WALLMAN-TYPE REALCOMPACTIFICATIONS

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1. Introduction.

In this paper we extend our results in the area of Wallman-type (as first discussed by Orrin Frink in [4]) compactifications to include realcompactifications (see [1] and [2]). The theory of realcompact spaces is in many ways analogous to the theory of compact spaces. To a very large extent the realcompact spaces play the same role in the theory of C(X) (the ring of continuous real valued functions on a Tichonov topological space X) that the compact spaces do in the theory of C*(X) (that subring of C(X) consisting of the bounded functions). Since compact spaces have pleasant properties, their study has led to the study of compactifications. Usually a compactification of a space can give interesting and worthwhile information about the space itself. For example, Frink (see, [4]) used a compactification to show that every T_1 space with a normal base is a completely regular T_1 space (a Tichonov space). In doing this, he gave an elegant internal characterization of Tichonov spaces.

Since realcompact spaces and compact spaces play similar roles, a study of realcompactifications is also worthwhile. We use a variation of Frink's notion of normal base to construct realcompactifications of Tichonov spaces. We call our bases strong delta normal bases. It follows from our results that a space is Tichonov if and only if it has a strong delta normal base.

2. Definitions and Basic Results,

A family Z of subsets of a non-empty set X is a RING of sets if it is closed under finite unions and finite intersections. It is said to be a DELTA RING if it is a ring that is also closed under countable intersections. If X is a T_1-topological space, then Z is said to be DISJUNCTIVE if for any closed set F and for any point x not in F there is a 2 of Z that contains x and is disjoint from F; Z is said to be NORMAL if any two disjoint members A and B of Z are subsets respectively of disjoint complements C^1 and D' of members of Z. The family Z is COMPLEMENT GENERATED if for every member Z of Z there is a sequence of complements $(C^1_n)_{n \in N}$ of members of Z such that Z is their intersection.

The family Z is a NORMAL BASE for the T_1-topological space X if it is a disjunctive normal ring of sets that also Jbrms a base for the closed subsets of X. It is a STRONG NORMAL BASE if it is complement generated. It is a STRONG DELTA NORMAL BASE if it is a strong normal base that is also a delta ring of sets.

One of the most important examples of a strong delta normal base is the collection of all zero-sets of a Tichonov space X. A subset Z of X is a ZERO-SET if there is a continuous real valued function f defined on X such that Z is the set of all points x in X for which $f(x) = 0$. The complement of a zero-set is called a COZERO-SET. Gillman and Jerison in [5] have shown that this collection is a delta ring, that it satisfies the requirements for a normal base, and that each of its members is a countable intersection of cozero-sets (that is it is complement generated). Thus every Tichonov space has a strong delta normal base.

A topological space is called PERFECTLY NORMAL if it is normal and if each closed subset is a G_δ. Thus in every perfectly normal space the family of closed subsets is a strong delta normal base.

A proper subset of a ring of sets Z is a Z-FILTER if it is closed under finite intersections, contains every superset in Z of each of its member, and does not contain the empty set. A Z-ULTRAFILTER is a maximal Z-filter. (in [3], a theory of Z-filters is given). A subfamily 3 of Z has the COUNTABLE INTERSECTION PROPERTY (c.i.p.) if any countable subcollection of Z has a non-empty intersection. If Z is the collection of all zero sets of the topological space X then X is REALCOMPACT if every Z-ultrafilter with the c.i.p. has a non-empty intersection. Since a Z-filter is a collection of closed sets with the finite intersection property every compact space is realcompact. A REALCOMPACTIFICATION of a topological space X is a realcompact space Y which contains X densely.

In studying realcompact spaces the following notion is very useful. A subset A of X is Q-CLOSED in X if for every point p not in A there is a G_δ set G that contains p and is disjoint from A. The Q-CLOSURE of a subset A of X (denoted by $cl_Q A$) is the set of points p in X such that every G_δ set G containing p meets A. In general the Q-ciosure of a set need not be closed. Indeed, the Q-closure of any open interval of the real line is the open interval. However, the Q-closure of a set always contains the set and is contained in the ordinary closure of the set. S. G. Mrowka in [6] has shown the following two theorems which signals the importance of this notion and which we will need for our main result.

<u>Theorem 2.1</u>. Every Q-closed subset of a realcompact space is realcompact.

<u>Theorem 2.2</u>. A topological space X is realcompact if and only if it is Q-closed in its Stone-Cech compactification, #X.

As a corollary to these theorems we observe that:

<u>Corollary 2.3</u>. A topological space X is realcompact if and only if it is Q-closed in some compactification of X.

<u>Proof</u>. Necessity of the condition is obvious by the second theorem above. On the other hand, since every compact space is realcompact, the first theorem shows that the condition is also sufficient*

In compactifying a Tichonov space X, Frink. (see [4]) utilized the set to(Z) of all Z-ultrafilters for some normal base Z on X. A compact Hausdorff topology for co(Z) was obtained by assigning to each Z in Z the set Z^{ω} of all Z-filters 3 in co(Z) that contain Z. The collection Z^{40} of all Z^{CO} for Z in Z served as a base for the closed sets in a>(Z) . Each x in X is represented in co(Z)$_3$ since the set <p(x) of all Z in Z that contain x. is a member of co(Z) . Through this map <p, X is homeomorphic to a dense subset of to(Z) .

Equivalently a base for the open sets can be defined. For each U contained in X such that the complement of U, U», is a member of Z, assign the set U^{ω} of all 3 in co(Z) for which there is some Z in 3 that is contained in U. This collection of sets U^{ω} is an open base for the same topology on X.

3. Wallman Realcompactifications.

Let Z be any strong delta normal base on a Tichonov space X. For our realcompactification of X we consider the subspace p(Z) of to(Z) which consists of all Z-ultrafilters with the c.i.p. A base for the closed subsets of this subspace topology will be the collection of subsets *2?* of all 5 in p(Z) that contain Z. The sets <p(x) defined above are also members of p(Z). Thus X is homeomorphic to a dense subset of p(Z) via the map <p.

In [3], Alo and Shapiro have given some basic properties of this space. In particular the following lemma, which will be •needed in the next theorem, is proved there.

Lemma 3.1. Let Z be a delta ring of sets which is a base for the closed subsets of the topological space X and let 3 be a Z-ultrafilter with the c.i.p. If $(A_n)_{n\in N}$ is a sequence of sets in *&,* then the intersection A of the s6ts A_n is in 3.

Theorem 3.2. Let Z be a strong delta normal base of a Tichonov space X. Then X is homeomorphic to a dense subspace of a real-compact space p(Z).

Proof. The theorem will follow from the Corollary 2.3, if we show that p(Z) is Q-closed in its Frink-type compactification co(Z) . Let 3» be any member of £d(Z) that does not have the c.i.p. and let $(Z_i)_{i \in N}$ be a collection of subsets of 3 such that their co(Z)# Let 5 be any member of oo(Z) that does not have the c.i.p. and let $(Z_i)_{i \in N}$ be a collection of subsets of 3» such that their intersection Z is empty. Since Z is a strong delta normal base, each Z_i is generated by a sequence of open subsets $(A_i^n)_{n \in N}$ whose complements are in Z. Let G be the intersection of the basic open sets $(A_i^{i,n})_{i,n \in N}$. Since Z_i is contained in $A_i^{i,n}$ for each n, the G_δ set G contains 5# On the other hand by the lemma no Z-ultrafilter with the c.i.p. can be contained in G. Hence G does not meet p(Z) and therefore p(Z) is Q-closed in co(Z) . This completes the proof.

Corollary 3.3. If Z is the strong delta normal base of all zero sets of a Tichonov space X, then p(Z) is precisely the Hewitt realcompactification *uX*.

Theorem 3.4. If Z is a strong delta normal base on a Tichonov space X, then p(Z) is the Q-closure of <p(X) in co(Z) .

Proof. If 5 is any element of co(Z) without the c.i.p., then the proof of Theorem 3.2 exhibits a G_δ^* set G that contains 3* and misses p(Z). Therefore G misses the subset <p(X) of p(Z). Hence the Q-closure of <p(X) in co(Z) is contained in p(Z). To show the other direction it is sufficient to consider only G_δ sets G which are the intersection of basic open sets A_n^ω where the complement of A_n is in Z. If G is such a set which contains a member 5 of p(Z) then for each n

there is a Z_n in Z such that $Z_n{}^c A...{}_n^\omega$ Since *3ep(Z) ,* we can choose an x in the intersection of the Z_n. Then <p(x) is in the intersection of G and <p(X) and therefore p(Z) is contained in the Q-closure of <p(X) in co(Z) . This completes the proof of the theorem.

Since the Q-closure of a set is Q-closed, Theorem 3.2 can be deduced from Theorem 3.4 using Theorem 2.1. However, our approach above is justified by the importance of the construction.

In considering strong delta normal bases on realcompact spaces the following example is interesting. Let X be a discrete space of cardinality c (or any uncountable discrete space with non-measurable'cardinal [see Gillman and Jerison, p. 163]). Let Z_1 be the collection of all subsets A of X such that A or the complement of A, A^T, is at most countable. It is easy to verify that Z_1 is a strong delta normal base. Now $p(Z_1)$ is not equal to <p(X) for there is a member 5 of $p(Z_1)$ that is not in <p(X) . In fact let 3 be the Z^filter that is the collection of all subsets of X whose complement is at most countable. It is a Z_1-ultrafilter since the complement $Z^!$ of Z is in # for any member Z of a filter containing # where Z is not in 3. if (z_n) ^^ is any sequence of sets in 3^ then their common intersection Z is not empty since the complement of Z is at most countable and hence not equal to X. This shows that # has the countable intersection property. Finally X - {x} is in 5 for each x in X and the common

intersection of these sets is empty. It follows that the intersection of all the sets Z in 3 is empty. Thus 5 is not in <*p*(X).

Another interesting example is to consider the same space X with the strong delta normal base Z_2 of all subsets of X. It is clear that P(z 2^ = *vX* = ^(x) by Corollary 3.3 and by the fact that X is realcompact if and only if X = *vX* (see Gillman and Jerison p. 116). Hence, in this case we have Z_1 a subfamily of Z_2 and p(Z^) is, homeomorphically, a proper subset of $p(Z_1)$.

4. Conclusions.

Our examples above show that for subfamilies of the collection of all zero sets of a realcompact space X,p(Z) need not be homeomorphic to X. Also different strong delta normal bases on a space X give different realcompactifications of the space. The question immediately arises as to whether every realcompactification of a space can be obtained in this manner. That is if Y is a real-compactification of a Tichonov space X, does there exist a strong delta normal base Z on X such that Y is homeomorphic to p(Z)?

In contrast, we consider the following condition on a delta normal base Z.

(Q) If $Z_1 \Rightarrow Z_2 \Rightarrow \ldots \supset Z_n \Rightarrow \ldots$ is a nested sequence of members of Z then there is a sequence $(A_n)_{n \in N}$ of complements of members of Z such that $Z_n \subset A_n \subseteq Z_{n-1}$, for each n.

If we replace the condition of complement generated in our strong normal base by condition Q, similar results can be obtained.

That is Theorem 3.2, Corollary 3.3, and Theorem 3.4 can be obtained for delta normal bases satisfying condition Q.

In closing we note that most examples of Wallman realcompactifications that we know use subfamilies of the collection £ of all zero sets of X as their strong delta normal base. In this respect the following question is raised. If p(Z) is a Wallman realcompactification of a space X with strong delta normal base Z (which is not a subfamily of £) does there exist a strong delta normal base £* cz £ such that p(£*) is homeomorphic to p(Z)?

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