# EXTENSIONS OF UNIFORMITIES

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## 1. Introduction.

In this paper we are concerned with extending a certain type of uniformity from a subspace of a topological space to the entire space. This is a generalization of extending to the whole space a particular class of continuous real valued functions defined on the subspace. In the case of normal spaces our results can be applied to give new characterizations of these spaces.

In particular the uniformities that we wish to consider are those generated by totally bounded continuous pseudometrics. A pseudometric is totally bounded if for every positive real number $\epsilon$ there is a finite subset $G$ such that $X$ is the union of spheres of radius $\epsilon$ with centers in $G$. In [1], the authors considered extensions to the topological space $X$ of totally bounded continuous pseudometrics defined on a subspace $S$ of $X$. Subspaces $S$ for which every such pseudometric extends were said to be T-EMBEDDED in $X$. It was shown that T-embedded subspaces are the same as those subspaces for which every bounded continuous real valued function extends to the whole space. Since pseudometrics are intrinsic to the structure of uniform spaces corresponding notions must be considered. Using Andre Weil's definition of a uniformity we can say that every collection of pseudometrics on a space generates a uniformity and conversely for every uniformity there is a collection of pseudometrics that generate it. The case for the use of pseudometrics is strong since they yield a rich supply of continuous functions. Conversely every real valued function f defined on X determines a pseudometric on X by

$$\Psi_f(x,y) = |f(x) - f(y)|.$$

We say that a uniformity is generated by a class of real valued functions if it is GENERATED by pseudometrics $\Psi_f$ for f in the class.

Hence we can show that the subspaces S for which every uniformity generated by a collection of totally bounded continuous pseudometrics extends to X (such subspaces are called $\mu_T$-EMBEDDED) are the same as those subspaces for which every precompact uniformity extends (such subspaces are called u*-EMBEDDED). A uniformity is PRECOMPACT if it is generated by a collection of bounded continuous real valued functions.

## 2. Definitions and basic results.

Our notation and terminology coincides with that of J. L. Kelley in [5]. In particular our definition of a uniformity is that of Weil's as stated in Kelley.

A BASE $\mathcal{B}$ for a uniformity $\mathcal{U}$ on a non-empty set X is a subcollection of $\mathcal{U}$ such that for each U in $\mathcal{U}$ there is a B in $\mathcal{B}$ and $B \subset U$. A SUBBASE $\mathcal{S}$ for $\mathcal{U}$ is a subcollection of $\mathcal{U}$ such that finite intersections of members of $\mathcal{S}$ form a base for $\mathcal{U}$.

Given any collection $\mathcal{P}$ of pseudometrics on a non-empty set X, a unique uniformity on X can be obtained. The subbase for this uniformity is the subsets of $X \times X$ which are inverse images under the maps d in $\mathcal{P}$ of the real interval $[0,\epsilon)$ where $\epsilon$ ranges over all positive real numbers. Such a uniformity is called the UNIFORMITY GENERATED BY $\mathcal{P}$. In particular for each real valued function f on a non-empty set X there is a natural pseudometric $\Psi_f$ ASSOCIATED WITH f defined by $\Psi_f(x,y) = |f(x) - f(y)|$ for x and y in X. If $\mathcal{A}$ is a collection of real valued functions on X then the UNIFORMITY GENERATED BY $\mathcal{A}$ is the uniformity generated by the collection of pseudometrics $\Psi_f$ associated with f in $\mathcal{A}$. A PRECOMPACT UNIFORMITY is a uniformity $\mathcal{U}$ on X generated by a collection of bounded continuous real valued functions defined on X.

Every uniformity $\mathcal{U}$ on a non-empty set X yields a unique topology $T(\mathcal{U})$. This topology is obtained by taking as a base for the open sets the collection of sets U[x] for U in $\mathcal{U}$ and x in X. If $T(\mathcal{U})$ agrees with the original topology $\mathcal{J}$ of the topological space $(X,\mathcal{J})$ then $\mathcal{U}$ is said to be an ADMISSIBLE uniformity.

If S is a subset of X and if $\mathcal{G}$ is a collection of subsets of X then $\mathcal{G}|S$ is the collection of traces $G \cap S$ for G in $\mathcal{G}$. If $\mathcal{U}$ is a uniformity on S and if $\mathcal{V}$ is a uniformity on X then $\mathcal{V}$ is an EXTENSION OF $\mathcal{U}$ in case $S|S \times S = \mathcal{U}$.

Let $(F_\alpha)_{\alpha\in I}$ be a family of subsets of a topological space $(X,\mathcal{J})$. The family is DISCRETE if for each x in X there is a neighborhood of $x$ that meets at most one $F_\alpha$. Now if $(X,\mathfrak{U})$ is a uniform space, the family is UNIFORMLY DISCRETE in $X$ if there is a $U$ in $\mathfrak{U}$ such that the family of subsets $(U[F_\alpha])_{\alpha\in I}$ is pairwise disjoint.

In considering extensions of uniformities, it is necessary to consider extensions of continuous pseudometrics, that is pseudometrics which are continuous in the product topology. To analyze the situation the following definitions are required. For $\gamma$ an infinite cardinal number, a pseudometric $d$ is $\gamma$-SEPARABLE if there is a subset $G$ of $(X,\mathfrak{J})$ of cardinality at most $\gamma$ which is also dense in the topology $\mathfrak{J}_d$ generated by d. (This is just the topology obtained by taking as a base for the open sets the d-spheres of radius $\epsilon$). The pseudometric is TOTALLY BOUNDED if for every positive real number $\epsilon$ there is a finite subset $G$ of $X$ such that $X$ is the union of d-spheres of radius $\epsilon$ with centers from G. For a subset $S$ of X, and a pseudometric $d$ on S, an EXTENSION of $d$ is a pseudometric $r$ on $X$ such that $r|S \times S = d$. The subset $S$ is $P^\gamma$-EMBEDDED in $X$ if every $\gamma$-separable continuous pseudometric on $S$ can be extended to a $\gamma$-separable continuous pseudometric on X. The subset is T-EMBEDDED in $X$ if every totally bounded continuous pseudometric on $S$ extends to a totally bounded continuous pseudometric on X. It is P-EMBEDDED if every continuous pseudometric on $S$ extends to a continuous pseudometric on X.

It is clear that a P-embedded subset is $P^{\gamma}$-embedded and a $P^{\gamma}$-embedded subset is $P^{\aleph_o}$-embedded. Every totally bounded pseudometric is ${}_o$-separable, hence a $P^{\aleph_o}$-embedded subset is T-embedded. In [4], Gilman and Jerison define a subset S to be C (respectively C*)-EMBEDDED in X if every continuous (respectively bounded continuous) real valued function on S extends to a continuous (respectively bounded continuous) real valued function on X. In [6], it was shown that a C-embedded subset is $P^{\aleph_o}$-embedded, while in [3], the converse was shown. Hence the two notions are equivalent. In [1], the notions of T-embedding and C*-embedding were shown to be equivalent.

Corresponding notions for uniformities can now be defined. If $\gamma$ is an infinite cardinal number as before, then S is $U_{\gamma}$-EMBEDDED in X if every admissible uniformity on S generated by a collection of $\gamma$-separable continuous pseudometrics can be extended to an admissible uniformity on X. The subset is $\mu$*-EMBEDDED (respectively $\mu$-EMBEDDED) if every admissible uniformity on S generated by a collection of bounded continuous (respectively continuous) real valued functions can be extended to an admissible uniformity on X. It is $\mu_o$-EMBEDDED in X if every admissible uniformity on S can be extended to an admissible uniformity on X.

These notions were defined and discussed in [3]. Their relationships with the various forms of pseudometric embeddings were established also. In particular it was shown that: a $\mu_o$-embedded subset is necessarily P-embedded, a $\mu_{\gamma}$-embedded subset is necessarily $P^{\gamma}$-embedded, a $\mu$-embedded subset is necessarily C-embedded, a $\mu^*$-embedded subset is necessarily C*-embedded; also a $\mu_o$-embedded subset is necessarily $\mu_{\gamma}$-embedded, a $\mu$-embedded subset is necessarily $\mu^*$-embedded and finally that the subspaces which are $\mu_{\aleph_o}$-embedded are the same as those which are $\mu$-embedded. It was also shown that none of the implications, except for the implication that a $\mu_o$-embedded subset is $\mu_{\gamma}$-embedded, could be reversed.

It is now possible to define a subspace S to be $\mu_T$-EMBEDDED in the topological space X if every admissible uniformity on S generated by a collection of totally bounded continuous pseudometrics on S has an admissible extension to X.

The following results will be needed. It will be assumed throughout that S is a subspace of a topological space $(X,\mathfrak{T})$.

Theorem 2.1 (Gantner [3]). *If an admissible precompact uniformity on S has an admissible extension then it has an admissible precompact extension.*

Theorem 2.2 (Shapiro [7]). *If a $\gamma$-separable continuous pseudometric on S has an extension to a continuous pseudometric on X then it can be extended to a $\gamma$-separable continuous pseudometric on X.*

Theorem 2.3 (Alo and Shapiro [1]). *If a totally bounded continuous pseudometric on S can be extended to a continuous pseudometric on X then it can be extended to a totally bounded continuous pseudometric on X.*

Lemma 2.4. *If d is a totally bounded continuous pseudometric on X and if $S \subset X$, then $d|S \times S$ is a totally bounded continuous pseudometric on S.*

Proof: Clearly $d|S \times S$ is a continuous pseudometric. If S is finite then it is immediate that d is totally bounded. On the other hand if S has an infinite number of points and if $\epsilon > 0$ then there is a finite set $F \subset X$ such that X is the union of $\epsilon/2$ spheres centered at points of F. At least one sphere D must contain an infinite number of the points of F. Select any point x of $F \cap D$. Then the $\epsilon$ sphere centered at x covers D. Continuing this process for any $\epsilon/2$ sphere D, it follows that $d|S \times S$ is totally bounded.

It should be remarked that the usual definition of a precompact uniformity is not the one that we have given. In fact, a precompact uniformity is usually defined as one whose completion is compact. However, it can be shown that this is equivalent to our definition (see Gillman and Jerison [ 4 ], 15I.1 and 15I.2).

3. Main results.

Throughout $(X,\mathfrak{J})$ or just X will mean a topological space and S will be a subspace.

Proposition 3.1. *If f is a bounded continuous real valued function on X, then $\Psi_f$ is a totally bounded continuous pseudometric on X.*

Proof: Using [ 4 , 15E.1] it is sufficient to prove that for each positive real number $\epsilon$, X is a finite union of zero sets of diameter at most $\epsilon$. Thus for $\epsilon > 0$, choose an integer $k$ such that $f(x) \leqq (k + 1)\cdot\epsilon$ for all $x$ in $X$. The finite number of zero sets $Z_n = \{x \in X : n\epsilon \leqq f(x) \leqq (n + 1)\cdot\epsilon\}$ are of $\Psi_f$-diameter at most $\epsilon$ and $X$ is their union. It follows that $\Psi_f$ is totally bounded.

Theorem 3.2. *If $(X, \mathcal{U})$ is a precompact Hausdorff uniform space, then $\mathcal{U}$ is generated by the collection of all bounded real valued functions on $X$ that are uniformly continuous on $X$.*

This theorem is due to Gaal and appears in [ 2 ].

Theorem 3.3. *If $X$ is a completely regular, $T_1$ space then the following are equivalent.*

(1) $S$ *is* $\mu_T$-*embedded in* $X$.

(2) $S$ *is* $\mu^*$-*embedded in* $X$.

Proof. (1) *implies* (2). This implication is immediate by 3.1. In fact if $\mathcal{U}$ is a uniformity generated by a collection of bounded continuous real valued functions $\mathcal{G}$ then $\mathcal{U}$ is generated by $(\Psi_f)_{f\in\mathcal{G}}$ which is a collection of totally bounded continuous pseudometrics.

(2) *implies* (1). If $\mathcal{U}$ is an admissible uniformity on $S$ generated by totally bounded continuous pseudometrics on $S$, then $\mathcal{U}$ is generated by the collection $\mathcal{P}$ of all totally bounded uniformly continuous pseudometrics on $(S, \mathcal{U})$ (see Kelley [5 ], 6.15). By Proposition 3.1 for every bounded real valued uniformly continuous function $f$ on $(S, \mathcal{U})$, $\Psi_f$ is a member of $\mathcal{P}$.

The uniformity $\mathcal{V}$ generated by such uniformly continuous functions is precompact. Hence by assumption $\mathcal{V}$ will extend to a precompact uniformity if it is admissible. Since $\mathcal{U}$ is admissible, it is clear that the topology $T(\mathcal{V})$ generated by $\mathcal{V}$ is contained in the original topology $\mathcal{J}$ on S which is generated by $\mathcal{U}$. On the other hand, for $x \in G \in \mathcal{J}$ Weil (see [ 8 ]) has shown that there is a bounded real valued uniformly continuous function f on $(S,\mathcal{U})$ for which $f(x) = 0$ and $f(CG) \subset \{1\}$. The set $W = \{y \in X : f(x) < 1\}$ in $T(\mathcal{V})$, contains x and is contained in G. Hence G belongs to $T(\mathcal{V})$ and $\mathcal{V}$ is admissible.

Let $\mathcal{V}^*$ be an admissible precompact uniformity on X which is an extension of $\mathcal{V}$. By Theorem 3.2, $\mathcal{V}^*$ is generated by the collection of all bounded real valued uniformly continuous functions on $(X,\mathcal{V}^*)$. Let $\mathcal{U}^*$ be the uniformity on X generated by the collection $\mathcal{P}^*$ of all totally bounded continuous pseudometrics on X which are extensions of members d of $\mathcal{P}$. Recall that S is $u^*$-embedded in X implies that S is $C^*$-embedded in X. In [1 , Theorem 3.8 ] it was shown that S being $C^*$-embedded in X is equivalent to S being T-embedded in X. This with 3.1 shows that $\mathcal{P}^*$ is non empty. However $\mathcal{U}^*|S \times S$ is generated by $\mathcal{P} = \mathcal{P}^*|S \times S$ so $\mathcal{U}^*|S \times S = \mathcal{U}$. It remains to show that $\mathcal{U}^*$ is an admissible uniformity on $(X,\mathcal{J})$.

Since $\mathcal{U}^*$ is generated by a collection of continuous pseudometrics, the topology $T(\mathcal{U}^*)$ generated by $\mathcal{U}^*$ is contained in $\mathcal{J}$. Yet if f is any bounded real valued continuous function on $(X,\mathcal{V}^*)$, then $\Psi_f$ is a totally bounded continuous pseudometric. Hence $\mathcal{V}^* \subset \mathcal{U}^*$, $\mathcal{J} = T(\mathcal{V}^*) \subset T(\mathcal{U}^*)$ and $\mathcal{U}^*$ is an admissible extension of $\mathcal{U}$ to X, that is S is $U_T$-embedded in X. This completes the proof.

Now using Theorem 3.4, it is possible to state some new equivalent conditions for a completely regular space to be normal. The definitions that are necessary are the following.

Definition 3.4. For X a completely regular $T_1$ space, $\mathcal{U}_T(X)$ is the uniformity on X generated by the collection of all totally bounded continuous pseudometrics on X. Similarly let $\mathcal{U}_o(X)$, $\mathcal{U}_\gamma(X)$, $\mathcal{C}(X)$, and $\mathcal{C}^*(X)$ denote the admissible uniformities on X generated by the collections of all continuous pseudometrics on X, all $\gamma$-separable continuous pseudometrics on X (for $\gamma$ an infinite cardinal number), all continuous real valued functions on X, and all bounded continuous real valued functions on X respectively.

Proposition 3.5. *If* $(X,\mathcal{J})$ *is a completely regular* $T_1$ *space then the uniformity* $\mathcal{U}_T(X)$ *is admissible.*

Proof. Since $\mathcal{U}_T(X)$ is generated by a collection of continuous pseudometrics it follows that the topology $T(\mathcal{U}_T)$ generated by $\mathcal{U}_T(X)$ is contained in $\mathcal{J}$. On the other hand for $x \in G \in \mathcal{J}$ there is a bounded uniformly continuous real valued function f for which $f(x) = 0$ and $f(\complement G) \subset \{1\}$ (see [9]). Hence by 3.1 and as in the proof of Theorem 3.4, $\mathcal{J} \subset T(\mathcal{U}_T)$. This completes the proof.

It is known that S *is* P-*embedded in* X *if and only if* $\mathcal{U}_o(X)|S \times S = \mathcal{U}_o(S)$; S *is* C-*embedded in* X *if and only if* $\mathcal{C}(X)|S \times S = \mathcal{C}(S)$; *and* S *is* C*-*embedded in* X *if and only if* $\mathcal{C}^*(X)|S \times S = \mathcal{C}^*(S)$. Moreover, if X is completely regular then clearly $C(X) \subset \mathcal{U}_{\aleph_0}(X)$. However, it is not known if the converse implication is true. Similarly, if $X$ is completely regular then by 3.1, $C^*(X) \subset \mathcal{U}_T(X)$. We do not even know if $\mathcal{U}_T(X) \subset C(X)$.

It is now possible to state some preliminary results that will help in characterizing normal spaces.

<u>Theorem</u> 3.6. <u>If</u> $(X, \mathfrak{J})$ <u>is a completely regular</u> $T_1$ <u>space and if</u> $S \subset X$, <u>then the following are equivalent</u>.

(1) $S$ <u>is</u> T-<u>embedded in</u> $X$.

(2) $\mathcal{U}_T(X) | S \times S = \mathcal{U}_T(S)$.

(3) $\mathcal{U}_T(S)$ <u>has an admissible extension to</u> $X$.

<u>Proof</u>. (1) <u>implies</u> (2). Since $\mathcal{U}_T(X) | S \times S$ is an admissible uniformity generated by a collection of totally bounded continuous pseudometrics on $S$ and since $\mathcal{U}_T(S)$ is generated by all totally bounded continuous pseudometrics on $S$, it follows that $\mathcal{U}_T(X) | S \times S \subset \mathcal{U}_T(S)$. Conversely for $d$ a totally bounded continuous pseudometric on $S$ there is a pseudometric extension $e$ of $d$ to $X$. Hence for $\epsilon > 0$, the member of $\mathcal{U}_T(S)$ of the form $\{(x,y) \in S \times S : d(x,y) < \epsilon\} = \{(x,y) \in X \times X : e(x,y) < \epsilon\} \cap S \times S$ belongs to $\mathcal{U}_T(X) | S \times S$.

(2) <u>implies</u> (3). This implication is obvious.

(3) <u>implies</u> (1). Let $\mathcal{V}$ be an admissible uniformity on $X$ such that $\mathcal{V} | S \times S = \mathcal{U}_T(S)$. If $d$ is a totally bounded continuous pseudometric on $S$, then $d$ is uniformly continuous on $(S, \mathcal{U}_T(S))$. Since $S$ is a uniform subspace of $X$ in [ 7 , Theorem 1], it was shown that $d$ has a continuous pseudo-

metric extension to X. Using Theorem 2.3, it follows that S is T-embedded in X. The proof is now complete.

Corollary 3.7. If $(X,\mathfrak{J})$ is a completely regular topological space and if S is a $\mu_T$-embedded subset of X then S is a T-embedded subset of X.

Theorem 3.8. (See [3] Theorem 7.31 and [6] Theorem 3.9).

*If X is a completely regular space, then the following statements are equivalent.*

(1) *X is normal.*

(2) *Every closed subset of X is $\mu^*$-embedded in X.*

(3) *Every closed subset of X is T-embedded in X.*

(4) *Every finite family of pairwise disjoint closed subsets of X is uniformly discrete in $(X, C^*(X))$.*

Theorem 3.9. *If X is a completely regular topological space then the following statements are equivalent.*

(1) *X is normal.*

(2) *Every closed subset of X is $\mu_T$-embedded in X.*

(3) *For every closed subset F of X, $\mathcal{U}_T(X)|F \times F = \mathcal{U}_T(F)$.*

(4) *For every closed subset F of X, $\mathcal{U}_T(F)$ has an admissible extension to X.*

(5) *Every finite discrete family of closed subsets of X is uniformly discrete in $(X, \mathcal{U}_T(X))$.*

Proof. By 3.3 and 3.8, (1) is equivalent to (2). By 3.7, (1) implies that every closed subset is T-embedded in X. Thus by (3.6), (1) implies (3). Obviously (3) implies (4). Using

Theorems 3.6 and 3.8, it follows that (4) implies (1). Since discrete families of subsets are pairwise disjoint, Theorem 3.8 also shows that (1) implies (5). It remains to show that (1) follows from (5). Suppose $F_1$ and $F_2$ are disjoint closed subsets of X. Then since $\{F_1, F_2\}$ is discrete there is a $U \in \mathcal{U}_T(X)$ such that $U[F_1] \cap U[F_2] = \emptyset$. Moreover since $\mathcal{U}_T(X)$ is asmissible, U may be taken to be open. Hence X is normal and the proof is complete.

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