -* U_1(F,C) is continuous. For i « 2, suppose f : T_2(E,C) -• (F,J) is a function. Then f is continuous iff for each £ e C we have f£ x f£ - f(£ x £) e J and this happens iff f : (E,C) •* U_2(F,J) is continuous. The case i * 3 is immediate from the preceding two cases.

Cprpllary 4*3 For structures A and B on the set E, and for i * 1,2,3 we have:

(a) T±A $ B <»> A $ U±B

(b) TJUJB $ B atvd A ^< U±T±A

(c) A « U±B if and only if A « V±T±k.

Proof, (a) follows from the definition of adjoint, and (b) and (c) follow immediately from (a).

5. Axioms for Separation, Uniformity, and Regularity

It is useful to digress somewhat and to provide a list of axioms that may hold in a limit space (E,q). So let £ and <G be two filters on E and let x,y, and z be points of E.

TQ : x q y and y q x *> x • y

T^ : x q y •> x « y

T^ : £ q x and F q y •> x * y

5Q : £ q y ami x q y »> £ q x

s_1 : Z q y and £ q y and £ q x => Z. ^ x

RQ : z q y and z q x »> x q y

R, : £ q x and £ q y «> x q y

R_2 : £ q x «> IT q x

where, in axiom R_2, IT - {rF: F e £}+ and

IT • {z e E : 3 £(G e £E and £ q z and F e G)}.

The first three axioms are readily recognized as beinp the usual separation axioms from general topology expressed in convergence form. Axioms S_o and S_1 express uniformity conditions and will be used in section 6. The R axioms are those of Davis [6]. We note that R_2 is the usual separation axiom T_3, relabeled by Davis in view of its connection with R_1 and RQ to be given below. Biesterfeldt [1] has shown that axiom R_2 is equivalent to the regularity condition of Cook and Fischer [5]. We remark that axiom R_0 implies that the relation R defined on E by x R y <«> $\dot{x}$ q y is an equivalence relation.

Proposition 5.1

T_2 - T_1 - T_0	R_0 and T_O -> T_x
S_i -> S_O	R_1 and T_O -> T_2
"2 "* R_1 0> R_0	R_1 and S_O -> S,

Proof. The only implications not immediate from the definitions are given here. First, SQ »> RQ, for suppose SQ to be satisfied and that $\dot{z}$ q y and $\dot{z}$ q x. Since $\dot{x}$ q x and $\dot{y}$ q y we have $\dot{x}$ q z and $\dot{y}$ q z by SQ. Another application of SQ gives $\dot{x}$ q y so that R_n holds. Secondly, R_2 => R_1, for if q satisfies R_2, then whenever £ q x we also have $\dot{x}$ $ IT (since for each IT e IT we know x e IT because IT q x)• Now suppose that £ q x and F q y (to show: $\dot{x}$ q y). By R_2 we know IT q y and since $\dot{x}$ $ IT we have $\dot{x}$ q y. We omit constructions of counterexamples to absent implications, noting however that in a topological space we do have R_2 => S_n.

6.iv] Uniformization Theorems

Proposition^£.JL Let (E,C) be a Cauchy space. Then U^C is a limit structure on E satisfying axiom SQ.

Proof. Suppose £ v ẏ and ẋ V ẏ are in C. By condition Cau_3 of the definition of a Cauchy structure, since ẏ $ £ v ẏ and ẏ $ ẋ V ẏ, we see that £ V ẋ V ẏ e C. But since ẋ (_F Vẋ < _F Vẋ Vẏ, we have £ V ẋ e C, by Cau_2.

Prpppsltlqn^ 6^2 I£ q is a limit structure on E satisfying axiom ^0» then T^q « {£ V ẋ : £ q x}, rather than just being generated by this set.

Proof. We show that {£ V ẋ : F q x} is a Cauchy structure. Cau_1 and Cau2 are obviously satisfied and so we proceed to verify Cau^. Suppose £ q x, G q y, and H. q z with F V̇x ^ G V̇y and F Vẋ $ H Vż. Then ẋ ^ G Vẏ and hence ẋ q y. Similarly ẋ q z and thus by axiom $S_{Aj>}$ G q x and H q x. Now also we know that ẏ qy and ż q z so, by axiom SQ, ẏ q x and ż q x. Hence (G \^H Vẏ Vż) qx and thus G V H V ẏ Vż Vẋ is in the set {F V ẋ : £ q x} and by condition Cau_2 we easily obtain G VH Vẏ Vż is in this set also.

Theorem 6^3. Let (E,q) be a limit space. There exists a Cauchy structure C on E such that U-iC •» q iff q satisfies axiom S_n.

Proof. Proposition 6.1 shows the condition to be necessary. Sufficiency is a result of Proposition 6.2 and Corollary 4.3.

Proposition 6^t Let (E,J) be a uniform limit space. Then $U_?J$ * $(U_?J)^-$.

Proof. Suppose G £ F for F e U.J. Then F x F e J and since J is a dual filter we see that £ x £ e J so that £ e U_2J.

<u>Definition 6,5</u> A Cauchy structure C on a set E is said to be <u>uniformizable</u> iff there exists a uniform limit structure J on E such that C~ - U_2J.

<u>Theorem 6.6</u> <u>For a Cauchy space</u> (E,C) <u>the following conditions are equivalent</u>!

(a) C <u>is uniformizable</u>;

(b) C~ <u>is a Cauchy structure</u>;

(c) V_F V_G (F,G e C and _F A_G 4 W* •» F_V £ e C) .

<u>Proof</u>* (a) »> (b) follows from Proposition 6.4 and the remarks which follow Definition 4.1 For (b) »> (c), suppose £,£ e C and that £ A £ is not the null filter. Then F ∧.G e C~ and _F A £ ^ Z»£» Since C~ is a Cauchy structure we know that JF V £ e C~ and then an application of condition Cau_2 gives us J£ V j]re C. To show that (c) «> (a), we note that the set {$_ V...V$_ ; n finite and F. e C} forms a dual filter basis of the uniform limit structure T_2C under the hypothesis of (c) by Lemma 2.3. We will show that the Cauchy filters of T_2C are those of C" (i.e., that U_2T^C " O. By Corollary 4.3 we know that C ^ U?T2C so that if jF e C"* then _F, is finer than a filter of U_2T_2C, and hence £ e U_2T_2C since this structure is initial. So suppose £ is a Cauchy filter of T_2C, i.e., H x £ $ *F V...V $ F for a finite number of filters |\,...,F_n in C. Since $_ V 4>_ $ *_ F by Lemma 2.3, we may replace $_V v 4>

L± Ij ^ V ij ^L 4

by Φ F_i – if F.A F. is a proper filter and maintain the same form (for in this case, ^v F._j e C by (c)). So now suppose that F_i A ZA is null filter for all i *i* j. Then there exist n sets A.^ e J^,...,^ *z* J'n which are pairwise disjoint and there exists a set F e £ such that F x F *d* LJ (A, * A.) O A. But if F x F c A then F « {x} for some x e E and then £ « x e C, so that £ e C~. If F x F ^ A then F * F c $\bigcup_{1\leq j\leq n}$ (AJ x A.) and hence F *C* A_i for some particular i. It is now easy to show that £ $ $£_i$ so that £ *z* C" in this case as well.

Corollary 6.7 If E is a finite set and C is a Cauchy structure on E, then C is uniformizable.

Proof. We show that condition (c) of the previous theorem is satisfied. So let £,j5 e C, and let H « £ /\ G. We suppose H is a proper filter and we let H^f • f)(H : H e H}. By the filter property (since there are only finitely many sets in the power set of E), H^f *£* 0, and so let x *z* H^f. Then we have x $ H^ $ £ and x $ TH £ iG, so that by the condition Cau_3 we have £ V £ e C.

We now wish to investigate conditions on a limit space (E,q)» under which there is a uniform limit structure J on E with U^J • q. In such a case we say that q is uniformizable.

Proposition 6.8 If J is a uniform limit structure on E then

U_3^J * U£,x) : F x i e J (,

Proof. It suffices to prove £ x x e J iff (£ v x) x (£ y x) e J. Since £ x £ ≤ (£ v x) x (£ v x) and J is a dual filter we have the "if" part. Conversely, (£ V x) x (£ • x) « (£ x £) y (x * F) y (£ x x) V (x x x) and if £ x x e J we easily have the four filters on the right in J by applying conditions Uni^, Uni_2, and Uni^ of Definition 3.2.

We remark that this proposition shows the functor U_3 to be the same as that (implicitly) defined by Cook and Fischer [A, Theorem 1].

Proposition 6.9 If (E,J) is a uniform limit space, then the limit structure $U_1 J$ satisfies axiom S_1.

Proof. Suppose $\dot{F} \times \dot{y}$, $\dot{G} \times \dot{y}$, and $G \times \dot{x}$ are in J. By condition Uni_2 we have $\dot{y} \times \dot{G} \in J$ and then $F \times \dot{x} = (F \times \dot{y}) * (\dot{y} \times G) * (G \times \dot{x})$, which is in J by condition Uni_3.

Lemma 6.10 Suppose C is a Cauchy structure on E and that C" is also a Cauchy structure (cf. Theorem 6.6), Then $U_1 C = U_1(C^{\sim})$.

Proof. $U_1 C = \{(F,x) : F \vee \dot{x} \in C\} = \{(F,x) : F \vee \dot{x} \subseteq F'$, for some $F' \in C\} = U_1(C^{\sim})$.

Lemma 6.11 If q is a limit structure on E satisfying axiom S_1, then $(T_1 q)^{\sim}$ is a Cauchy structure on E.

Proof, Since q satisfies axiom S_1 and hence S_0, $T_1 q = \{F \vee \dot{x} : F\, q\, x\}$ and this set is a Cauchy structure by 6,2. We note that $(T_1 q)^{\sim}$ is then the set of all filters which q-converge to some point of E. Conditions Cau^1 and Cau^2 obviously hold for $(T_1 q)^{\sim}$ and so for Cau^3 suppose $F \supseteq G$ and $F \supseteq H$ for convergent filters $F\, q\, x$, $G\, q\, y$, and $H\, q\, z$, Then $F\, q\, y$ and $F\, q\, z$ so that $G\, q\, z$ by axiom S_1. Hence $G \vee H$ converges to z by Lim^3 and $G \vee H \in (T_1 q)^{\sim}$.

Theorem 6.12 A limit structure q on a set E is uniformizable iff q satisfies axiom S_1.

Proof. Necessity follows from Proposition 6.9. For sufficiency, if q satisfies S_1, then $T_1 q$ is uniformizable by Lemma 6.11 and Theorem 6.6. So $(T_1 q)^{\sim} = U_2 J$ for some uniform limit structure J. Applying the functor U_1 we have $U_1((T_1 q)^{\sim}) = U_1 U_2 J$ and by Lemma 6.10, $U_1 T_1 q = U_1((T_1 q)^{\sim}) = U_1 U_2 J = U_3 J$. By Theorem 6.3, $q = U_1 T_1 q$ and so we have $q = U_3 J$.

We remark that Corollary 4.3 allows us to assert that q « U$_3$T^q.

Corollary 6JJL3 <u>If the limit structure</u> q <u>on a set</u> E <u>satisfies the</u> '$\frac{1}{2}$ <u>8^{e}P^ration axiom, then</u> q <u>is uniformizable</u>.

Proof. By proposition 5.1, T_9 *=> S_7.

Thus, for example, the real numbers with their usual (metric) topology may be uniformized as a uniform limit space. Letting q denote the convergence given by this topology, we may describe T^$_3$q as follows. For each point p € A let S_p be the set of all squares in the plane with center p. Letting $\$_p$ « {M U A : M e S_p}$^+$, we see that the dual filter generated by {$*_p$: p e A} is T_3q, since •p^{*5}'*^ " $\$_p$ V $\$_q$ by Lemma 2.3.

Corollary 6.13 was first proven by Cochran [3] as a corollary to a sufficient condition for the uniformizability of a limit space.

We conclude with some remarks concerning Kowalsky\s work [9]. Kowalsky introduced an initial set C^ of filters on E with D̲E̲ C $C._K$. A filter J̲F̲ e C_k is called a <u>fundamental filter</u> iff for each £ and H̲ in C^*_K, if F̲ $\$$ G A !i» then £ VH̲ e C_k. He then required that for each F̲e C_k there must exist a fundamental filter J̲F̲^{1} e C^ such that £ $ H*, and that all $\dot{x}$ be fundamental, for x e E. With these definitions it follows that the fundamental filters of C_k form a Cauchy structure C_f in our sense and that C^ » (C^)"". In fact, C_f- is the coarsest Cauchy structure finer than C_{fc}. Furthermore, if C is a Cauchy structure and we let C_k • C"", then every filter I̲[e C is a fundamental filter of C_k, so that C $ $(C")_f$. We might also remark that if f : (E_1,C_1) -> (E_2,C_2) is a continuous function, then fCC^) CC_2", but it is not necessarily true that $f((C_1")_f)$ C$(C_2$~$)_f$. Insofar as Cauchy filters are concerned, the theory of uniform limit structures gives rise to Cauchy structures C in which C - C", and thus one cannot distinguish between Cauchy filters and fundamental filters. That such is not always the case has been remarked by Kowalsky [9, p. 322].

References

[1] Biesterfeldt, H.J.: Regular convergence spaces. Indagationes Mathematicae 2&, 605-607 (1966).

[2] Bourbaki, N.: Topologie g£n6rale. Livre III, Chap. I-II, 4e ed. Paris: Herman Press. 1965.

[3] Cochran, A.: On Uniform Convergence Structures and Convergence Spaces. Doctoral dissertation, University of Oklahoma. 1966.

[4] Cook, C.H., and H.R. Fischer: Uniform convergence structures. Mathematische Annalen 173, 290-306 (1967).

[5] Cook, C.H.,and H.R. Fischer: Regular convergence spaces. Mathematische Annalen 174, 1-7 (1967).

[6] Davis, A.S.: Indexed systems of neighborhoods for general topological spaces* American Mathematical Monthly 68, 886-893 (1961).

[7] Fischer, H.R.: Limesräume. Mathematische Annalen 137, 269-303 (1959).

[8] Freyd, P.: Abelian Categories. New York: Harper and Row. 1964.

[9] Kowalsky, H.J.: Limesr'dume und Komplettierung. Mathematische Nachrichten 12, 301-340 (1954).

[10] Mitchell, B.: Theory of Categories. New York: Academic Press. 1965

[11] Ramaley, J.F., and 0. Wyler: Cauchy spaces II. Regular completions and compactifications. To appear.

[12] Ramaley, J.F.: Completion and Compactification Functors for Cauchy Spaces. Doctoral dissertation, University of New Mexico. 1967.

[13] Wagner, E.: On the convergence structure of Mikusiński operators. Studia Mathematicae 27, 39-48 (1966).

[14] Wallman, H.: Lattices and topological spaces, Annals of Mathematics 39, 112-126 (1938).

[15] Wyler, 0.: On categories in general topology. Technical Report 67-10. Department of Mathematics, Carnegie-Mellon University. 1967.

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