# APPROXIMATION THEOREMS ON SOME CLASSES OF AUTOMATA

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ABSTRACT

This paper considers a machine as a pair (G,M) where G is a group or a semi group and where M is a state space. The first part of the paper called reversible-state machines considers the case where G is a locally compact group and M is any locally compact group and M is any locally compact space. The essential requirement is that $(x,p) \to x(p)$ be continuous where $x \in G$, $p \in M$ and $x(p) \in M$; i.e., we require that the next state function be continuous. The notion of projective limit is discussed and a criterion is given as to when G is the projective limit of some of its quotient groups. Next an infinitesimal element is defined. An identification is then made near the respective identities of G and the set of infinitesimal operations.

The second part of the paper treats the case when G is a so called amenable semi group, having a representation of bounded operators on a Hilbert space. In the case in which the representation is an isometry, weakly continuous, a decomposition theorem is given. On a particular subspace the representation turns out to be a direct sum of finite dimensional operations. Diverse characterizations of that space are given. Next the notion of coordinates of a representation is defined and two orthogonality theorems are stated.

The whole paper might be considered as an attempt at giving approximation theorems on essentially infinite automata.

## I. INTRODUCTION.

A classical way to view a machine is as follows: let M be any set which represents the status of the machine and let G be a group or semi-group of transformations on M. The pair (G,M) constitutes a machine [1], [3], [8], [10]. It is then meaningful to study purely algebraic properties of G [2], [5], [6], [11]. In this paper a machine is considered as a pair (G,M) where G is a topological group or semi-group and M is a topological space. It is then meaningful to talk about topological properties of groups and semi-groups. This paper brings into focus such relevant properties. The techniques are well known to people working in topological groups [4], [7], [9].

## II. "REVERSIBLE-STATE" MACHINES.

### 1. Definitions.

a. The state-space.

Let $M$ be a set with a topology $t$ defined on it. Assume $M$ is locally compact in this topology. $M$ will be called a state-space.

Examples: 1) Let $M$ be finite, and let $t$ be the discrete topology.

2) Let $M$ be a set of n-tuples with the usual topology.

3) Let $M$ be a set of $n \times n$ matrices with the usual norm topology.

b. The tape-group.

Let $G$ be a group of transformations on the space $M$. Suppose $G$ is topologized so as to be locally compact. (I.e. $G$ consists of continuous maps of the space $M$ into itself; $G$ has an algebraic group structure; the group operations are continuous on $G$; moreover, $G$ is locally compact in that topology). $G$ will then be called a tape group on $M$.

c. "Reversible-State" Machines.

A reversible state machine will be a pair $(G,M)$ where $M$ is a state-space and $G$ is a tape-group on $M$. We, in addition shall require that the map $(x,p) \to x(p)$, where $x \in G$ and $p \in M$, be continuous.

d. Physical Interpretation.

A reversible state machine is determined if one knows its set of states and how a series of inputs acts on a particular state. G can be thought of as a set whose elements are a series of inputs with the obvious composition law. Because elements of G are continuous maps on M, this means that "if a series of inputs is slightly modified, then the resulting state is also slightly modified." The topologies on M and G of course give meaning to the concept of "slight" in G and M. A machine of the above type will be called transitive if, given p and q in M, there exists x in G such that $x(p) = q$.

## 2. Quotient State Spaces.

a. Let H be a closed subgroup of G (not necessarily normal). Consider the map $\pi$ which maps any element x of G into the set xH (all elements of form x multiplied by an element of H). Pick the open sets in G/H to be the ones whose inverse by $\pi$ are open in G. Then $\pi$ is a continuous map of G into G/H. Moreover, $\pi$ maps open sets of G into open sets of G/H. G/H is a locally compact Hausdorff space. All these statements follow by definition.

b. We can state the following:

Theorem:

Let H be a closed subgroup of G, then (G,G/H) is a transitive, reversible-state machine.

Before proving this theorem, let us give a physical interpretation. Starting with a machine which has a given state space it is possible to construct in general a new state space on which $G$ is a tape group. Moreover, the new state space is related to the first state space as follows: If two tape elements $x_1$ and $x_2$ are such that $x_1$ can be obtained by composition of $x_2$ with an element of $H$ then $\pi(x_1)$ and $\pi(x_2)$ are identical states in the new state space.

Proof:

Define $\rho_y(xH) = yxH$.

Then, given $x_1H$ and $x_2H$,

$\rho x_2 x_1^{-1}(x_1H) = x_2H$ so the machine is transitive.

Now $(x,y) \to xy$ is continuous and $(xy) \to (xy)H$ is continuous. The map $(x,y) \to (x,yH)$ preserves open sets hence $(x,yH) \to (xy)H$ is continuous. Q.E.D.

c. Consider the machine $(G,M)$. Let $p$ be an element of $M$. Let $G_p$ be the set of tape elements of $G$ which leave the state $p$ invariant. Clearly $G_p$ is a closed group. We can then form the state space $G/G_p$. Now we can define a natural map $\psi\ G/G_p \to M$ by $\psi\ (xG_p) = x(p)$: Now let us assume that $(G,M)$ is a transitive machine. What can we say about $\psi$?

Let us verify first of all that $\psi$ is a well defined map. If $xG_p = yG_p$ then $y^{-1}xG_p$ so $y^{-1}x(p) = p$ so $x(p) = y(p)$. This also shows that $\psi$ is 1:1. $\psi$ is onto $M$ because we have a transitive machine. Let $\varphi(x) = x(p)\ (x \in G)$. By definition, $\varphi$ is continuous. $\pi : G \to G/G_p$ is an open map, hence $\psi$ is a continuous map. If we know that $\varphi$ is open, then $\psi$ is a homeomorphism (i.e. the spaces $M$ and $G/_{G_p}$ are topologically equivalent). It is natural then to ask the following question: under what conditions on $G$ and $M$ is $G/_{G_p}$ homeomorphic to $M$?

Theorem:

Suppose $(G,M)$ is a transitive machine. Assume $G$ can be covered by countably many translates of each neighborhood of the identity. (This is the case if $G$ is separable). Then $\psi$ is a homeomorphism.

Note that this theorem essentially gives a criterion as to when $G/G_p$ is the same state space as $M$. The state space $G/G_p$ is obtained from $G$ and $M$ as follows. Each $x \epsilon G$ is mapped in such a fashion that tape elements which 'cancel' each other out on the state $p$ are mapped into the same element. Then the natural quotient topology is introduced on that new set. Now we proceed to the proof of the theorem:

Proof:

We remark that a locally compact space is not a countable union of nowhere dense sets.

Now we shall prove that $x \to \varphi(x) = x(p)$ is an open map. Let $x_0 \epsilon V \leq G$ where $V$ is open. We will show that $\varphi(V)$ contains an open neighborhood about $\varphi(x_0)$. Pick a compact neighborhood $U$ of $G$ such that $U = U^{-1}$ and $x_0 U^2 \leq V$. By hypothesis there exists a sequence $\{x_n\}$ such that $G = \bigvee_n x_n U$ whose image by $\varphi$ is $\bigvee_n \varphi(x_n U) = M$. By the above remark one of the $\varphi(x_n U)$ is not nowhere dense, so contains an open set. Thus $\varphi(U)$ has an interior point $U_0 p$ $(u_0 \epsilon U)$.

So $\varphi(V)$ has $x_0(p)$ as an interior point. Q.E.D.

## 3. Decomposition Theorem.

a. Consider the tape-group G let $\rho_\alpha$ be a continuous, open homomorphism of G onto $G_\alpha$ which is also a tape group. Let $k_{er}\rho_\alpha$ = the kernel of $\rho_\alpha$. The problem we shall consider is the following: What is G topologically equivalent to the projective limit of the $G_\alpha$? A criterion of this problem will give us a way to look at the tape group G as a projective limit of homomorphic tape groups, the projective limit being topologically equivalent to the group tape G.

b. Now let A be a directed set (i.e. for $\alpha,\beta$ in A there exists $\lambda\epsilon A$ such that $\lambda > \alpha, \lambda > \beta$). To each $\alpha$ associate a tape group $G_\alpha$ (in this paragraph we are not interested in the state-spaces). To each pair $\alpha,\beta$ of A such that $\beta > \alpha$ we associate an open homomorphism $\pi_{\beta\alpha}$ of $G_\beta$ on $G_\alpha$ satisfying: if $\alpha < \beta < \lambda$ then $\pi_{\lambda\alpha} = \pi_{\lambda\beta}\pi_{\beta\alpha}$ (the convention here is to read from left to right). Form $\tilde{G} = \pi_{\alpha\epsilon A} G_\alpha$ (i.e. G is the product group with the natural topology on it). Now let $G = \{x\epsilon G/X_\alpha = \pi_{\beta\alpha}(x_\beta)$ whenever $\beta > \alpha\}$ G is a subgroup of G. Let us topologize G by the relative topology respectively to G. It is easy to check the following:

1) An open basis at $e\epsilon G$ is defined by the sets of the form $\{x = \{x_\beta\}/x_\alpha\epsilon$ open neighborhood of e in $G_\alpha\}$ ($\alpha$ fixed)

2) G is closed in $\tilde{G}$.

Now let $\varkappa_\alpha$ be the projection of G in $G_\alpha$.

If $\varkappa_\alpha$ is onto $G_\alpha$, then G is the projective limit of the tape groups $G_\alpha$. (Note that $\varkappa_\alpha$ is an open map).

<u>Theorem</u>:

Let $G$ be a tape-group. $N_\alpha$ where $\alpha \in A$ is a collection of normal subgroups of $G$ satisfying the following conditions:

1) If $\alpha, \beta \in A$ then there exists $\gamma \in A$ such that $N_\gamma \leq N_\alpha \wedge N_\beta$.
2) If $U$ is any neighborhood of $e$, there exists $\gamma \in A$ such that $N_\gamma \leq U$.
3) At least one of the $N_\alpha$ is compact.

Then we can form $G'$ the tape group which is the projective limit of the $G_\alpha = G/N_\alpha$.

Further let $\Phi$ map $G$ into $G'$ by $\Phi(g) = \{\pi_\alpha(g)\}$ then $\Phi$ is an isomorphism onto $G'$.

.Let us comment on this theorem before giving the proof. We have here a criterion as to 'how thin' we must choose a set of tape subgroups such that the tape group $G$ may be reconstituted as a projective limit of its quotient tape groups. $G$ will be topologically equivalent to this projective limit. The physical interpretation of the projective limit is tentatively the following:

For each $\alpha$ form the tape group $G/N_\alpha$. Then consider the state space which is the cartesian product of the state spaces of $G/N_\alpha$. Then a state in that product is given as a sequence of states. The $\alpha^{th}$ element of the sequence represents a state of the state space of $G/N_\alpha$. Then topologically the tape group can be considered as acting on that new state space. The transformation of such a sequence is done 'component wise', i.e. $g$ is identified with $\{g_\alpha\}$ and

$$g(p_1, p_2, \ldots p_\alpha \ldots) = (g_1(p_1), g_2(p_2) \ldots g_\alpha(p_\alpha), \ldots).$$

Now we prove the theorem:

Proof:

First of all $\Phi$ is 1:1 and continuous. The verification is trivial. Now we have to show that $\Phi$ is open. Let $g \in G$ and $g \in V$ which is open in G.

Pick a neighborhood U of e such that $gU^2 \leq V$.

Select $\alpha \in A$ such that $N_\alpha \leq U$.

Consider $\{x \in G' / x_\alpha \in \pi_\alpha(gU)\}$.

This is an open set in the relative topology for the range of $\Phi$. Indeed, if this set contains $\Phi(h)$ we will show that $h \in V$.

We have $\pi_\alpha(h) \in \pi_\alpha(gU)$, so $h \in gUN_\alpha \leq gU^2 \leq V$.

Now if we show that Range $\Phi = G'$ we will have proven the theorem.

Let $x = \{x_\alpha\} \in G'$, $x_\alpha \in G/N_\alpha$, $x_\alpha = s_\alpha N_\alpha (s_\alpha \in G)$.

Consider $\alpha_1, \ldots, \alpha_n$. Pick $\beta > \alpha_i$ for $i = 1, \ldots, n$.

Then $N_\beta \leq N_{\alpha_i}$

so $s_\beta N_\beta \leq S_{\alpha_i} N_{\alpha_i}$ (because $\pi_{\beta\alpha_i}(x_\beta) = x_{\alpha_i}$).

Because $N_\alpha$ is compact for some $\alpha$, there exists $g \in \bigwedge_\alpha s_\alpha N_\alpha$ therefore $\pi_\alpha(g) = x_\alpha$. Q.E.D.

## 4. Infinitesimal tape elements.

a. In this paragraph V will denote a finite dimensional vector space/R. M(V) = all linear transformations which map V into V. GL(V) = all non singular linear transformations of M(V). Hence GL(V) is a group for multiplications. Define

$$\exp(T) = \sum_{k=0}^{\infty} \frac{1}{k!} T^k$$

where

$$T \in M(V).$$

Then it is easily seen that exp. maps topologically a neighborhood of O in M(V) onto a neighborhood of I in GL(V). Physically we will keep in mind that V is the state space, and GL(V) is a group-tape on V. Now let G be a closed subgroup of GL(V). $x \in M(V)$ is called an infinitesimal operation on V if there exists a sequence

$$A_n \in G, \epsilon_n \searrow 0 \quad \text{and} \quad \frac{A_n - I}{\epsilon_n} \quad X.$$

Convergence here is defined in the norm sense of M(V). L(G) is the set of all infinitesimal operations (respectively to G). (Note that L(G) is nothing else but the Lie algebra of G). Now if G is a closed subgroup of GL(V) and $x \in L(G)$, then $\exp(x) \in G$. This is a trivial conclusion using the fact that G is closed and that $A_n^{[\frac{1}{\epsilon_n}]} \longrightarrow \exp(x)$ where $[1/\epsilon_n]$ is the smallest integer above $1/\epsilon_n$.

Now in L(G) introduce a new operation [A,B] = AB - BA. Where A and $B \in L(G)$ it can be shown then that L(G) is a linear subspace of M(V) closed under [ , ] operation.

b. <u>Theorem</u>.

exp maps topologically some neighborhood of O in L(G) onto some neighborhood of I in G.

Let us comment now on this theorem. Here we are interested in tape elements which are not necessarily part of a tape group, i.e., the infinitesimal elements. The action of two infinitesimal elements on the state space results in the usual vector addition. On the other hand, we are interested in the action of a tape group.

The theorem says that if the tape elements and infinitesimal operations are close enough to the respective identities, then the infinitesimal operations are topologically equivalent to the tape elements. (This is really a local identification near the identities).

We prove this theorem in the following manner:

Proof:

All we have to show is that exp is an open map. So assume that $\exp[L(G)]$ does not contain a neighborhood of $I$ in $G$. Let $N$ be the complementary subspace to $L(G)$ in $M(V)$.

$$M(V) = L(G) + N$$

$$N \cap L(G) = (0)$$

So there exists $\{A_n\}$ such that:

$$A_n \in G \quad \text{for all} \quad n$$

$$A_n \to I$$

$$\text{Log } A_n \notin L(G) \quad \text{for all} \quad n$$

$$\text{Log } A_n = x_n + y_n$$

$$x_n \in L(G)$$

$$y_n \in N$$

$y_n \neq 0$ for all $n$, for if $y_n$ could equal zero then

$$\text{Log } A_n \in L(G), \text{ a contradiction.}$$

$$\text{Log } A_n \to 0$$

so

$$x_n \to 0 \quad \text{and} \quad y_n \to 0.$$

Consider $y_n/\|y_n\|$. The unit sphere is compact, so some subsequence converges to $y$. Therefore $\|y\| = 1$.

$y \in N$, so $\exp \lambda y \neq I$ for small $\lambda$, because for small values of $\lambda$, exp is 1:1; and since $\lambda y \neq 0$, $\exp \lambda y \neq I$. If now we show that $y \in L(G)$ we will have demonstrated the contradiction.

$$\left\| \frac{1}{\|y_n\|} \{A_n - \exp x_n - y_n\} \right\| = \frac{1}{\|y_n\|} \|\exp(x_n + y_n) - \exp x_n - y_n\|$$

$$= \frac{1}{\|y_n\|} \left\| \sum_{k=2}^{\infty} \frac{1}{k!} ((x_n + y_n)^k - x_n{}^k) \right\|$$

$$\leq \frac{1}{\|y_n\|} \sum_{2}^{\infty} \frac{1}{k!} [(\|x_n\| + \|y_n\|)^k - \|x_n\|^k]$$

$$= \frac{1}{\|y_n\|} [\exp(\|x_n\| + \|y_n\|) - \exp\|x_n\| - \|y_n\|]$$

$$= \exp\|x_n\| \rightarrow 1 \quad \frac{\exp\|y_n\| - 1}{\|y_n\|} \rightarrow 0$$

$$\overline{\qquad\qquad}$$
$$0$$

$$\lim \frac{y_n}{\|y_n\|} = \lim \frac{A_n - \exp x_n}{\|y_n\|}$$

$$= \lim \exp x_n \left( \frac{(\exp(-x_n) A_n - I}{\|y_n\|} \right.$$

$$= \lim \frac{\exp(-x_n) A_n - I}{\|y_n\|} \in L(G)$$

so $y \in N \wedge L(G)$ so $y = 0$; contradiction. Q.E.D.

c. <u>Theorem</u>.

Let $G_1$ and $G_2$ be closed subgroups of $GL(V_1)$ and $GL(V_2)$. Let $\varphi$ be a continuous homomorphism of $G$ into $Q$. Then there exists a homomorphism $d\varphi$ of $L(G)$ into $L(Q)$ such that

$$\begin{array}{ccc} L(G) & \xrightarrow{d\varphi} & L(Q). \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow[\varphi]{} & Q \end{array}$$

# III. REPRESENTATION THEOREMS FOR 'NON-REVERSIBLE MACHINES'

## 1. Definitions.

### a. Non-reversible machine.

A 'non-reversible machine' will be defined as a pair (G,M) where G is an 'amenable' semi-group; i.e., G has the algebraic structure of monoid (it is closed under composition and the identity is present). Let C(G) be the set of all bounded and continuous functions. G is said to be left amenable if there exists a linear functional defined on C(G) which is positive, left invariant, and normalized; the same definition applies for right- nable. G is amenable if it has a left and a right mean. (Note that the left mean is not necessarily equal to the right mean.) Formally, a non-reversible machine will then be the pair (G,M) where G is a semi-group with identity and where $\lambda_1$, $\lambda_2$ are linear functionals such that

$\lambda_i(I) = 1$ $(i = 1,2)$ (I = the identity element of C(G)

$\lambda_1 f(x) = \lambda_1 f(ax)$ for all $a \in G$ $(f \in C(G))$ $(x \in G)$

$\lambda_2 f(x) = \lambda_2 f(xb)$ for all $b \in G$

$\lambda_i f \geq 0$ if $f \geq 0$ $(i = 1,2)$

Examples: Any abelian semi group is amenable.

Any compact group is amenable.

### b. Almost convergence.

We will define now a so-called 'almost convergence'. Let $f \in C(G)$ and let $\alpha$ be a number then we say f converges almost to $\alpha$ if $\lambda f = \alpha$ for all $\lambda$ where $\lambda$ is a mean (left or right).

We denote this fact as $f \xrightarrow{(a)} \alpha$. It can be shown that if $G = \{1,2,3,\ldots\}$, in order that $f \in C(G)$ satisfies $f \xrightarrow{(a)} \alpha$ it is necessary and sufficient that

$\frac{1}{n} \sum_{k=1}^{n} f(k+m) \rightarrow \alpha$ uniformly in $m$. (Ergodic theorem).

Now we assume that we have a Hilbert space $h$, and that the tape semi-group $G$ has a representation in $L(h)$ as a set of bounded linear operators on $h$; i.e., if $x \in G$ then $x \rightarrow T_x \in L(h)$ and $T_{xy} = T_x T_y$.

Roughly speaking we assume now that each tape element (which is an element of a semi-group $G$) can be represented as a linear operator over the Hilbert space $h$. ($h$ in general is infinite dimensional.) We make the hypothesis that the semi-group $G$ is amenable. We want to study the properties of such a representation, and hence, hopefully, have a different way to look at elements of $G$. A representation $T_x$ is called bounded if $\|T_x\| \leq k$ for all $x \in G$. A representation $T_x$ is called weakly continuous if $x \rightarrow (T_x \zeta, \varkappa)$ is a continuous function on $G$ for all $\zeta, \varkappa \in h$.

## 2. Bounded, Isometric Representations.

### a. Theorem.

Let $u_x$ be a bounded, weakly continuous representation of $G$ on $h$. Then there exists $Q \in L(h)$ such that

$$(u_x \zeta, \varkappa) \xrightarrow{(a)} (Q\zeta, \varkappa) \text{ for all } \zeta, \varkappa \in h.$$

Moreover if $u_x$ is an isometry, $Q$ = projection on $\{\zeta / u_x \zeta = \zeta$ for all $x \in G\}$. We proceed to prove the theorem as follows:

Fix $\lambda$ a left or right mean. Then there exists $Q_x$ such that $\lambda(u_x\zeta,\varkappa) = (Q_\lambda\zeta,\varkappa)$ for all $\zeta,\varkappa\in h$. This is merely the Riez theorem.

Observe that $Q_\lambda\zeta\in$ closed convex hull of $\{u_x\zeta/x\in G\}$ (use the separation property for convex sets).

Let now $\lambda_1$ be a right mean. Claim $Q_{\lambda_1}u_x = Q_x$. Indeed,

$$\begin{aligned}(Q_{\lambda_1}u_x\zeta,\varkappa) &= \lambda_1(u_y\zeta,\varkappa)\\ &= (Q_{\lambda_1},\zeta,\varkappa).\end{aligned}$$

If $\lambda$ is a left mean it is equally trivial to check that $u_xQ_x = Q_x$.

Now to show that $Q_\lambda = Q_{\lambda_1}$. By the same token we will have shown that $Q_\lambda$ is independent of $\lambda$.

Consider $\zeta\in h$

$x_1,\ldots x_n\in G$

$\alpha_i,\ldots\alpha_n$ such that $\sum_1^n \alpha_i = 1$

$$\begin{aligned}Q_{\lambda_1}(\Sigma\alpha_i u_{x_i}(\zeta)) &= \Sigma\alpha_i Q_{\lambda_1}u_{x_i}\zeta\\ &= \Sigma\alpha_i Q_{\lambda_1}\zeta\\ &= Q_{\lambda_1}\zeta\end{aligned}$$

Since $Q_\lambda\zeta\in$ closed convex hull of $\{u_x\zeta/x\in G\}$, then $Q_\lambda\zeta$ or $Q_{\lambda_1}Q_\lambda = Q_\lambda$.

Now
$$\begin{aligned}(Q_{\lambda_1}Q_\lambda\zeta,\varkappa) &= \lambda_1(u_xQ_\lambda\zeta,\varkappa)\\ &= (Q_\lambda\zeta,\varkappa)\end{aligned}$$

so

$$Q_\lambda, Q_\lambda > 0$$
$$Q_{\lambda_1} = Q_\lambda.$$

Now to show the second part of the theorem. Let E be the projection described in the theorem.

$$\begin{aligned}(Q\zeta, E\varkappa) &= \lambda[u_x\zeta, E\varkappa) \\ &= \lambda[Eu_x\zeta, \varkappa] \\ &= \lambda(u_x E\zeta, \varkappa) \\ &= (E\zeta, \varkappa)\end{aligned}$$

so $EQ = E$.

Now let us check that $EQ = Q$. This will show $E = Q$.

For all $x \in G$ we have $u_x Q\zeta = Q\zeta$. $Q\zeta$ fixed under $u_x$ implies $Q\zeta \in$ range of E. So $EQ\zeta = Q\zeta$ therefore $EQ = Q$. Q.E.D.

b. <u>Theorem</u>.

There exists a positive self-adjoint linear map $\Phi$ of $G_{h,s}$ into $G_{h,s}$ such that

1) $(Au_x\zeta, u_x\varkappa) \xrightarrow{(a)} (\Phi(A)\zeta, \varkappa)$ (for all $A \in G_{h,s}$)

2) $\Phi(A)u_x = u_x\Phi(A)$

3) $\Phi \geq 0$ and self adjoint.

I.e., $A^* = A \quad \Phi(A) = \Phi(A^*)$.

Let us explain some of the terminology and conditions involved.

First of all $u_x$ is assumed to be a weakly continuous isometric representation of G.

$G_{h,s}$ represents the so-called Hilbert-Schmidt class of operators, i.e. $A \in G_{h,s}$ means $A : h \to h$ and if $e_\alpha$ is a complete orthonormal basis for h, then

$$\sum_\alpha \|Ae_\alpha\|^2 < \infty .$$

Here $\sum_\alpha$ means sup over all finite sums. It can be shown that $G_{h,s}$ is closed under * (adjoint) and is an ideal in $L(h)$. $G_{h,s}$ is also a Hilbert space in its own right under the inner product

$$[A,B] = \sum_\alpha (Ae_\alpha, Be_\alpha)$$

(here $\sum_\alpha$ makes sense since only a countable number of terms are $\neq 0$). Consider $G_{h,s}$ as a Hilbert space (with [ , ] as an inner product). Consider the map

$$x \rightarrow U_x^* AU_x = T_x(A) \quad (A \in G_{h,s})$$

$$\begin{aligned}\|T_x(A)\|^2 &= [U_x^*AU_x, U_x^*AU_x] \\ &= \sum_\alpha \|U_x^*AU_x e_\alpha\|^2 \\ &= \sum_\alpha \|Au_x e_\alpha\|^2 \\ &\leq \|\sum_\alpha Ae_\alpha\|^2 \\ &= [A,A].\end{aligned}$$

So $T_x$ is an operator on $G_{h,s}$ of norm less than or equal to 1.

$$\begin{aligned}T_{xy}(A) &= [(U_{xy})^*AU_{xy}] \\ &= (U_y^*(U_x^*AU_x)U_y \\ &= T_yT_x(A)\end{aligned}$$

Hence $T_\alpha$ is a bounded anti-rep of $G$ on $G_{h,s}$.

Claim: This anti-rep is weakly continuous. It suffices to show that $x \rightarrow [T_xA,B]$ is continuous for $A$, and that $B \in$ dense subspace of $G_{h,s}$; i.e., for finite rank.

So let

$$A\zeta = (\zeta,\sigma)\sigma$$

$$B\zeta = (\zeta\tau)\tau'$$

$$\sigma,\ \sigma',\ \tau,\ \tau' \epsilon h$$

$$[T_x A,B] = \sum_\alpha (U_x^* A U_x e_\alpha, B e_\alpha)$$

$$= \sum_\alpha ((U_x e_\alpha,\sigma)U_x^*\sigma', (e_\alpha,\tau)\tau').$$

Claim: This series converges uniformly in x. Let $\epsilon > 0$. There exists a finite set, call it f such that

$$\left| \sum_{\alpha \notin f} (U_x e_\alpha,\sigma)(e_\alpha,\tau)(U_x^*\sigma',\tau') \right|^2$$

$$\leq |(U_x^*\sigma',\tau')|^2 \sum_{\alpha \notin f} |(U_x e_\alpha,\sigma)|^2 \sum_{\alpha \notin f} |(\tau_\zeta e_\alpha)|^2$$

$$|(U_x^*\sigma',\tau')|^2 \leq \|\sigma'\|^2 \ \|\tau'\|^2$$

$$\sum_{\alpha \notin f} |(U_x e_\alpha,\sigma)|^2 \leq \|\sigma\|^2$$

$\sum_{\alpha \notin f} |(\tau_\zeta e_\sigma)|^2$ can be rendered $\leq \epsilon/\|\sigma\|^2\|\sigma'\|^2\|\tau'\|^2$.

Given any $\epsilon > 0$, such an f can be chosen. This proves the claim. $[T_x A,B]$ is hence weakly continuous as a uniform limit. By the previous theorem, there exists a bounded operator $\Phi$ such that

$$[T_x A,B] \xrightarrow{(a)} [\Phi(A),B].$$

Choose B of the form

$$B\rho = (\rho,\zeta)\varkappa$$

$$\|\zeta\| = 1.$$

Imbed $\zeta$ is an orthonormal basis with $\zeta = e_{\alpha_0}$

$$[\Phi(A),B] = \sum_\alpha (\Phi(A)e_\alpha, Be_\alpha)$$

$$= (\Phi(A)\zeta,\varkappa)$$

Likewise

$$[T_x A,B] = (U_x^* A U_x \zeta, \varkappa)$$

We have $(U_x^* A U_x \zeta, \varkappa) \xrightarrow{(a)} (\Phi(A)\zeta, \varkappa)$ for all $\zeta$, $\eta$.

This proves 1).

Now let $\lambda$ be a right mean:

$$(\Phi(A)_y \zeta, U_y \varkappa) = \lambda(U_x^* A U_x U_y \zeta, U_y \varkappa)$$

$$= \lambda(U_x^* A U_x \zeta, \varkappa)$$

$$= (\Phi(A)\zeta, \varkappa)$$

so

$$U_y^* \Phi(A) U_y = \Phi(A)$$

This shows 2).

Now let $A \geq 0$.

Then $(AU_x \zeta, U_x \zeta) \geq 0$ for all $x, \zeta$

$$\lambda(U_x^* A U_x \zeta \ \zeta) \geq 0$$

so

$$(\Phi(A)\zeta, \zeta) \geq 0.$$

So $\Phi$ is a positive map, and 3) is proven.

3. <u>Peter-Weyl Theory on Amenable Semi-Groups</u>.

a. In this paragraph $u_x$ will be assumed to be an isometric, weakly continuous representation of G in L(h). We will state the essential results. The representation $u_x$ will be said to be reducable over h, if there exists a subspace m of h (non-trivial) such that $u_x(m) \leq m$ and $u_x(m^{\perp}) \leq m$ for all $x \in G$.

Theorem.

Let $\varkappa = \{\zeta \in h / \Phi(A)\zeta = 0$ for all $A \in G_{h,s}\}$ ($\Phi$ as defined above). Then the following statements hold:

1) $\varkappa$ reduces the representation.

2) $u_x|_{\varkappa}$ (restricted to $\varkappa$) has no finite dimensional subrepresentation.

3) $u_x|_{\varkappa}$ is the direct sum of finite dimensional subrepresentations.

This theorem may be regarded as a sort of localization theorem. It states how $u_x$ behaves on a certain subspace of $h$, and namely $u_x$ is a direct sum of finite dimensional subrepresentations on $\varkappa$. The theorem ties $\varkappa$ and the Hilbert-Schmidt class of operators.

In the proof we shall make use of the following elementary fact: let $m$ be a proper closed subspace of the Hilbert space $h$; then the rep reduces $m$ if and only if $U_x$ for each $x$ commutes with $P$ which is the associated projection on $m$. By Zorn, let $P_\alpha$ be the maximal collection of non zero, finite dimensional, orthogonal projections each commuting with all $U_x$. (This family could be empty.)

Let $P = \sum_\alpha P_\alpha$.

Then $P$ commutes with all $U_x$.

Let $m$ be the space on which $I - P$ projects.

Claim: $U_x$ restricted to $M$ has no finite dimensional subrep. In fact, let $Q \neq 0$ be a finite dimensional projection which commutes with all $U_x$.

Claim: $Q \leq P$. If not $Q(I - P) \neq 0$ so $(I - P)Q(I - P) \neq 0$ and $\in G_{h,s}$ since the latter is a two-sided ideal and any operation of finite rank $\in G_{h,s}$. This operation is self adjoint and compact, so by the spectral representation:

$$(1 - P)A(1 - P) = \sum_{\lambda \neq 0} \lambda P_{\lambda}$$

where $P_{\lambda}$ are finite dimensional, orthogonal projections. It follows that each $P_{\lambda}$ commutes with $U_x$ and at least on $P_{\lambda} \neq 0$; also $P_{\lambda} \leq I - P$. This contradicts the maximality of $\{P_{\alpha}\}$.

Now the claim is that Range $(I - P) = \varkappa$. The theorem then will be established. Suppose $\zeta \in$ Range $(I - P)$ and $\Phi(A)\zeta \neq 0$ for some $A \in G_{h,s}$. Without loss of generality we may assume that $A$ is self adjoint. We have seen that $\Phi(A)$ commutes with all $U_x$ and is a non-zero, self-adjoint, compact operator (compact because it is the uniform limit of operators of finite rank).

Hence

$$\Phi(A) = \Sigma \lambda P_{\lambda}$$

where for some $\lambda \neq 0$, $P_{\lambda}\zeta \neq 0$.

As before $P_{\lambda}$ commutes with all $U_x$ so $P_{\lambda}U_x$ is a finite dimensional subrepresentation of $U_x$.

Now Range $P_{\lambda} \leq$ Range $P$ (by maximality)

$P\zeta \neq 0$ since $P_{\lambda} \leq P$

$(1 - P)\zeta \neq \zeta$

so $\zeta \notin$ Range $(I - P)$ which is a contradiction.

Conversely let $\zeta \notin$ Range $(I - P)$ then there exist $\alpha$ such that $P_{\alpha}\zeta \neq 0$, since $P_{\alpha}$ is finite dimensional then $P_{\alpha} \in G_{h,s}$ and $P_{\alpha}$ commutes with all $U_x$ so:

$$(\Phi(P_\alpha)\zeta,\eta) = \lambda(P_\alpha U_x\zeta, U_x\eta)$$
$$= (U_x^{*}P_\alpha U_x\zeta,\eta)$$
$$= \lambda(P_\alpha\zeta,\eta)$$
$$= (P_\alpha\zeta,\eta)$$

so $\Phi(P_\alpha) = P_\alpha$

therefore $\Phi(P_\alpha)\zeta = P_\alpha\zeta \neq 0$

and $\zeta \notin m$.

In the process we have proven the following:

Theorem.

$\eta$ R Range of $1 - P$ where $P$ is the sup of all projections (finite dimensional) which commute with all $U_x$ as $x \in G$.

Theorem.

If $m_\lambda = \{\zeta \in h / \lambda|(U_x\zeta, t)| = 0$ for all $t \in h\}$ then $\eta = m_\lambda$.

Say $\zeta \in \eta$, then $(\Phi(A)\zeta; t) = 0 = \lambda(Au_x\zeta, u_x t)$ for all $A \in G_{h,s}$.

Take $A\zeta = (\zeta,\rho)\sigma$

$$0 = \lambda[(u_x\zeta,\rho)\overline{(u_x t,\sigma)}]$$

put $t = \zeta$, $\rho = \sigma$

then $\lambda(u_x\zeta,\rho)^2 = 0$ for all $\rho$.

$$\lambda|(u_x\zeta,\rho))|1 \leq \sqrt{\lambda|(u_x\zeta,\rho)|^2} \quad \sqrt{\lambda(1)^2} = 0$$

so $\zeta \in m_\lambda$.

Now $\zeta \in m_\lambda$ to show $\zeta \in \eta$

$$\lambda|(u_x\zeta,t)| = 0 \text{ for all } t$$

$$|\lambda(u_x\zeta,t)(u_x t,\sigma)| \leq \lambda|(u_x\zeta,t)||(u_x t,\sigma)|$$
$$\leq \|t\|\|\sigma\|\lambda|(u_x\zeta,t)| = 0$$

if $A\zeta = (\zeta,\sigma)\sigma$ then $(\Phi(A)\zeta, t) = 0$ so $\Phi(A)\zeta = 0$.

Hence $\Phi(A)\zeta = 0$ for any operator of finite rank; hence, because the latter set is dense in $G_{h,s}$, $\Phi(A)\zeta = 0$ for all $A \in G_{h,s}$.

In case $G$ is a compact group $\eta = (0)$ and we have the statement of the Peter-Weyl theorem for compact groups. A representation is called unitary if $u_x$ is unitary for all $x \in G$. (i.e., $u_x^* u_x = I$.)

Now let us consider the situation where the tape semi-group has different representations in different Hilbert spaces. The following theorems will tell us how this affects the representations.

<u>Theorem</u> (Schur).

Let $u_x$ and $v_x$ be continuous, finite dimensional, irreducable, unitary representation of a amenable semi-group on Hilbert spaces $L(h_u)$ and $L(h_r)$. If $T$ is any Hilbert-Schmidt operator from $h_u$ to $h_r$, then there exists $\Phi$ which is a linear operator on the space of Hilbert-Schmidt operators from $h_u$ to $h_r$ such that

$$\Phi(T)u_x = V_x \Phi(T)$$

$$(Tu_x\zeta, V_x\eta) \xrightarrow{(a)} (\Phi(T)\zeta, \eta) \text{ for all } \zeta \in h_u \text{ and all } \eta \in h_r.$$

Moreover if $u_x$ is a representation non-equivalent to $v_x$, then $\Phi(T) = 0$.

If $u_x = v_x$ then $\Phi(T) = \frac{1}{n}$ (trace $T$) $I$ where $n = \dim h_u$.

b. Let us define now the term coordinate. A coordinate (respective to a given representation) is a function belonging to $C(G)$ of the form $(u_x h'', h')$ where $h''$ and $h'$ are fixed vectors of $h$.

Theorem.

Coordinates of non-equivalent representations are orthogonal respectively to any mean, i.e.

$$(u_x h, h'')\ \overline{(v_x k, k')} \xrightarrow{(a)} 0.$$

Now consider $h_u$ a finite dimensional vector space. Then let $e_1, \ldots e_n$ be an orthogonal basis for $h_u$. Then

Theorem.

$(u_x e_j, e_i)$ are orthogonal respectively to any mean. $(1 \leq i,j \leq n)$.

Conclusion.

We have now decomposition and orthogonality theorems for certain types of automata, essentially for these where the underlying semi-group of inputs is a amenable group. The theorems are algebraic as well as topological in their nature. A possible way to look further into the matter is to take specific examples of groups with a known mean and see what are then the results of applying these theorems. The idea is to break down these types of machines into easier parts. Another direction open for investigation is to explore topological groups, not necessarily amenable, but having some other special conditions and consider their decompositions.

The theorems on bounded isometric representations essentially say that an infinite machine acts like a finite one when restricted to certain subsets of the set of states. The theorems give a characterization of the zone in which the machines seem to act as finite ones. The orthogonality theorems essentially give averaging processes on automata where a mean can be constructed on the semi-groups of inputs.

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