CAUCHY SPACES II. REGULAR COMPLETIONS AND COMPACTIFICATIONS

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1. Introduction

Uniform limit spaces were introduced by Cook and Fischer [5] as a generalization of the uniform spaces of Weil and Bourbaki [3]. Wyler [12] and Biesterfeldt [1] obtained completions of uniform limit spaces, but not a satisfactory completion and compactification theory. The present paper attempts to present such a theory, in a wider setting.

Kowalsky [9] seems to have been the first to discuss completions using only Cauchy filters, and not a uniform structure furnishing them. The junior author of the present paper took up this lead in his thesis [10]. He gave an axiomatic characterization of Cauchy spaces, i.e. spaces with Cauchy filters having reasonable properties, he studied the relations between these spaces and uniform limit spaces on one hand, and limit spaces (as defined in [9] and [7]) on the other hand, and he obtained some regular completions and compactifications of Cauchy spaces, with the appropriate universal mapping properties. The first part of this work has been published in [11], the second part is presented here, with some generalizations and simplifications due to the senior author.

We have become convinced that regularity is an essential part of a completion or compactification with sufficiently strong properties. Regularity for limit spaces was defined by Fleischer [8] and Cook and Fischer [6] by a partial converse of the iterated limit theorem. Biesterfeldt [2] has shown that this is equivalent to the obvious generalization of the topological axiom T_3 to limit spaces. Thus regularity for limit spaces is fairly well understood.

For Cauchy spaces, there are several possible generalizations of T_3. These generalizations are not equivalent, and it seems that there is no single regularity concept for Cauchy spaces which is appropriate for all occasions. This has led us to present an axiomatic theory of regularity for Cauchy spaces. Using this theory, we obtain a unified theory of various regular completions and compactifications of Cauchy spaces.

There is at present no satisfactory regularity concept, and hence no satisfactory completion theory, for uniform limit spaces.

One advantage of Cauchy spaces over uniform limit spaces, already pointed out by Kowalsky [9], is the possibility to distinguish between T_1 spaces and completions, and T_2 spaces and completions. Correspondingly, we obtain distinct T_1 and T_2 completions and compactifications with appropriate universal mapping properties. Modifying a compactification of limit spaces obtained by Cochran [4], we also obtain a Stone-Čech compactification for limit spaces.

We use the notations and terminology of [11], but otherwise there is not much overlap between this paper and [11]. We wish to point out the following departures from frequently used terminology. $S_1 \leqslant S_2$ is used consistently for structures such as filters, topologies, limit structures, to mean that S_1 is <u>finer</u> than S_2, regardless of whether this means $S_2 \subset S_1$ or $S_1 \subset S_2$. This simplifies the formalism a great deal, as pointed out e.g. in [9]. We have replaced the term "convergence space" of [5] and others by the original term "limit space" of [9] and [7], in order to preserve the term "convergence space" for the more general spaces studied in [8] and elsewhere. To be consistent, we replaced the term "uniform convergence structure" of [5] by "uniform limit structure".

Several unsolved problems and open questions remain. One has already been stated; some of the others follow. We do not know just how our completions and compactifications are related to the corresponding constructs for topological spaces, and when the universal mappings defining our completions and compactifications are in fact embeddings. Except for some results in [4] concerning abelian groups, no completion theory for groups or other algebraic systems with compatible limit structures has been obtained.

2. Preliminaries

We denote by $\underline{F}\,E$ the set of all filters on a set E, by $\underline{U}\,E$ the set of all ultrafilters on E, and by $\underline{D}\,E$ the set of all fixed ultrafilters $\dot{x}$, $x \in E$. As in [9] and [11], $\mathcal{F} \leq \mathcal{G}$ will mean that $\mathcal{F}$ is finer than $\mathcal{G}$, i.e. $\mathcal{G} \subset \mathcal{F}$, and filters $\mathcal{F} \vee \mathcal{G}$ and $\mathcal{F} \wedge \mathcal{G}$ consist respectively of all sets $A \cup B$ and all sets $A \cap B$, with $A \in \mathcal{F}$, $B \in \mathcal{G}$. If $\mathcal{B}$ is a filter basis, then $\mathcal{B}^+$ denotes the filter generated by $\mathcal{B}$.

For a set C of filters on E, we denote by C^+ and C^- the sets of all filters on E respectively coarser or finer than some filter in C. We call C a Cauchy structure on E if C satisfies the following three conditions.

Cau_1. $\underline{D}\, E \subset C \subset \underline{F}\, E$.

Cau_2. $C^+ \cap C^- = C$.

Cau_3. If $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are in C and $\mathcal{F} \leq \mathcal{G}$, $\mathcal{F} \leq \mathcal{H}$, then $\mathcal{G} \vee \mathcal{H} \in C$.

We call (E,C) a <u>Cauchy space</u> if C is a Cauchy structure on E .

Examples have been given in [11]. We recall that $\underline{D}\, E$ and $\underline{F}\, E$ are Cauchy structures on E , and that A^+ is a Cauchy structure for any set A of filters on E such that $\underline{D}\, E \subset A$, but that C^- is in general not a Cauchy structure even if C is one. A Cauchy structure C is called <u>uniformizable</u> [11, Thm. 6.6] if C^- is a Cauchy structure, and <u>initial</u> if $C^- = C$.

The following two statements follow easily from the axioms.

2.1. If $(C_i)_{i \in I}$ is a family of Cauchy structures on the same set E , then $\bigcap C_i$ is a Cauchy structure on E .

2.2. If $f : E \longrightarrow E'$ is a mapping of sets and C' a Cauchy structure on E' , then

$$f^*(C') = \{\mathcal{F} \in \underline{F}\, E : f\mathcal{F} \in C'\}$$

defines a Cauchy structure $f^*(C')$ on E .

Thus the general theory of [11, sec. 3] is applicable to Cauchy spaces, and we shall use this theory in the present paper.

If (E,C) is a Cauchy space, let

$$q_C = \{(\mathcal{F}, x) : \mathcal{F} \vee \dot{x} \in C\} . \tag{2.3}$$

This is a limit structure on E by [11, Prop. 6.1], and q_C satisfies Axiom S_o of [11, sec. 5]. We shall attribute topological properties of q_C to C . Thus we shall say that (E,C) is a T_i space $(i = 1, 2)$, and C a T_i structure, if q_C

satisfies T_1 , and we call (E,C) compact if (E,q_C) is compact. We call a Cauchy space (E,C) separated if it satisfies T_1 and is uniformizable, i.e. if C^- is the set of Cauchy filters of a separated uniform limit structure on E . A separated Cauchy space satisfies T_2 ; the converse is not true in general.

If we define a term or concept for Cauchy spaces, we shall use it without further ado for Cauchy structures, and vice versa.

3. Regular and biregular Cauchy spaces

3.1. We define a regularity for Cauchy spaces as a correspondence r which assigns to every Cauchy space (E,C) a set $r(E,C)$ of filters on E , with the following two properties.

Reg_1. $\underline{D}\,E \subset r(E,C) \subset \underline{F}\,E$.

Reg_2. If $f : (E,C) \longrightarrow (E',C')$ is a continuous map of Cauchy spaces, then f maps $r(E,C)$ into $r(E',C')$.

Thus a regularity is in fact a functor from Cauchy spaces to sets, a subfunctor of the functor given by $(C,E) \longmapsto \underline{F}\,E$. In the following, we shall use lower case greek letters to denote elements of $r(E,C)$, and capital greek letters to denote filters on $r(E,C)$.

3.2. Examples and remarks. The most useful regularities are given by

$$p(E,C) = C \ , \qquad p^-(E,C) = C^- \ ,$$

$$d(E,C) = \underline{D}\,E \ , \qquad u(E,C) = \underline{U}\,E \ .$$

p^- is the appropriate regularity for limit spaces; p seems to be more appropriate for Cauchy spaces. d is needed for the construction of T_1 spaces, and u and related regularities are useful for compactifications.

If r' and r" are regularities, then

$$r(E,C) = r'(E,C) \cap r''(E,C)$$

defines a regularity r . If a set correspondence R satisfies

$$\underline{D}\,E \subset R\,E \subset \underline{F}\,E \quad , \qquad R\,E' \subset \{f\mathcal{F} : \mathcal{F} \in R\,E\} \, ,$$

for all sets E and mappings $f : E \longrightarrow E'$, then $r(E,C) = R\,E$ defines a regularity r . In particular, $r^*(E,C) = r(E,\underline{F}\,E)$ defines a regularity r^* if r is a regularity.

If r is a regularity, then $r(E,C) \subset r(E,C')$ for Cauchy structures C , C' on a set E such that $C \leq C'$. In particular, $r(E,C) \subset r(E,\underline{F}\,E)$.

3.3. Definitions. Let r be a regularity for Cauchy spaces and (E,C) a Cauchy space. We define the r-closure $\Gamma^r(A,C)$ of a set $A \subset E$ by

$$\Gamma^r(A,C) = \{x \in E : (\exists \varphi)(\varphi \in r(E,C) \, , \ A \in \varphi \, , \ \varphi \vee \dot{x} \in C)\} \, .$$

The r-closure $\Gamma^r(\mathcal{F},C)$ of a filter $\mathcal{F}$ on E is defined by

$$\Gamma^r(\mathcal{F},C) = \{\Gamma^r(A,C) : A \in \mathcal{F}\}^+ \, .$$

We say that (E,C) is r-regular if always

$$\mathcal{F} \in C \implies \Gamma^r(\mathcal{F},C) \in C \, ,$$

and that (E,C) is r-biregular if always

$$\mathcal{F} \in C \iff \Gamma^r(\mathcal{F},C) \in C \, .$$

If r and r' are regularities, we say that r' is finer than r , and write

$r' \leqslant r$, if always $\Gamma^{r'}(A,C) \subset \Gamma^{r}(A,C)$, for a Cauchy space (E,C) and a set $A \subset E$. We call the regularities r and r' <u>equivalent</u> if $r' \leqslant r$ and $r \leqslant r'$, i.e. if always $\Gamma^{r'}(A,C) = \Gamma^{r}(A,C)$.

<u>3.4. Remarks</u>. The following properties of r-closure are easily verified.

$$\Gamma^r(\emptyset,C) = \emptyset \ , \quad A \subset \Gamma^r(A,C) \ ,$$

$$A \subset B \implies \Gamma^r(A,C) \subset \Gamma^r(B,C) \ .$$

It follows from these properties that $\Gamma^r(\mathcal{F},C)$ is a filter on E , for a Cauchy space (E,C) and any filter $\mathcal{F}$ on E , and that

$$\mathcal{F} \leq \Gamma^r(\mathcal{F},C) \ , \qquad \mathcal{F} \leq \mathcal{G} \implies \Gamma^r(\mathcal{F},C) \leq \Gamma^r(\mathcal{G},C) \ ,$$

for filters $\mathcal{F}, \mathcal{G}$ on E .

The four regularities of 3.2 satisfy $d \leqslant p \leqslant \bar{p} \leqslant u \leqslant \bar{p}$, and closure with respect to $\bar{p}$ is closure with respect to the limit structure q_C of (2.3).

<u>Proposition 3.5</u>. <u>Let</u> r <u>be a regularity for Cauchy spaces</u>.

(i) <u>If</u> $(C_i)_{i \in I}$ <u>is a family of</u> r-[<u>bi-</u>]<u>regular Cauchy structures on the same set</u> E , <u>then</u> $\bigcap C_i$ <u>is</u> r-[<u>bi-</u>]<u>regular</u>.

(ii) <u>If</u> $f : E \to E'$ <u>is a mapping and</u> C' <u>an</u> r-[<u>bi-</u>]<u>regular Cauchy structure on</u> E' , <u>then</u> $f^*(C')$ <u>is</u> r-[<u>bi-</u>]<u>regular</u>.

<u>Proof</u>. We need the following set inclusions. If $C = \bigcap C_i$, then

$$A \subset \Gamma^r(A,C) \subset \Gamma^r(A,C_i) \subset \Gamma^r(\Gamma^r(A,C),C_i) \ ,$$

for $A \subset E$ and all $i \in I$. If $C = f^*(C')$, then

$$f(A) \subset f(\Gamma^r(A,C)) \subset \Gamma^r(f(A),C') \subset \Gamma^r(f(\Gamma^r(A,C)),C') \ .$$

These inclusions are easily verified, and they carry over to filters on E . Using the first group of inclusions, we have

$$\mathcal{F} \in C \implies \Gamma^r(\mathcal{F}, C_i) \in C_i \text{ for all } i \in I$$
$$\implies \Gamma^r(\mathcal{F}, C) \in C_i \text{ for all } i \in I \implies \Gamma^r(\mathcal{F}, C) \in C$$

if all C_i are r-regular and $C = \bigcap C_i$, and

$$\Gamma^r(\mathcal{F}, C) \in C \iff \Gamma^r(\mathcal{F}, C_i) \in C_i \text{ for all } i \in I$$
$$\iff \mathcal{F} \in C_i \text{ for all } i \in I \iff \mathcal{F} \in C$$

if all C_i are r-biregular. This proves (i). One obtains (ii) in the same way, using the second group of inclusions.

Proposition 3.6. *If r and r' are regularities such that $r' \leq r$, then every r-[bi-]regular Cauchy space (E,C) is also r'-[bi-]regular.*

Proof. $A \subset \Gamma^{r'}(A,C) \subset \Gamma^r(A,C) \subset \Gamma^r(\Gamma^{r'}(A,C),C)$ for $A \subset E$. From these set inclusions, the proof proceeds in the same way as that of 3.5.

The following results will be needed later.

Proposition 3.7. *Let (E,C) be an r-regular Cauchy space, and let $\varphi \vee \dot{x}$ and $\varphi \vee \mathcal{F}$ be in C , for $\varphi \in r(E,C)$, $x \in E$, $\mathcal{F} \in C$. Then $\mathcal{F} \vee \dot{x} \in C$.*

Proof. If $A \in \varphi \vee \mathcal{F}$, then $A \in \varphi$, $\varphi \vee \dot{x} \in C$, hence $x \in \Gamma^r(A,C)$. Thus $\varphi \vee \mathcal{F} \leq \varphi \vee \mathcal{F} \vee \dot{x} \leq \Gamma^r(\varphi \vee \mathcal{F}, C)$, and $\varphi \vee \mathcal{F} \vee \dot{x} \in C$ by r-regularity of C and Cau_2. As $\dot{x} \leq \mathcal{F} \vee \dot{x} \leq \varphi \vee \mathcal{F} \vee \dot{x}$, also $\mathcal{F} \vee \dot{x} \in C$.

Corollary 3.8. *If (E,C) is an r-regular Cauchy T_1 space, and if $\varphi \vee \dot{x}$ and $\varphi \vee \dot{y}$ are in C for $\varphi \in r(E,C)$ and x, y in E , then $x = y$.*

4. Some reflective classes of Cauchy spaces

4.1. A class $\mathcal{S}$ of Cauchy spaces is called reflective if the corresponding full subcategory of Cauchy spaces is reflective, i.e. for every Cauchy space (E,C) there is a map $h : (E,C) \longrightarrow (E_1,C_1)$ with codomain (E_1,C_1) in $\mathcal{S}$, and with the universal property that for every map $f : (E,C) \longrightarrow (E',C')$ of Cauchy spaces with $(E',C') \in \mathcal{S}$ there is exactly one map $f_1 : (E_1,C_1) \longrightarrow (E',C')$ such that $f = f_1 h$.

We call a reflective class $\mathcal{S}$ strictly reflective if the universal map $h : (E,C) \longrightarrow (E_1,C_1)$ can always be constructed so that $E_1 = E$ and $h = 1_E$.

Proposition 4.2. *A class $\mathcal{S}$ of Cauchy spaces is strictly reflective if and only if $\mathcal{S}$ satisfies the following two conditions.*

(i) *If $(C_i)_{i \in I}$ is a family of Cauchy structures on a set E such that all spaces (E,C_i), $i \in I$, are in $\mathcal{S}$, then the space $(E, \bigcap C_i)$ is in $\mathcal{S}$.*

(ii) *If $f : E \longrightarrow E'$ is a mapping and C' a Cauchy structure on E' such that $(E',C') \in \mathcal{S}$, then $(E, f^*(C')) \in \mathcal{S}$.*

Proof. If $\mathcal{S}$ is strictly reflective, and if $(E,C_i) \in \mathcal{S}$ for all $i \in I$, let $1_E : (E, \bigcap C_i) \longrightarrow (E,C')$ be the universal map for $\mathcal{S}$. It follows that $1_E : (E,C') \longrightarrow (E,C_i)$ is continuous, and thus $\bigcap C_i \leq C' \leq C_i$, for all $i \in I$. But then $C' = \bigcap C_i$, and $(E, \bigcap C_i) \in \mathcal{S}$. (ii) is verified in the same way.

Conversely, let $\mathcal{S}$ satisfy (i) and (ii). For a Cauchy space (E,C), let C_1 be the intersection of all Cauchy structures C^* on E such that $C \leq C^*$ and $(E,C^*) \in \mathcal{S}$. Then $C \leq C_1$, and $(E,C_1) \in \mathcal{S}$ by (i). If $f : (E,C) \longrightarrow (E',C')$ is a map with $(E',C') \in \mathcal{S}$, then $C \leq f^*(C')$, and $(E, f^*(C')) \in \mathcal{S}$

by (ii). But then $C_1 \leq f^*(C')$ by the construction of C_1, and $f : (E,C_1) \longrightarrow (E',C')$ is continuous. This shows that $1_E : (E,C) \longrightarrow (E,C_1)$ has the desired universal property.

4.3. Examples and remarks. If r is a regularity for Cauchy spaces, then r-regular and r-biregular Cauchy spaces form strictly reflective classes by 3.5.

Uniformizable and initial Cauchy spaces form strictly reflective classes.

If $\mathcal{S}$ is the intersection of a family of strictly reflective classes of Cauchy spaces, then $\mathcal{S}$ is a strictly reflective class.

Strictly reflective classes can be defined in any concrete category. 4.2 and 4.4 clearly remain true, with virtually no changes, for any category of the type discussed in [11, sec. 3]. 4.6 remains true for topological spaces and limit spaces, except that the use of d-regularity has to be replaced by another device.

A strictly reflective class $\mathcal{S}$ is replete, i.e. if $u : (E,C) \longrightarrow (E',C')$ is an isomorphism and $(E,C) \in \mathcal{S}$, then $(E',C') \in \mathcal{S}$. This is a special case of the following result.

Proposition 4.4. Let $\mathcal{S}$ be a strictly reflective class. If $f : E \longrightarrow E'$ is a surjective mapping, and if $(E,f^*(C')) \in \mathcal{S}$ for a Cauchy structure C' on E', then $(E',C') \in \mathcal{S}$.

Proof. $f\,h = 1_{E'}$ for a mapping $h : E' \longrightarrow E$, and then $h^*(f^*(C')) = C'$. Thus $(E',C') \in \mathcal{S}$ by 4.2, (ii).

Lemma 4.5. A Cauchy space (E,C) is d-biregular (see 3.2) if and only if $C = q^*(C_1)$ for some T_1 space (E_1,C_1) and a surjection $q : E \longrightarrow E_1$.

Proof. $\Gamma^d(A,C_1) = A$ for any $A \subset E_1$ if (E_1,C_1) is a T_1 space, and thus any T_1 space is d-biregular. By (3.5), (ii), (E,C) is d-biregular if $C = q^*(C_1)$ for a mapping $q : E \rightarrow E_1$ and a T_1 structure C_1 on E_1.

Conversely, $x \delta y \Longleftrightarrow \dot{x} \vee \dot{y} \in C$ defines an equivalence relation δ on E for any Cauchy space (E,C). Let $E_1 = E/\delta$ be the quotient set and $q : E \rightarrow E_1$ the quotient mapping. Then $\Gamma^d(A,C) = q^{-1}(q(A))$ for $A \subset E$, and $q^{-1}(\mathcal{G}) = \{q^{-1}(B) : B \in \mathcal{G}\}^+$ defines a filter $q^{-1}(\mathcal{G})$ on E if $\mathcal{G}$ is a filter on E_1. Let C_1 be the set of all filters $\mathcal{G}$ on E_1 such that $q^{-1}(\mathcal{G}) \in C$. The formal properties of q^{-1} show immediately that C_1 satisfies axioms Cau_2 and Cau_3. Let now (E,C) be d-biregular. Then

$$q(\mathcal{F}) \in C_1 \Longleftrightarrow \Gamma^d(\mathcal{F},C) \in C \Longleftrightarrow \mathcal{F} \in C$$

for a filter $\mathcal{F}$ on E, since $q^{-1}q$ is d-closure. Thus C_1 also satisfies Cau_1 in this case, and $C = q^*(C_1)$. If $u = q(x)$ and $v = q(y)$ are points of E_1 such that $\dot{u} \vee \dot{v} = q(\dot{x} \vee \dot{y})$ is in C_1, then $\dot{x} \vee \dot{y} \in C$, so that $x \delta y$ and $q(x) = q(y)$. Thus C_1 is a T_1 structure.

<u>Theorem 4.6</u>. <u>If $\mathcal{S}$ is a strictly reflective class of Cauchy spaces, then the T_1 spaces and the separated spaces in $\mathcal{S}$ form reflective classes.</u>

<u>Proof</u>. Let $\mathcal{S}_1$ be the class of d-biregular spaces in $\mathcal{S}$, or the class of uniformizable d-biregular spaces in $\mathcal{S}$. In either case, $\mathcal{S}_1$ is strictly reflective. The T_1 spaces in $\mathcal{S}_1$ are the T_1 spaces in $\mathcal{S}$ or the separated spaces in $\mathcal{S}$ respectively. Let $\mathcal{T}$ denote this class.

For a Cauchy space (E,C), let $1_E : (E,C) \longrightarrow (E,C^*)$ be the universal map for (E,C) and $\mathcal{S}_1$. By 4.5, $C^* = q^*(C_1)$ for a surjective mapping $q : E \rightarrow E_1$ and a T_1 structure C_1 on E_1. By 4.4, $(E_1,C_1) \in \mathcal{T}$.

Let now $f : (E,C) \longrightarrow (E',C')$ be continuous, with $(E',C') \in \mathcal{T}$. Then $(E',C') \in \mathcal{S}_1$, and $f : (E,C^*) \longrightarrow (E',C')$ is continuous. If $q(x) \doteq q(y)$, for x, y in E, then $\dot{x} \vee \dot{y} \in C^*$, and thus $f(\dot{x}) \vee f(\dot{y}) \in C'$. Since C' satisfies T_1, this implies $f(x) = f(y)$. It follows that $f = f_1 q$ for a unique mapping $f_1 : E_1 \longrightarrow E'$. As q is surjective, $C_1 = q_*(C^*)$ by [11, 3.14], and thus $f_1 : (E_1, C_1) \longrightarrow (E',C')$ is continuous by [11, 3.17]. This shows that $q : (E,C) \longrightarrow (E_1,C_1)$ is the desired universal mapping for $\mathcal{T}$.

5. Quasicompletions

5.1. Let r be a regularity for Cauchy spaces, and let (E,C) be a fixed (but arbitrary) Cauchy space. For $A \subset E$, let

$$A^r = \{\varphi \in r(E,C) : A \in \varphi\} .$$

In particular, $E^r = r(E,C)$ and $\emptyset^r = \emptyset$. We note that

$$\dot{x} \in A^r \iff x \in A \quad \text{and} \quad (A \cap B)^r = A^r \cap B^r ,$$

for $x \in E$ and subsets A, B of E.

For filters $\mathcal{F}$ on E and Φ on E^r, we put

$$\mathcal{F}^r = \{A^r : A \in \mathcal{F}\}^+ , \qquad \Phi_r = \{A \subset E : A^r \in \Phi\} .$$

This defines filters $\mathcal{F}^r$ on E^r and Φ_r on E, and one sees easily that

$$(5.2) \qquad \Phi \leq \mathcal{F}^r \iff \Phi_r \leq \mathcal{F} .$$

It follows from (5.2) that $\mathcal{F} \mapsto \mathcal{F}^r$ preserves meets, and $\Phi \mapsto \Phi_r$ joins, of filters.

For $U \subset E^r$, let $U_r = \bigvee\{\varphi : \varphi \in U\}$. This is a filter on E, and $A \in U_r \iff A \in \varphi$ for all $\varphi \in U \iff U \subset A^r$. For a filter Φ on E^r, it follows that $\Phi_r \leq U_r$ for all $U \in \Phi$. On the other hand, if $A \in \Phi_r$, then $A \in U_r$ for $U = A^r$ in Φ. Thus $\Phi_r = \bigwedge\{U_r : U \in \Phi\}$. We have proved

(5.3) $$\Phi_r = \bigwedge_{U \in \Phi} \bigvee_{\varphi \in U} \varphi .$$

This connects Φ_r with the compression operator κ of Kowalsky.[9]. In particular, $\Phi_r = \kappa\Phi$ if $r(E,C) = \underline{F}\, E$.

5.4. We define a natural injection $j : E \longrightarrow E^r$ by putting $j(x) = \dot{x}$ for $x \in E$. Thus $j(A) = A^r \cap \underline{D}\, E$ for $A \subset E$. From this and from basic filter properties, it follows immediately that

$$A \subset B \iff A^r \subset B^r \iff j(A) \subset j(B)$$

for subsets A, B of E. We also note that

$$(\mathcal{F}^r)_r = (j(\mathcal{F}))_r = \mathcal{F} , \quad (\dot{\varphi})_r = \varphi ,$$

$$\varphi \leq \mathcal{F} \iff \dot{\varphi} \leq \mathcal{F}^r ,$$

for filters $\mathcal{F}$ on E and $\varphi \in E^r$. This follows easily from the definitions of $\mathcal{F}^r$ and Φ_r, and from (5.2).

5.5. Definitions. We denote by C^r the finest Cauchy structure on E^r which contains all filters $j(\mathcal{F})$ and $\mathcal{F}^r$ for $\mathcal{F} \in C$, and all filters φ^r for $\varphi \in r(E,C)$. We call the Cauchy space (E^r, C^r) the r-<u>quasicompletion</u> of the space (E,C). We note that $j : (E,C) \longrightarrow (E^r, C^r)$ is continuous.

We denote by C_r the finest Cauchy structure on E which contains all filters $\mathcal{F} \in C$ and $\varphi \in r(E,C)$.

Proposition 5.6. *If* $\Phi \in C^r$, *then* $\Phi_r \in C_r$. *If* $C^r \subset C^-$, *then* $C_r = \{\mathcal{F} \in \underline{F}\, E : \mathcal{F}^r \in C^r\}$.

Proof. For the first part, consider the set $C^1 = \{\Phi \in \underline{F}\, E^r : \Phi_r \in C_r\}$. By 5.4, the filters which generate C^r are in C^1, and C^1 satisfies Cau_1. C^1 clearly satisfies Cau_2. Thus $C^r \subset C^1$ if C^1 satisfies Cau_3. If $\Phi \leq \Psi$, $\Phi \leq \Omega$, with Φ_r, Ψ_r, Ω_r in C_r, then $(\Psi \vee \Omega)_r = \Psi_r \vee \Omega_r$ is in C_r by Cau_3 for C_r. Thus C^1 satisfies Cau_3.

For the second part, consider the set $C_1 = \{\mathcal{F} \in \underline{F}\, E : \mathcal{F}^r \in C^r\}$. The filters which generate C_r are in C_1, and C_1 satisfies Cau_1 and Cau_2. $C_1 \subset C_r$ by 5.4 and the first part. Thus $C_1 = C_r$ if C_1 satisfies Cau_3. If $\mathcal{F} \leq \mathcal{G}$, $\mathcal{F} \leq \mathcal{H}$, for filters $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ in C_1, then $\mathcal{G} \vee \mathcal{H} \in C_r$ since $C_1 \subset C_r$. If $\mathcal{G} \vee \mathcal{H} \leq \mathcal{K}$, $\mathcal{K} \in C$, then $\mathcal{F}^r \leq (\mathcal{G} \vee \mathcal{H})^r \leq \mathcal{K}^r$, and hence $(\mathcal{G} \vee \mathcal{H})^r \in C^r$. Thus C_1 satisfies Cau_3 if $C_r \subset C^-$.

5.7. Definition. We say that (E,C) is r-complete if for every filter $\varphi \in r(E,C)$ there is a point $x \in E$ such that $\varphi \vee \dot{x} \in C$. In other words, we require that every filter $\varphi \in r(E,C)$ converges for the limit structure q_C.

Examples (see 3.2): p-completeness is completeness in the usual sense: every Cauchy filter converges. $\bar{p}$-completeness is equivalent to p-completeness. u-completeness is compactness: every ultrafilter converges.

We note that $r(E,C) \subset C^-$ if (E,C) is r-complete. If (E,C) is r-complete and r-regular, then the condition $C_r \subset C^-$ of 5.6 is satisfied by 3.7 (see also [11], concluding remarks).

Theorem 5.8. *Let* $f : (E,C) \longrightarrow (E',C')$ *be continuous. If* (E',C') *is an* *r-regular and r-complete T_1 space, then there is a unique continuous mapping* $f^r : (E^r, C^r) \longrightarrow (E', C')$ *such that* $f = f^r j$.

Proof. For $\varphi \in E^r$, we have $\dot{\varphi} \leq j(\varphi) \vee \dot{\varphi} \leq \varphi^r$, so that $j(\varphi) \vee \dot{\varphi} \in C^r$. If f^r exists and $f^r(\varphi) = y$, then $f^r(j(\varphi) \vee \dot{\varphi}) = f(\varphi) \vee \dot{y}$ must be in C'. Since $f(\varphi) \in r(E', C')$ and C' is an r-complete T_1 space, such a point y exists, and there is only one such point y in E' by 3.8. This shows that there is at most one mapping f^r with the desired properties.

We must show that the mapping f^r just constructed has these properties. If $f^r(\dot{x}) = y$, then $f(\dot{x}) \vee \dot{y} \in C'$ by the construction of f^r, and hence $f(x) = y$ by T_1 for C'. Thus $f^r j = f$. Let now $A \subset E$. If $\varphi \in A^r$ and $f^r(\varphi) = y$, then $f(A) \in f(\varphi)$, $f(\varphi) \in r(E', C')$, and $f(\varphi) \vee \dot{y} \in C'$. Thus $y \in \Gamma^r(f(A), C')$. Since $j(A) \subset A^r$, this shows that $f(A) \subset f^r(A^r) \subset \Gamma^r(f(A), C')$. For a filter $\mathcal{F}$ in C, it follows that

$$f(\mathcal{F}) = f^r(j(\mathcal{F})) \leq f^r(\mathcal{F}^r) \leq \Gamma^r(f(\mathcal{F}), C') ,$$

so that $f^r(j(\mathcal{F}))$ and $f^r(\mathcal{F}^r)$ are in C'. For $\varphi \in E^r$, $f^r(\varphi) = y$, and $A \in \varphi$, we have $y \in f^r(A^r)$ and $f(\varphi) \vee \dot{y} \in C'$. Thus

$$\dot{y} \leq f^r(\varphi^r) \leq \Gamma^r(f(\varphi), C') \leq \Gamma^r(f(\varphi) \vee \dot{y}, C') ,$$

so that $f^r(\varphi^r) \in C'$. With [11, 3.16], we conclude that f^r is continuous.

Proposition 5.9. *Let* (E, C) *be a Cauchy space such that for* $\varphi \in r(E, C)$ *there is at most one point* $y \in E$ *such that* $\varphi \vee \dot{y} \in C$. *Then a continuous mapping* $g : (E^r, C^r) \longrightarrow (E, C)$ *such that* $g j = 1_E$ *exists if and only if* (E, C) *is r-regular and r-complete.*

Proof. The hypothesis implies that C is a T_1 structure. Thus g exists, by 5.8, if C is r-regular and r-complete.

Assume now that g exists. By the proof of 5.8 and the hypothesis of 5.9, $\varphi \vee \dot{y} \in C \iff y = g(\varphi)$ for $\varphi \in E^r$ and $y \in E$. Thus C is r-complete, and $g(A^r) = \Gamma^r(A, C)$ for $A \subset E$. If $\mathcal{F} \in C$, then $g(\mathcal{F}^r) = \Gamma^r(\mathcal{F}, C)$ is in C, and thus C is r-regular.

6. Regular completions

We believe that a general construction of completions for Cauchy spaces should furnish universal mappings $h : (E,C) \longrightarrow (E_1, C_1)$, with $(E_1, C_1) \in \mathcal{C}$, for a reflective class $\mathcal{C}$ of complete Cauchy spaces. The r-quasicompletion $j : (E,C) \longrightarrow (E^r, C^r)$ does not meet this requirement, despite Theorem 5.8, since (E^r, C^r) is in general neither r-regular nor a T_1 space. We shall correct this deficiency for the regularity p of 3.2 given by $p(E,C) = C$.

We need a general lemma and a specific result.

Lemma 6.1. Let C and C_1 be Cauchy structures on a set E, and let $C_2 = (C^+ \cap C_1)^- \cap C_1$. Then C_2 is a Cauchy structure on E. If C_1 is r-[bi-]regular for a regularity r, or uniformizable, or initial, then C_2 is r-[bi-]regular, or uniformizable, or initial respectively. If $C \leqslant C_1$ and C is p-complete, then $C \leqslant C_2$ and C_2 is p-complete.

Proof. C_2 obviously satisfies Cau_1 and Cau_2; we verify Cau_1. If $\mathcal{F} \leq \mathcal{G}$, $\mathcal{F} \leq \mathcal{H}$, for filters in C_2, let $\mathcal{G} \leq \mathcal{G}'$, $\mathcal{H} \leq \mathcal{H}'$, for filters $\mathcal{G}', \mathcal{H}'$

in $C^+ \cap C_1$. Then $\mathcal{G} \vee \mathcal{H} \leq \mathcal{G}' \vee \mathcal{H}'$, with both filters in C_1 and $\mathcal{G}' \vee \mathcal{H}'$ also in C^+ . Thus $\mathcal{G} \vee \mathcal{H} \in C_2$, and C_2 satisfies Cau_3.

For $\mathcal{F} \leq \mathcal{F}'$ or $\Gamma^r(\mathcal{F}, C_2) \leq \mathcal{F}'$, and $\mathcal{F}' \in C^+ \cap C_1$, we have

$$\mathcal{F} \leq \Gamma^r(\mathcal{F}, C_2) \leq \Gamma^r(\mathcal{F}, C_1) \leq \Gamma^r(\mathcal{F}', C_1) \ .$$

It follows that $\mathcal{F} \in C_2 \implies \Gamma^r(\mathcal{F}, C_2) \in C_2$ if C_1 is r-regular, and $\Gamma^r(\mathcal{F}, C_2) \in C_2 \implies \mathcal{F} \in C_2$ if C_1 is r-biregular.

It is easily seen that $C_2^- = (C^+ \cap C_1^-)^- \cap C_1^-$. Thus C_2 is uniformizable if C_1 is uniformizable, and initial if C_1 is initial.

Finally, $C \leq C_1$ clearly implies $C \leq C_2$. For $\mathcal{F} \in C_2$, choose $\mathcal{G} \in C_1$ and $\mathcal{H} \in C$ such that $\mathcal{F} \leq \mathcal{G}$, $\mathcal{H} \leq \mathcal{G}$. If $\mathcal{H} \vee \dot{x} \in C$, then $\mathcal{G} \vee \dot{x} \in C^+ \cap C_1$, and $\mathcal{F} \leq \mathcal{F} \vee \dot{x} \leq \mathcal{G} \vee \dot{x}$, so that $\mathcal{F} \vee \dot{x} \in C_2$. Thus C_2 is p-complete if C is p-complete and $C \leq C_1$.

<u>Proposition 6.2</u>. (E^p, C^p) is p-complete for every Cauchy space (E, C) .

<u>Proof</u>. Let $\Phi_p = \varphi$ for $\Phi \in p(E^p, C^p) = C^p$. Then φ is in $C_p = C = E^p$ by 5.6, and $\Phi \leq \Phi \vee \dot{\varphi} \leq \varphi^p$ by 5.2 and 5.4. Thus $\Phi \vee \dot{\varphi} \in C^p$.

<u>6.3</u>. In order to obtain a reasonably general and simple existence theorem for p-regular p-completions with a universal mapping property, we consider a class $\mathcal{S}$ of Cauchy spaces which meets the following two requirements.

(i) $\mathcal{S}$ is strictly reflective, and all spaces in $\mathcal{S}$ are p-regular.

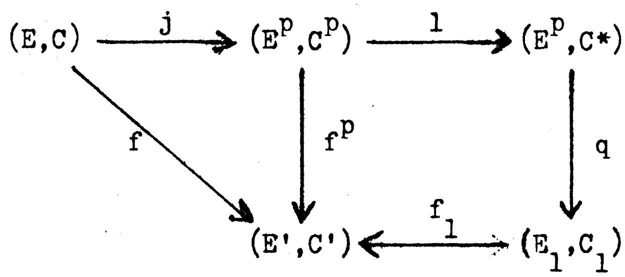
(ii) If C and C_1 are Cauchy structures on a set E such that $C \leq C_1$ and $(E, C_1) \in \mathcal{S}$, and if $C_2 = (C^+ \cap C_1)^- \cap C_1$, then $(E, C_2) \in \mathcal{S}$.

Condition (ii) is not as restrictive as it may seem. In fact, 6.1 shows that all examples of 4.3 satisfy (ii), and if $\mathcal{S}$ is an intersection of strictly reflective classes which satisfy (ii), then $\mathcal{S}$ satisfies (ii).

<u>Theorem 6.4</u>. <u>If a class $\mathcal{S}$ of Cauchy spaces satisfies the two requirements stated above, then the p-complete T_1 spaces in $\mathcal{S}$ form a reflective class of Cauchy spaces.</u>

<u>Proof</u>. Let $\mathcal{S}_1$ be the class of d-biregular spaces in $\mathcal{S}$ (see 4.5). This does not affect the T_1 spaces in $\mathcal{S}$, and $\mathcal{S}_1$ satisfies requirements (i) and (ii) by 6.1 and the remarks made above.

Now the following diagram illustrates our construction.



Let $j : (E,C) \longrightarrow (E^p,C^p)$ be the quasicompletion of sec. 5, and let C^* be the finest Cauchy structure on E^p such that $C^p \leq C^*$ and $(E^p,C^*) \in \mathcal{S}_1$. By 4.4 and 4.5, $C^* = q^*(C_1)$ for a surjection $q : E \longrightarrow E_1$ and a T_1 Cauchy structure C_1 on E_1, and $q : (E^p,C^p) \longrightarrow (E_1,C_1)$ is universal for the class of T_1 spaces in $\mathcal{S}$, by the proof of 4.6. We wish to show that $q\,j : (E,C) \longrightarrow (E_1,C_1)$ is universal for the class of p-complete T_1 spaces in $\mathcal{S}$.

If $C_2 = ((C^p)^+ \cap C^*)^- \cap C^*$, then $C^p \leq C_2 \leq C^*$, and $(E^p,C_2) \in \mathcal{S}_1$ by requirement (ii), used for $\mathcal{S}_1$. Thus $C_2 = C^*$ by the construction of C^*, and C^* is p-complete by 6.2 and 6.1. If $\mathcal{G} \in C_1$, then $q^{-1}(\mathcal{G}) \in C^*$, and if $q^{-1}(\mathcal{G}) \vee \dot{\varphi} \in C^*$, with $\varphi \in E^p$, then $q(q^{-1}(\mathcal{G}) \vee \dot{\varphi}) = \mathcal{G} \vee \dot{y}$ is in C_1 for $y = q(\varphi)$. Thus C_1 is p-complete, and (E_1,C_1) is in the right class of spaces.

Now let $f : (E,C) \longrightarrow (E',C')$ be continuous, with (E',C') a p-complete T_1 space in $\mathcal{S}$. Then $f = f^p j$ for a unique map $f^p : (E^p,C^p) \longrightarrow (E',C')$ by 5.8, and $f^p = f_1 q$ for a unique map $f_1 : (E_1,C_1) \longrightarrow (E',C')$ by the universal nature of q. Thus $q j$ is indeed the desired universal mapping.

7. Regular compactifications

7.1. A Cauchy structure C is compact if and only if it is u-complete (see 5.7). Thus compactifications are special completions. In order to be able to obtain regular T_1 compactifications which are not separated, we consider other regularities besides u. We shall require, however, that

$$r(E,C) \subset \underline{U}\, E$$

for every Cauchy space (E,C).

From now on, let r be such a regularity and (E,C) a Cauchy space. We say that (E,C) is r-precompact if (E^r,C^r) is compact.

Lemma 7.2. $(A \cup B)^r = A^r \cup B^r$ *for subsets* A, B *of* E, *and* $(\mathcal{F} \vee \mathcal{G})^r = \mathcal{F}^r \vee \mathcal{G}^r$ *for filters* $\mathcal{F}, \mathcal{G}$ *on* E.

Proof. An ultrafilter $\mathcal{U}$ on E is characterized as such by the fact that $A \cup B \in \mathcal{U}$ iff $A \in \mathcal{U}$ or $B \in \mathcal{U}$. As $r(E,C)$ consists of ultrafilters, the first part of 7.2 follows. The second part follows immediately from the first part.

Lemma 7.3. *If* Ω *is an ultrafilter on* E^r, *then* Ω_r *is an ultrafilter on* E.

Proof. Using 7.2, we see that $A \cup B \in \Omega_r \iff (A \cup B)^r = A^r \cup B^r$ is in $\Omega \iff A^r \in \Omega$ or $B^r \in \Omega \iff A \in \Omega_r$ or $B \in \Omega_r$. This shows that Ω_r is an ultrafilter.

Lemma 7.4. $C_r = \{\mathcal{F} \in \underline{F}\, E : \mathcal{F}^r \in C^r\}$..

Proof. Let $C_1 = \{\mathcal{F} \in \underline{F}\, E : \mathcal{F}^r \in C^r\}$. By the proof of 5.6, it is sufficient to show that C_1 satisfies Cau_3. If $\mathcal{F}^r, \mathcal{G}^r, \mathcal{H}^r$ are in C^r, and $\mathcal{F} \leq \mathcal{G}$, $\mathcal{F} \leq \mathcal{H}$, then $(\mathcal{F} \vee \mathcal{G})^r = \mathcal{F}^r \vee \mathcal{G}^r$ by 7.2, and $\mathcal{F}^r \vee \mathcal{G}^r \in C^r$. Thus C_1 does satisfy Cau_3.

Theorem 7.5. A Cauchy space (E,C) is r-precompact if and only if for every ultrafilter $\mathcal{U}$ on E there is a filter $\varphi \in r(E,C)$ such that $\mathcal{U} \vee \varphi \in C_r$.

Proof. If Ω is an ultrafilter on E^r, then Ω_r is an ultrafilter on E by 7.3. If $\Omega_r \vee \varphi \in C_r$, with $\varphi \in E^r$, then (using 7.2)

$$\dot{\varphi} \leq \dot{\varphi} \vee \Omega \leq \varphi^r \vee (\Omega_r)^r = (\varphi \vee \Omega_r)^r .$$

With 7.4, it follows from this that $\dot{\varphi} \vee \Omega \in C^r$.

Conversely, if $\mathcal{U}$ is an ultrafilter on E, let $\Omega \leq \mathcal{U}^r$ for an ultrafilter Ω on E^r. Then $\Omega_r \leq \mathcal{U}$, and hence $\Omega_r = \mathcal{U}$. If $\Omega \vee \dot{\varphi} \in C^r$, with $\varphi \in E^r$, then $(\Omega \vee \dot{\varphi})_r = \mathcal{U} \vee \varphi$ is in C_r by 5.6.

Corollary 7.6. Every Cauchy space (E,C) is u-precompact.

Proposition 7.7. If (E,C) is compact, then (E,C) is r-precompact and r-complete. Conversely, if (E,C) is r-precompact and r-complete, and if $C_r \subset C^-$, then (E,C) is compact.

Proof. The first part is obvious. If C is r-precompact and r-complete, let $\mathcal{U}$ be an ultrafilter on E. Then $\mathcal{U} \vee \varphi \in C_r$ for some $\varphi \in r(E,C)$, and $\varphi \vee \dot{x} \in C$ for some $x \in E$. It follows that $\mathcal{U} \vee \dot{x} \in C_r$. If $C_r \subset C^-$, this implies that $\mathcal{U} \vee \dot{x}$ is in $C^- \cap C^+ = C$, and thus C is compact.

Theorem 7.8. Let $\mathcal{S}$ be a strictly reflective class of r-regular Cauchy spaces. The compact T_1 spaces in $\mathcal{S}$ form a reflective class for the category of r-precompact Cauchy spaces.

Proof. Let (E,C) be r-precompact, let $j : (E,C) \longrightarrow (E^r,C^r)$ be the r-quasicompletion, and let $q : (E^r,C^r) \longrightarrow (E_1,C_1)$ be the universal mapping for (E^r,C^r) and the class of all T_1 spaces in $\mathcal{S}$. Since (E^r,C^r) is compact and q surjective (by the proof of 4.6), (E_1,C_1) is compact. Now if $f : (E,C) \longrightarrow (E',C')$ for a compact T_1 space (E',C') in $\mathcal{S}$, then $f = f^r j$ for a unique map $f^r : (E^r,C^r) \longrightarrow (E',C')$ by 5.8, since (E',C') is r-complete by 7.7. Also $f^r = f_1 q$ for a unique map $f_1 : (E_1,C_1) \longrightarrow (E',C')$ by the universal nature of q. Thus $q\,j$ is the desired universal mapping.

7.9. Remark. A uniform limit space E has been called precompact in [5] if every ultrafilter of E is a Cauchy filter. Carried over to uniformizable Cauchy spaces, this condition becomes $\underline{U}\,E \subset C^-$.

If we introduce a regularity v by putting $v(E,C) = \underline{U}\,E \cap C^-$ for every Cauchy space (E,C), then v is equivalent to the regularity p^- of 3.2, $C^- \subset C_v$ for every Cauchy space, and $C^- = C_v$ if C is uniformizable. One sees easily that the following three statements are logically equivalent for a uniformizable Cauchy space (E,C). (i) $\underline{U}\,E \subset C^-$. (ii) $C_u = C^-$. (iii) C is v-precompact.

8. The Stone-Čech compactification for limit spaces

We regard a compactification of a limit space (E,q) as a universal mapping for (E,q) and a class of compact limit spaces, and not as an embedding of (E,q) into a compact limit space. From this viewpoint, the following result justifies the title of this section.

Theorem 8.1. *The class of regular compact T_2 limit spaces is reflective in the category of limit spaces.*

Before proving this result in 8.6, as an application of 7.8, we must adapt a construction of [11, sec. 4] to our present needs.

8.2. If (E,q) is a limit space, we denote by $S\,q$ the finest initial Cauchy structure on E which contains the set $\{\mathcal{F} \vee \dot{x} : \mathcal{F}\, q\, x\}$ of filters on E, and we put $S(E,q) = (E,Sq)$. If $f : (E,q) \longrightarrow (E',q')$ is a map of limit spaces, we put $S\,f = f : (E,Sq) \longrightarrow (E',Sq')$. If $\mathcal{F}\, q\, x$ and $y = f(x)$, then $f(\mathcal{F})\, q'\, y$, and $f(\mathcal{F} \vee \dot{x}) = f(\mathcal{F}) \vee \dot{y}$. Thus $S\,f$ is continuous by [11,3.16], and we have defined a functor S from limit spaces to initial Cauchy spaces. We shall need the following properties of this functor.

Proposition 8.3. $q \leqslant q_{Sq}$ *for a limit structure* q, *and* $S\,q_C \leqslant C$ *for an initial Cauchy structure* C. *Moreover*, $S\,q_C = \{\mathcal{F} \vee \dot{x} : \mathcal{F}\, q_C\, x\}^{-}$.

Proof. The first two statements follow immediately from the definitions. For the third statement, it is sufficient to prove that $C' = \{\mathcal{F} \vee \dot{x} : \mathcal{F}\, q_C\, x\}^{-}$ defines a Cauchy structure. Axioms Cau_1 and Cau_2 for C' are obvious. Suppose now $\mathcal{F} \leq \mathcal{G} \leq \mathcal{G}' \vee \dot{x}$ and $\mathcal{F} \leq \mathcal{H} \leq \mathcal{H}' \vee \dot{y}$, with $\mathcal{G}' \vee \dot{x}$ and $\mathcal{H}' \vee \dot{y}$ in C.

Then $\mathcal{F} \leq \mathcal{G} \vee \mathcal{H} \leq \mathcal{G}' \vee \mathcal{H}' \vee \dot{x} \vee \dot{y}$, and $\mathcal{G}' \vee \mathcal{H}' \vee \dot{x} \vee \dot{y} \in C$ by Cau_3 for C. Thus $\mathcal{F} \vee \mathcal{G} \in C'$, and C' does satisfy Cau_3.

<u>Proposition 8.4</u>. <u>If</u> q <u>is a</u> T_2 <u>limit structure, then</u> $S\,q$ <u>is a separated Cauchy structure, and</u> $q = q_{Sq}$. <u>If</u> C <u>is a compact initial Cauchy structure, then</u> $C = S\,q_C$.

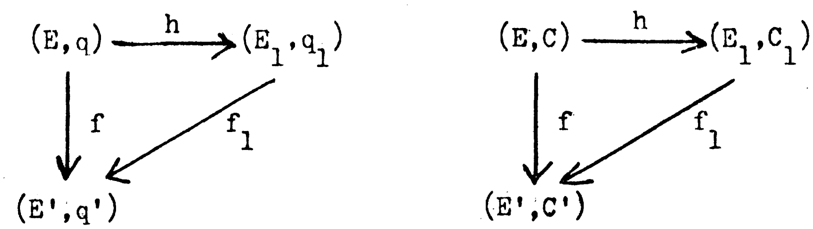
<u>Proof</u>. For the first part, we note that axiom T_2 implies axioms S_o and S_1 of [11, sec. 5]. Thus $S\,q = (T_1\,q)^-$ in the notation of [11], by [11, 6.11], and then $q_{Sq} = U_1\,(T_1\,q)^- = U_1\,T_1\,q = q$ by [11, 6.3 and 4.3], if q satisfies T_2. It follows that $S\,q$ is separated.

For the second part, let $\mathcal{F} \in C$. Then $\mathcal{U} \leq \mathcal{F}$ for an ultrafilter $\mathcal{U} \in C$, and $\mathcal{U} \vee \dot{x} \in C$ for some point x since C is compact. But then $\mathcal{F} \vee \dot{x} \in C$ by Cau_3, and thus $\mathcal{F} \in S\,q_C$. With 8.3, this shows that $S\,q_C = C$.

<u>Proposition 8.5</u>. <u>If</u> C <u>is a</u> u-<u>regular initial Cauchy structure, then</u> q_C <u>is a regular limit structure</u>. <u>If</u> q <u>is a regular</u> T_2 <u>limit structure, then</u> $S\,q$ <u>is a</u> u-<u>regular Cauchy structure</u>.

<u>Proof</u>. From the definitions, $\Gamma^u(A,C)$ is the closure of a set A for q_C. Thus q_C is regular if C is u-regular. If q is a T_2 limit structure, then $q = q_{Sq}$ by 8.4, and thus closure for q is u-closure for $S\,q$. If $\mathcal{F} \in S\,q$, then $\mathcal{F} \leq \mathcal{G} \vee \dot{x}$ for a filter $\mathcal{G} \vee \dot{x}$ such that $\mathcal{G}\ q\ x$, and $\Gamma^u(\mathcal{F}, S\,q) \leq \Gamma^u(\mathcal{G} \vee \dot{x}, S\,q)$. But $\Gamma^u(\mathcal{G} \vee \dot{x}, S\,q)\ q\ x$ if q is regular, and it follows that $\Gamma^u(\mathcal{F}, S\,q) \in S\,q$. Thus $S\,q$ is u-regular if q is regular.

<u>8.6</u>. We are ready now for the proof of 8.1. Our proof is illustrated by the following two diagrams, with limit spaces at left, initial Cauchy spaces at right.



Let $\mathcal{S}$ be the class of separated u-regular compact initial Cauchy spaces, and $\mathcal{T}$ the class of regular compact T_2 limit spaces. Let (E,q) be a limit space, let $C = S\,q$, let $h : (E,C) \longrightarrow (E_1,C_1)$ be the universal mapping for (E,C) and $\mathcal{S}$, and let $q_1 = q_{C_1}$. h exists by 7.6 and 7.8, $(E_1,q_1) \in \mathcal{T}$ by 8.4 and 8.5, and $h : (E,q) \longrightarrow (E_1,q_1)$ is continuous since $q \leq q_C$. If $f : (E,q) \longrightarrow (E',q')$ is continuous, with $(E',q') \in \mathcal{T}$, let $C' = S\,q'$. Then $(E',C') \in \mathcal{S}$, and $f : (E,C) \longrightarrow (E',C')$ is continuous. Thus $f = f_1 h$ for a unique map $f_1 : (E_1,C_1) \longrightarrow (E',C')$. Since also $q' = q_{C'}$ and $C_1 = S\,q_1$, by 8.4, a mapping $f_1 : E_1 \longrightarrow E'$ is (q_1,q')-continuous iff f_1 is (C_1,C')-continuous. Thus $f = f_1 h$ for a unique map $f_1 : (E_1,q_1) \longrightarrow (E',q')$, and $h : (E,q) \longrightarrow (E_1,q_1)$ is the desired universal mapping for (E,q) and $\mathcal{T}$.

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