# PARACOMPACT SUBSPACES

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## 1. Introduction.

The definition of paracompactness for a topological space $(X,\mathfrak{J})$ states that every open cover of the space have a locally finite open refinement. When applying the definition to a subspace $S$ of $(X,\mathfrak{J})$, the phrases 'open cover' and 'open refinement', of course, refer to the relative topology on $S$. Locally finite also is in reference to open sets in $S$. Immediate questions arise when these phrases are used in reference to the topology for the whole space $X$. That is, one can consider a subspace $S$ of $(X,\mathfrak{J})$ to be STRONGLY PARACOMPACT if every cover of $S$ by members of $\mathfrak{J}$ has a locally finite in $X$ refinement by members of $\mathfrak{J}$. (In [1], such subspaces were called $\alpha$-paracompact.)

It is clear that every strongly paracompact subspace of a topological space is paracompact. However, the two notions are not equivalent. This follows from the example of a completely regular, non-normal space due to Niemytzki. Let $X$ be that subset of the plane $R \times R$ consisting of the points $(x,y)$ for $y \geq 0$. Let $S$ be that subset of $X$ consisting of the points $(x,0)$ of $R$. The relative product topology $\mathfrak{J}$ on $X$ is enlarged to include also as neighborhoods of the points in $S$ the $\epsilon$-spheres tangent to the points $(x,0)$ together with $\{(x,0)\}$ for all $\epsilon > 0$. That is, the sets

$$K(x,\epsilon) = \{(x,0) \cup \{(u,v)\epsilon S : (u-x)^2 + (v-\epsilon)^2 < \epsilon^2\}\}$$

are also included for all $\epsilon > 0$. With this new topology $S$ is a closed discrete subspace of X and hence is paracompact. However, D is not strongly paracompact. For the cover $(K(x,1/2))_{x\in D}$ is an open in X cover of S that has no refinement and is not locally finite in X. For, any $\mathfrak{J}$ neighborhood N of a point (x,0) in S will meet an infinite number of the sets K(x,1/2). As our main theorem will show this subspace S is not strongly paracompact since it is not P-embedded in X.

In particular, it will be shown that for normal Hausdorff spaces a subspace is strongly paracompact if and only if it is paracompact and P-embedded. A subset S is P-EMBEDDED in a topological space X if every continuous pseudometric on S extends to a continuous pseudometric on X. This concept was studied in [3] and [5]. Its relationship to paracompactness was studied in [4] and [6]. The notion of P-embedding generalizes the notion of C-embedding where extensions of continuous real valued functions are required.

The importance of P-embedding can be readily realized from the fact that it plays the same role in collectionwise normal spaces as C-embedding plays in normal spaces. In particular a space is <u>collectionwise normal if and only if every closed subset is</u> P-<u>embedded</u> (see [3]). Hence the notion of strongly paracompact, from the results in this paper, will have applications in such spaces.

The notion of strongly paracompact can also be generalized by making use of infinite cardinal numbers $\gamma$. Such techniques have already been applied to paracompactness and P-embedding.

Hence we will relate this new notion of 'strongly $\gamma$-paracompact' to the notions of $\gamma$-paracompact and $P^{\gamma}$-embedded. Throughout this paper emphasis will be placed on the notions in the definitions of paracompactness that deal with the topologies of the subspace and of the space. Hence we will refer to, for example, the local finiteness of a cover as being either locally finite in S (the subspace) or locally finite in X. Also $(X,\mathfrak{J})$ or X will mean a topological space with topology $\mathfrak{J}$.

2. <u>Definitions and Basic Results</u>. Let S be a subset of the topological space $(X,\mathfrak{J})$ and let $\gamma$ be an infinite cardinal number. The relative topology on S will be denoted by $\mathfrak{J}_S$. The subset S is $\gamma$-PARACOMPACT if every cover by members of $\mathfrak{J}_S$ of cardinality at most $\gamma$ has a locally finite in S refinement by members of $\mathfrak{J}_S$. The subset is STRONGLY $\gamma$-PARACOMPACT if every cover by members of $\mathfrak{J}$ of cardinality at most $\gamma$ has a locally finite in X refinement by members of $\mathfrak{J}$. In the special case of $\gamma = \aleph_0$ then $\gamma$-paracompact is the usual notion of countably paracompact. A pseudometric d on S is $\gamma$-SEPARABLE if there is a subset G of S whose cardinality is at most $\gamma$ and such that G is dense in S relative to the pseudometric topology $\mathfrak{J}_d$. The subset S is $P^{\gamma}$-EMBEDDED in X if every $\gamma$-separable continuous pseudometric on S has a continuous extension to X.

The subspace S is PARACOMPACT (respectively STRONGLY PARACOMPACT) if every cover of S by members of $\mathfrak{J}_S$ (respectively $\mathfrak{J}$) has a locally finite in S (respectively in X) refinement by members of $\mathfrak{J}_S$ (respectively $\mathfrak{J}$). It is P-EMBEDDED if every continuous pseudometric on $S$ extends to a continuous pseudometric on $X$. This is equivalent to saying that $S$ is paracompact (respectively strongly paracompact, respectively P-embedded) if and only if $S$ is $\gamma$-paracompact (respectively strongly $\gamma$-paracompact, respectively $P^{\gamma}$-embedded) for all cardinal numbers $\gamma$. By an extension $\mathfrak{U}^*$ of a cover $\mathfrak{U}$ on $S$ is meant a cover of $X$ such that the trace of $\mathfrak{U}^*$ on $S$ is $\mathfrak{U}$.

The definitions of other terms used in this paper can be found in [3]. The following results will be needed in the main part of this paper.

Theorem 2.1 (see [3]). If $S$ is a subspace of $X$ and if $\gamma$ is an infinite cardinal number then the following statements are equivalent:

(1) $S$ is $P^{\gamma}$-embedded in $X$.

(2) Every locally finite in $S$ normal cover of $S$ by members of $\mathfrak{J}_S$ of power at most $\gamma$ has a refinement by members of $\mathfrak{J}_S$ that can be extended to a locally finite in $X$ normal cover of $X$ by members of $\mathfrak{J}$.

(3) Every locally finite in $S$ normal open cover of $S$ by members of $\mathfrak{J}_S$ of power at most $\gamma$ has a refinement by members of $\mathfrak{J}_S$ that can be extended to a locally finite in $X$ normal cover of $X$ by members of $\mathfrak{J}$.

Theorem 2.2 (see for example [1]). If $X$ is a topological space then the following statements are equivalent:

(1) $X$ is normal.

(2) Every locally finite open cover of $X$ is normal.

Lemma 2.3 (Tukey, see [7]). Let $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ be a cover by members of $\mathfrak{T}_S$ of a subset $S$ of a topological space $X$ and let $\mathfrak{V} = (V_\beta)_{\beta \in J}$ be a locally finite open cover of $X$ such that the trace of $\mathfrak{V}$ on $S$ refines $X$. Then there is a locally finite open cover $\mathfrak{W} = (W_\alpha)_{\alpha \in I}$ of $X$ such that $W_\alpha \cap S \subset U_\alpha$ for each $\alpha \in I$.

3. Main Results.

Theorem 3.1. If $X$ is a topological space and if $S$ is a normal $\gamma$-paracompact $P^\gamma$-embedded subset of $X$, then $S$ is strongly $\gamma$-paracompact.

Proof. Let $\mathfrak{U}^*$ be a cover of $S$ by members of $\mathfrak{T}$ with cardinality at most $\gamma$. Since $S$ is $\gamma$-paracompact, the trace $\mathfrak{U}$ of $\mathfrak{U}^*$ on $S$ has a locally finite in $S$ refinement $\mathfrak{V}$ by members of $\mathfrak{T}_S$. By Lemma 2.3, $\mathfrak{V}$ can be assumed to have cardinality at most $\gamma$. Normality of $S$ implies that $\mathfrak{V}$ is normal in $S$ by Theorem 2.2. Moreover, since $S$ is $P^\gamma$-embedded in $X$, Theorem 2.1 states that $\mathfrak{V}$ has a refinement that extends to a locally finite in $X$ normal cover $\mathfrak{W}^*$ of $X$ by members of $\mathfrak{T}$. Now the family $\{W \cap V\colon\ W \in \mathfrak{W}^* \text{ and } V \in \mathfrak{U}^*\}$ is a locally finite in $X$ refinement of $\mathfrak{U}$ by members of the topology $\mathfrak{T}$ on $X$. Hence $S$ is strongly $\gamma$-paracompact.

Theorem 3.2. If $(X, \mathfrak{T})$ is a normal topological space and if $S$ is a closed subset of $X$, then the following statements are equivalent:

(1) $S$ is $\gamma$-paracompact and $P^{\gamma}$-embedded in $X$.

(2) $S$ is strongly $\gamma$-paracompact.

Proof. Since $S$ is normal, (2) follows from (1) by Theorem 3.1. On the other hand, if $S$ is strongly $\gamma$-paracompact then $S$ is $\gamma$-paracompact. Theorem 2.1 is now used to show that $S$ is $P^{\gamma}$-embedded in $X$. In fact if $\mathfrak{U}$ is a locally finite in $S$ cover of $S$ by members of $\mathfrak{J}_S$ of cardinality at most $\gamma$ then a cover $\mathfrak{U}^*$ of $S$ by members of $\mathfrak{J}$ with cardinality at most $\gamma$ is obtained by selecting one $U \in \mathfrak{J}$ for each $U \cap S \in \mathfrak{U}$. By (2), there is a locally finite in $X$ cover $\mathfrak{W}^*$ of $S$ by members of $\mathfrak{J}$ with cardinality at most $\gamma$ such that $\mathfrak{W}^*$ refines $\mathfrak{U}^*$. Then the trace $\mathfrak{W}$ of $\mathfrak{W}^*$ on $S$ is the refinement of $\mathfrak{U}$ by members of $\mathfrak{J}_S$ needed in 2.1. It is clear that $\mathfrak{W}$ is locally finite in $S$. The family $\mathfrak{G} = \mathfrak{W}^* \cup \{X \setminus S\}$ is a locally finite in $X$ cover of $X$ by members of $\mathfrak{J}$ whose trace with $S$ is $\mathfrak{W}$. By 2.2 and the fact that $X$ is normal, $\mathfrak{G}$ is normal. Hence $\mathfrak{G}$ is the required extension of $\mathfrak{W}$ and $S$ is $P^{\gamma}$-embedded in $X$.

Corollary 3.3. If $X$ is a normal Hausdorff topological space and if $S$ is a subset of $X$, then the following statements are equivalent:

(1) $S$ is strongly paracompact.

(2) $S$ is paracompact and $P$-embedded in $X$.

Proof. In [1, Corollary 4] it is shown that a strongly paracompact subset is closed. Thus 3.2 shows that (1) implies (2). Since paracompact Hausdorff spaces are normal, Theorem 3.1 shows that (2) implies (1). This completes the proof.

Corollary 3.4. If $X$ is a topological space and if $S$ is a paracompact Hausdorff P-embedded subset of $X$, then $S$ is strongly paracompact in $X$.

Corollary 3.5. If $X$ is a topological space and if $S$ is a normal Hausdorff countably paracompact $P^{\aleph_0}$-embedded subset of $X$, then $S$ is strongly countably paracompact.

In [2], it was shown that a normal space $X$ is $\gamma$-collectionwise normal if and only if every closed subset is $P^{\gamma}$-embedded in $X$. In particular, it was shown that $X$ is normal if and only if every closed subset is $P^{\aleph_0}$-embedded in $X$. Using these results it is now possible to obtain the following three corollaries.

Corollary 3.6. If $S$ is a closed subset of a normal $\gamma$-collectionwise normal space $X$, then $S$ is $\gamma$-paracompact if and only if $S$ is strongly $\gamma$-paracompact.

Corollary 3.7. If $S$ is a closed subset of a normal space $X$ then $S$ is countably paracompact if and only if $S$ is strongly countably paracompact.

Corollary 3.8. If $S$ is a closed subset of a collectionwise normal space $X$, then $S$ is paracompact if and only if $S$ is strongly paracompact.

Suppose $F$ is a strongly $\gamma$-paracompact subset of $S$ and $S$ is a subset of a topological $X$. It is now possible to give some sufficient conditions for $F$ to be strongly $\gamma$-paracompact in $X$.

Theorem 3.9. Let $F$ be a strongly $\gamma$-paracompact subset of the closed subspace $S$ of the topological space $X$. If there is an open set $G$ in $X$ such that $F \subset G \subset S$ then $F$ is strongly $\gamma$-paracompact in $X$.

Proof. Let $\mathfrak{U}^*$ be a cover of $F$ by members of $\mathfrak{J}$ with cardinality at most $\gamma$. The trace $\mathfrak{U}$ of $\mathfrak{U}^*$ on $S$ is a cover of $F$ by members of $\mathfrak{J}_S$. By assumption $\mathfrak{U}$ has a locally finite in $S$ refinement $\mathfrak{V}^*$ by members of $\mathfrak{J}_S$. The trace $\mathfrak{V}$ of $\mathfrak{V}^*$ on $G$ is a locally finite in $S$ refinement of $\mathfrak{U}^*$ by members of $\mathfrak{J}_G$ and covers $F$. From $G \subset S$ and $G$ open in $X$ it follows that $\mathfrak{V}$ is open in $X$. Clearly $\mathfrak{V}$ is locally finite in $X$ since $S$ is closed.

Corollary 3.10. If $F$ is a strongly paracompact subset of the closed subspace $S$ of $(X,\mathfrak{J})$ and if there is an open set $G$ in $X$ such that $F \subset G \subset S$ then $F$ is strongly paracompact in $X$.

Corollary 3.11. If $F$ is a paracompact Hausdorff P-embedded subset of the closed subspace $S$ of $(X,\mathfrak{J})$ and if there is an open set $G$ in $X$ such that $F \subset G \subset S$ then $F$ is strongly paracompact in $X$.

Proof. By Corollary 3.4, $F$ is a strongly paracompact subset of the closed subspace $S$. Hence by Corollary 3.10 $F$ is strongly paracompact in $Y$.

Corollary 3.12 (see [1]). If $S$ is a closed paracompact subset of $X$ and if $F$ is a closed subset of $X$ that is contained in the interior of $M$, then $F$ is strongly paracompact in $X$.

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