Thermodynamics of Materials with Elastic Range

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Contents

Part I: The General Theory

1. Introduction.

Two papers by COLEMAN ([1],[2]), published in 1964, have created a great deal of interest in the study of thermodynamics of materials with memory. Subsequent papers by WANG and BOWEN [3], COLEMAN and MIZEL ([4],[5]), GREEN and LAWS [6], and GURTIN[7], have examined the basic structure of Coleman's theory and have led to generalizations of Coleman's results.

Coleman's results are of two types. Results of the first type rest on a knowledge of material response to suitably short (in time duration) continuations of any history of deformation. The generalized stress relation and generalized dissipation inequality (Theorem 1,[1]) are examples of results of this type. Results of the second type rest on a knowledge of material response to suitably long continuations and suitably slow time reparameterizations of any history. Coleman's results of the second type establish both the extremum properties of the free energy functional at constant deformation histories as well as the relation between the slow limit approximation to the general thermodynamic theory and the usual equations of equilibrium thermodynamics (Theorems 3, 4,5,[1]). In this paper I discuss only results of the first type.

In the papers [3],[5] and [7], results of the first type are consequences of two important properties of the response functionals: (1) The domain of the response functionals is closed under local linear continuations, and (2) on sufficiently smooth histories, the free energy functional possesses a chain-rule property. A continuation of a given history of deformation is a history obtained by adding a new segment to the given deformation path. (The new segment may represent a constant deformation, in which case the continuation is called a static continuation). A local linear continuation is a continuation of short duration performed at constant strain-rate and temperature-rate. The chain-rule property for isothermal deformations is a relation of the form

$$\dot{\psi} = D\psi \cdot \dot{F} + \delta\psi \qquad (*)$$

where $\dot{\psi}$ and $\dot{F}$ represent the current rates of change of free energy and deformation gradient respectively, and where the coefficients $D\psi$ and $\delta\psi$ are history dependent. In particular, each of the papers [1], [3],[5],[6] and [7] uses a form of (*) in which the coefficients $D\psi$ and $\delta\psi$ are independent of $\dot{F}$ and, furthermore, are continuous on local linear continuations of continuous histories.

In addition to a restriction on the entropy functional, the generalized stress relation implies that

$$T = \rho D\psi F^T$$

where T, F and ρ are the current values of Cauchy stress, deformation gradient, and density, respectively. The generalized dissipation inequality takes the form

$$\delta\psi + (\underline{q}\cdot\underline{g}/\rho\theta) \leq 0$$

where $\underline{q}\cdot\underline{g}$ is the inner product of the heat flux vector $\underline{q}$ and the temperature gradient vector $\underline{g}$. This inequality implies that $\delta\psi \leq 0$. (In Coleman's theory, $\delta\psi$ corresponds to the negative of the internal dissipation, i.e. $-\rho\delta\psi/\theta$ represents the current entropy production. It should be noted that, for Coleman's theory, the heat conduction inequality $\underline{q}\cdot\underline{g} \leq 0$ is not valid, in general but is replaced by the generalized dissipation inequality).

In this paper I establish counterparts of the generalized stress relation and generalized dissipation inequality for a class of materials which includes elastic-plastic materials. For such materials, the entropy production is not necessarily continuous on linear continuations. In fact, a plastic-elastic transition may involve a jump discontinuity in the entropy production. Thus, it is necessary to obtain a form of the chain rule property in which $\delta\psi$ may have jump discontinuities during linear continuations. In the theory which follows, a relation similar to (*) is derived, and the counterpart of the quantity $\delta\psi$ has the property that it vanishes during certain continuations of any history. This property along with an approximation property for the counterpart of $D\psi$ allows Coleman's main results to be proved for the class of materials discussed here.

More specifically, I discuss here a class of simple materials characterized by the existence of a collection of strain-temperature points called the <u>elastic range</u>. The elastic range depends upon a given strain-temperature history for a material point and is introduced here through the concept of the <u>elastic set</u> for the given history. The elastic set consists of all histories which are continuations of the given history and which give a special type of material response: <u>the free energy functional is locally path-independent on the elastic set</u>. (Histories in the elastic set are called <u>elastic continuations</u>.) Roughly speaking, this condition specifies that if an elastic continuation is itself continued in any direction for a short time, the free energy response functional behaves like that of a hyperelastic material. As indicated above, the entropy production $-\delta\psi$ vanishes during an elastic continuation of any history. In fact, I show in Section 8 that the quantity $\delta\psi$ reduces to negative the "plastic power production" in the case of an infinitesimal theory of elastic-plastic materials.

The proof of the generalized stress relation and the generalized dissipation inequality has three main parts. First, the analogue of (*) is derived. Next, the results are proved for elastic continuations of a fixed but arbitrary history. Lastly, the results are extended to the given history. The proof for elastic continuations relies on the fact that a form of the chain rule, with $\delta\psi$ identically zero and $D\psi$ continuous, holds during elastic continuations. The method of extension to an arbitrary history relies on the continuity of the stress and entropy functionals during certain continuations of any history.

The present theory is general enough to include rate effects during intervals of time when a given history is not an elastic continuation of any history (i.e., during the so-called loading periods). However, during elastic continuations, rate effects necessarily are absent. This last restriction can be removed without difficulty. In fact, the present theory can be extended to include rate effects at any time during a deformation. (To obtain the more general theory, one can introduce a viscoelastic range in place of the elastic range. Coleman's results would apply directly for "viscoelastic continuations" and the method used in the present theory would allow these results to be extended to arbitrary histories.)

In this paper, no assumptions of fading memory are made. Thus, the effect of events in the distant past on the present material response need not be small. One assumption is made which restricts the short-range memory of the materials in question: recently encountered states of strain and temperature are elastically accessible from the present state. In other words, elastic continuations to certain past states of deformation always are possible.

The present theory is a generalization of the theory of PIPKIN and RIVLIN [8] and of the theories of GREEN and NAGHDI ([9], [10], [11]). The assumption of the existence of an elastic range underlies each of these theories, including the present one. However, the present theory gives an analysis which is not limited to rate-independent materials and which does not depend upon the concepts of elastic and plastic strain. It is worth noting that the concepts of elastic and plastic strain arise naturally in the general theory presented here. In addition the generalized stress relation can be used to derive constitutive relations for the plastic strain rate. These remarks will be discussed in a future paper. The notions of elastic and plastic strain which are used in Section 8 of the present paper are the classical ones appropriate for the theory of elastic-plastic materials.

2. Preliminaries.

Let A be the set of all pairs $a = (E,\nu)$ where E is any second-order tensor in R^3 and ν is any real number. (Throughout this paper, R^n denotes an n-dimensional vector space.) The set A is given the structure of ten-dimensional Euclidean space with addition and scalar multiplication defined in the obvious way and with inner product and norm defined by the relations

$$a_1 \cdot a_2 = \mathrm{tr}(E_1 E_2^T) + \nu_1 \nu_2$$

$$|a| = [\mathrm{tr}(EE^T) + \nu^2]^{\frac{1}{2}},$$

respectively. A *positive pair* is defined to be a pair of the form (F,θ) where F has positive determinant and θ is positive. (In this presentation, F will represent the deformation gradient and θ the temperature at a material point.) The symbol A^+ denotes the set of all positive pairs, and it is clear that A^+ is an open subset of A.

A mapping $f: [0,\infty) \to A^+$ is called a *strain-temperature history* or, simply, a *history*. The set of all histories is denoted by $\mathcal{G}^+$. The difference between two histories in general is not a history. However, the difference is a mapping from $[0,\infty)$ into A. The symbol $\mathcal{G}$ represents the collection of all mappings with domain $[0,\infty)$ and with range a subset of A.

Following COLEMAN and MIZEL [4], it is convenient to define for each $\sigma\in[0,\infty)$ the *σ-section* of a history f by the relation

$$f_{(\sigma)}(s) = f(s + \sigma),$$

$s\in[0,\infty)$. The history $f_{(\sigma)}$ is one obtained from f by removing a recent piece of the strain-temperature path associated with f.

Let f and g be histories. If there exists $\sigma\in[0,\infty)$ such that $g_{(\sigma)} = f$, then g is called a *continuation of* f. If g is a continuation of f and a is a positive pair such that $g(0) = a$, then g is called *a continuation of* f *to* a. If $\sigma\in[0,\infty)$ and $b\in A$, *the* $[\sigma,b]$ *local linear continuation* of f is defined by the relation

$$f[\sigma,b](s) = \begin{cases} f(s-\sigma), & s \geq \sigma \\ f(0)+(s-\sigma)b, & 0 \leq s \leq \sigma. \end{cases}$$

Note that for σ sufficiently small, $f[\sigma,b]\in\mathfrak{A}^+$.

The derivative of a history evaluated at $s\in[0,\infty)$ is denoted by $\dot{f}(s)$ and the corresponding mapping of $[0,\infty)$ into A by $\dot{f}$. In Section 3 a discussion of smoothness requirements for histories is presented.

The following notation will be used for present values of thermodynamic quantities at a fixed material point:

ρ . . . density

θ . . . temperature

ϵ . . . internal energy per unit mass

η . . . entropy per unit mass

ψ . . . free energy per unit mass

r . . . heat supply

$\underline{b}$ . . . body force per unit mass

$\underline{g}$ . . . spatial temperature gradient

$\underline{q}$ . . . heat flux vector

F . . . deformation gradient

T . . . Cauchy stress

L . . . velocity-gradient tensor.

The present values of the time rates of change of ψ and η are denoted by $\dot{\psi}$ and $\dot{\eta}$, respectively. (The dot is used both to denote differentiation with respect to time and also with respect to the variable s. The particular use of the dot will be clear from the symbol above which it appears.) The pair $\Sigma \epsilon A$ is defined by the relation

$$\Sigma = (T(F^{-1})^T/\rho, -\eta)$$

and is called *the generalized stress*.

If f denotes the history of strain and temperature, the first and second laws of thermodynamics can be combined to give the inequality

$$\dot{\psi} + \Sigma \cdot \dot{f}(0) + \underline{q} \cdot \underline{g}/\rho\theta \leq 0.$$

This inequality will be referred to as the Clausius-Duhem inequality. The symbol "$\cdot$" denotes the inner product in n-dimensional Euclidean space. (The value of n should be clear from the context.) The sign appearing in front of the term in Σ is positive because $\dot{f}(0)$ represents a derivative with respect to the variable s. One can interpret the variable s as a backward measure of time with $s=0$ denoting the present instant.

In Section 7 Coleman and Noll's method [12] is used to determine the implications of the Clausius-Duhem inequality for a class of materials with memory. For the purpose of the thermodynamic analysis this technique regards the body force and heat supply as quantities determined through the laws of balance of momentum and balance of energy. Of course, for a problem in which the motion of a particular body is to be determined, the body force and heat supply must be treated as fixed quantities.

3. The Class of Histories.

Consider first the class of all functions $h \in \mathcal{G}$ such that h is continuous, bounded, and piecewise smooth with bounded derivative in $[0,\infty)$. This class is denoted by $\mathcal{PS}$. If $\dot{h}(s)$ does not exist in the usual sense, define $\dot{h}(s) = \dot{h}(s+)$. The fact that h is piecewise smooth guarantees (by definition) that the limit $\dot{h}(s+)$ exists. For each $h \in \mathcal{PS}$ the number $\|h\| = \sup_{s \in [0,\infty)} |h(s)|$ is finite.

The histories discussed in the sequel are restricted to belong to the class $\mathcal{PS}^+ \overset{def}{=\!=} \mathcal{PS} \cap \mathcal{G}^+$. The assumption of boundedness for the histories and their derivatives rules out histories such as those arising in steady viscometric flows. However, since the materials on which the present theory is based are not capable of undergoing such flows, the boundedness condition in the definition of $\mathcal{PS}^+$ is not unduly restrictive.

Since $\mathcal{PS}^+ \subset \mathcal{PS}$, the norm $\|\cdot\|$ induces a topology on $\mathcal{PS}^+$. The next two lemmas give some properties of this topology.

Lemma 1. *Let f be any history. Then* $\|f - f_{(\sigma)}\| \to 0$ as $\sigma \to 0$.

Proof. Let $\sigma, s \in [0,\infty)$ and assume that $\dot{f}$ is continuous in the interval $(s, s+\sigma)$. Applying the mean value theorem to each component of f, one obtains the inequality

$$|f_{(\sigma)}(s) - f(s)| < \sqrt{10}\, M_f \sigma$$

where $M_f \in (0,\infty)$ is chosen so that $|\dot{f}(s)| < M_f$ for every $s \in [0,\infty)$. Since $\dot{f}$ only has isolated jumps, the same estimate for the difference $|f_{(\sigma)}(s) - f(s)|$ applies for any interval $(s, s+\sigma)$. Hence $\|f_{(\sigma)} - f\| < \sqrt{10}\, M_f \sigma$ and the result follows immediately.

Lemma 2. Let $f \in \mathcal{P}\mathcal{S}^+$, $b \in A$ and $\sigma \in [0,\infty)$ be such that the local linear continuation $f[\sigma,b] \in \mathcal{P}\mathcal{S}^+$. Then $f[\sigma',b] \in \mathcal{P}\mathcal{S}^+$ for $0 \leq \sigma' \leq \sigma$ and $\|f - f[\sigma',b]\| \to 0$ as $\sigma \to 0$.

Proof. Clearly, if $f[\sigma,b] \in \mathcal{P}\mathcal{S}^+$ then $f[\sigma',b] \in \mathcal{P}\mathcal{S}^+$ for every $\sigma' \in [0,\sigma]$. The last statement in the lemma can be proved from the estimate

$$\begin{aligned}
\|f[\sigma',b] - f\| &\leq \sup_{\sigma' \leq s} |f[\sigma',b](s) - f(s)| \\
&\quad + \sup_{0 \leq s \leq \sigma'} |f[\sigma',b](s) - f(s)| \\
&\leq \|f - f_{(\sigma')}\| + \\
&\qquad \sup_{0 \leq s \leq \sigma'} \{ |f[\sigma',b](s) - f(0)| \\
&\qquad\qquad + |f(0) - f(s)| \}.
\end{aligned}$$

Lemma 1, the definition of $f[\sigma',b]$, and the continuity of f imply the desired result.

It is convenient to introduce some additional notation at this point. The set of all continuations of f is denoted by $C(f)$.

If $a \in A^+$, then the set of all continuations of f to the positive pair a is denoted by $C(f,a)$. In defining the elastic set, the histories which are continuations of a given history g and which are close to g play an important role. If $0 < \delta < \infty$, the set $C_\delta(g)$ is defined to be the collection of all continuations of g such that $\|h - g\| < \delta$ whenever $h \in C_\delta(g)$. Each element of $C_\delta(g)$ is called a δ-*continuation of* g. The concept of a δ-*continuation of* g *to* a, where a is any positive pair, is defined in the obvious way. The set of all such δ continuations of g to a is denoted by $C_\delta(g,a)$. It is worth noting that the number $\|g - h\|$ compares the histories g and h as functions on $[0,\infty)$. Consequently, for a fixed δ, a local linear continuation of $g[\sigma,b]$ in general will not be a δ-continuation of g if σ is too large. The same remark applies to static continuations of a fixed history.

4. The General Form for the Constitutive Relations.

The terms *thermodynamic process* and *admissible thermodynamic process* are used in the sense of COLEMAN and NOLL [12]. In the present context the constitutive relations take the form

$$\psi = \hat{\psi}(f)$$
$$\Sigma = \hat{\Sigma}(f)$$
$$\underline{q} = \hat{\underline{q}}(f, \underline{g})$$

where $\hat{\psi}$ maps $\mathcal{P}\mathcal{S}^+$ into the reals, $\hat{\Sigma}$ maps $\mathcal{P}\mathcal{S}^+$ into the set of pairs A , and $\hat{\underline{q}}$ maps $\mathcal{P}\mathcal{S} \times R^3$ into R^3.

Henceforth it is required that every admissible thermodynamic process satisfy the Clausius-Duhem inequality. If $\hat{\psi}$ and $\hat{\Sigma}$ are allowed to depend upon $\underline{g}$, then, with suitable smoothness assumptions, the Clausius-Duhem inequality rules out such dependence (see [1] for an analogous case). For convenience the dependence upon $\underline{g}$ is omitted from the outset.

The following smoothness assumptions on $\hat{\psi}$, $\hat{\Sigma}$ and $\hat{\underline{q}}$ are now set forth:

S1. $\hat{\Sigma}$ and $\hat{q}$ are continuous on $\mathcal{P}\mathcal{S}^+$ and $\mathcal{P}\mathcal{S}^+ \times R^3$, respectively.

S2. The limit

$$\dot{\hat{\psi}}(f) \overset{\text{def}}{=} \lim_{\sigma \downarrow 0} \frac{\hat{\psi}(f) - \hat{\psi}(f_{(\sigma)})}{\sigma}$$

exists for each $f \in \mathcal{P}\mathcal{S}^+$.

These assumptions require that the generalized stress, the heat flux vector, and the free energy functionals retain certain smoothness features of the histories on which they are evaluated.

5. The Elastic Set and the Elastic Range.

This section contains a detailed description of the materials of interest here.

Definition 1. Let $\gamma: \mathcal{G} \to R^m$ where $\mathcal{G} \subset \mathcal{PS}^+$ is an arbitrary set of histories. Then γ is said to be *path-independent on* $\mathcal{G}$ if $\gamma(g_1) = \gamma(g_2)$ whenever $g_1(0) = g_2(0)$ and $g_1, g_2 \in \mathcal{G}$.

The notion of path-independence arises in the study of hyperelastic materials. In fact, a hyperelastic material is one for which the free energy functional is path-independent on the set of all histories, i.e. on all of $\mathcal{PS}^+$.

The notion of path independence is the central idea in

Definition 2. Let f be a history and let g be a continuation of f with $g_{(\sigma)} = f$ and $\sigma > 0$. Then g is said to be an *elastic continuation of* f if the following conditions hold:

1. For each $\sigma' \in [0,\sigma)$ there exists $\delta = \delta(\sigma', g) > 0$ such that the free energy functional $\hat{\psi}$ is path independent on the set of all δ-continuations of $g_{(\sigma')}$.
2. $\lim_{\sigma' \downarrow 0} \delta(\sigma', g) > 0$.

If condition 1 holds with $\sigma' = \sigma$, then f is said to be an *elastic continuation of itself*. Before a discussion of this definition can be given, some additional terminology must be introduced. If g is an elastic continuation of f and $g(0) = a$, then g is said to be an *elastic continuation of* f *to* a. The set of all elastic continuations of f will be denoted by $\mathcal{E}(f)$ and called the *elastic set for* f. Note that f is included in $\mathcal{E}(f)$ in the event that f is an elastic continuation of itself. The set of all elastic continuations of f to a positive pair a, i.e. $\mathcal{E}(f) \cap \mathcal{E}(f,a)$, will be denoted by $\mathcal{E}(f,a)$.

The definition of elastic continuation presented here is related to the familiar description of hyperelastic materials: the constitutive functionals (in particular, the free energy) of a hyperelastic material are path-independent on $\wp\mathcal{S}^+$. The essential idea in developing a theory which includes elastic-plastic phenomena is that of requiring that the free energy functional be path-independent on certain proper subsets of $\wp\mathcal{S}^+$ rather than on the entire set. For each elastic continuation g of a history f, with $g_{(\sigma)} = f$, and for each $\sigma' \in [0,\sigma)$, there corresponds one such proper subset of $\wp\mathcal{S}^+$, namely, the set of all δ-continuation of $g_{(\sigma)}$, where $\delta = \delta(\sigma',g)$. In other words, sufficiently short continuations of $g_{(\sigma')}$ in <u>all</u> directions produce a path-independent response from the functional $\hat{\psi}$. It should be noted that one cannot expect, in general, that f will be an elastic continuation of itself. (Take the case of an elastic-plastic material when the current stress is on the yield surface.) For this reason $\sigma' \epsilon [0,\sigma)$, rather than $\sigma' \epsilon [0,\sigma)$, is written in part 1 of Definition 2. Condition 2 of Definition 2 guarantees that the sets $\mathcal{C}_\delta(g_{(\sigma')})$ on which Φ is path-independent not shrink in diameter to zero as $\sigma' \to 0$. If one were to assume that $\delta(\cdot,g)$ is a continuous function, then this condition automatically would be satisfied.

In the next formal assumption, it is required that, even if f is not an elastic continuation of itself, f must be a limit of elastic continuations of f to points $f(\sigma)$ as $\sigma \to 0$. Hence, a sequence of recently encountered strain-temperature points can be reached via elastic continuations which, themselves, differ only slightly from the history f.

A1. For each $f \epsilon \mathcal{P}\mathcal{S}^+$ the closure of the set $\bigcup_{\sigma \epsilon (0,\infty)} \mathcal{E}(f,f(\sigma))$ contains the history f.

In particular, this assumption implies that the elastic set for each history is non-empty.

In general, if $a \epsilon A^+$ and $g(0) = a$ for some elastic continuation of f, the pair a is said to be elastically accessible from f. The set of all pairs $a \epsilon A^+$ which are elastically accessible from f is denoted by $\mathbb{E}(f)$ and called the elastic range for f. Thus, for each history, the elastic set is itself a collection of histories, while the elastic range is a set of strain-temperature points, i.e. a set of positive pairs.

The next lemma describes properties of the sets $\mathcal{E}(f)$ and $\mathbb{E}(f)$.

Lemma 3. Let f be any history. Then

1. If g is an elastic continuation of f then $\mathcal{E}(g) \subset \mathcal{E}(f)$, and hence $\mathbb{E}(g) \subset \mathbb{E}(f)$.

2. The elastic range $\mathbb{E}(f)$ is an open subset of A^+.

Proof. The proof of 1 requires an examination of Definition 2. Suppose that $g \in \mathcal{E}(f)$ with $g_{(\sigma)} = f$. If $h \in \mathcal{E}(g)$ with $h_{(\sigma^*)} = g$, then $h_{(\sigma+\sigma^*)} = f$. Let

$$\delta^* (\sigma',h) = \begin{cases} \delta(\sigma',h), & \sigma' \in [0,\sigma^*) \\ \delta(\sigma'-\sigma^*,g), & \sigma' \in [\sigma^*,\sigma^*+\sigma). \end{cases}$$

Writing $\delta^* = \delta^*(\sigma',h)$, it follows that $\hat{\psi}$ is path-independent on $C_{\delta^*}(h_{(\sigma')})$ for every $\sigma' \in [0,\sigma^*+\sigma)$. Furthermore, $\underline{\lim}_{\sigma' \downarrow 0} \delta^*(\sigma',h) > 0$. Hence, $h \in \mathcal{E}(f)$.

The idea of the proof of 2 is to show that certain local linear continuations of each $g \in \mathcal{E}(f)$ are elastic continuations of f. The following estimate is needed: let $g \in \mathcal{P}\mathcal{S}^+$, $\epsilon > 0$ and $b \in A$ with $|b| = 1$; then

$$\|g - g[\epsilon,b]\| \leq \epsilon[2\sqrt{10}\, M_g + 1]$$

where $M_g = \sup_{s \in [0,\infty)} |\dot{g}(s)|$. Now, let $g \in \mathcal{E}(f)$, with $\delta = \delta(0,g)$ and $\bar{\epsilon} = \delta/[4\sqrt{10}\, M_g + 2]$. By the above estimate, if $\epsilon < \bar{\epsilon}$, then

$\|g - g[\epsilon,b]\| < \delta/2$, and $C_{\delta/4}(g[\epsilon,b]) \subset C_{\delta}(g)$ for every b such that $|b| = 1$. Hence $\hat{\Phi}$ is path-independent on the set $C_{\delta/4}(g[\epsilon,b])$, so that $g[\epsilon,b] \in \mathcal{E}(g)$ for each $\epsilon < \bar{\epsilon}$ and b such that $|b| = 1$. Since $g \in \mathcal{E}(f)$ it follows from the first part of this lemma that $g[\epsilon,b] \in \mathcal{E}(f)$. Hence all positive pairs of the form $g(0) + \epsilon b$, with $\epsilon < \bar{\epsilon}$ and $|b| = 1$ are elastically accessible from f. It follows that $E(f)$ is open.

6. Properties of $\hat{\psi}$.

Condition 1 in Definition 2 guarantees that $\hat{\psi}$ is path-independent on the set $C_\delta(g)$, where $\delta = \delta(0,g)$ and $g \in \mathcal{E}(f)$. Another way of stating this condition is to say that $\hat{\psi}$ is constant on the set $C_\delta(g,a)$ for each fixed $a \in A^+$. (If this set is empty, $\hat{\psi}$ trivially is constant.) For each $g \in \mathcal{E}(f)$ it is natural to define a function which is derived from the restriction of $\hat{\psi}$ to $C_\delta(g)$. Accordingly, if $g \in \mathcal{E}(f)$ and $a \in A^+$ are such that $C_\delta(g,a)$ is non-empty, define $\psi(a;g)$ by the relation

$$\psi(a;g) = \hat{\psi}(g^*)$$

where g^* is any element of $C_\delta(g,a)$. If $\pi_o : \mathcal{P}\mathcal{S}^+ \to A^+$ is the projection operator, i.e.

$$\pi_o(g) = g(0)$$

for each $g \in \mathcal{P}\mathcal{S}^+$, then the domain of $\psi(\cdot;g)$ is the open set $\mathbb{E}_o(g) \overset{\text{def}}{=\!=} \pi_o(C_\delta(g))$.

The next assumption gives important smoothness properties of $\psi(\cdot;g)$.

S3. 1. <u>If</u> g <u>is an elastic continuation of</u> f, <u>then</u> $\psi(\cdot;g)$ <u>is a</u> C^1 <u>function on</u> $\mathbb{E}_o(g)$.

2. <u>Let</u> $\nabla\psi(\cdot;g)$ <u>denote the gradient of</u> $\psi(\cdot;g)$. <u>Assume that</u>

$$D\psi(f) \overset{\text{def}}{=\!=} \lim_{n \to \infty} \nabla\psi(g_n(0);g_n)$$

exists for every sequence $\{g_n\} \subset \mathcal{E}(f)$ *for which* $g_n \to f$ *an* $n \to \infty$ *and that* $D\psi(f)$ *is independent of the particular sequence* $\{g_n\}$ *chosen from* $\mathcal{E}(f)$. *Then* $D\psi(f)$ *satisfies the following approximation property: there exists* $\epsilon > 0$ *such that*

$$\hat{\psi}(h) - \hat{\psi}(f) = D\psi(f)\cdot(h(0) - f(0)) + o(|h(0) - f(0)|)$$

whenever h *is both an elastic continuation and an* ϵ-*continuation of* f.

The smoothness assumptions guarantee that $\hat{\psi}$ (locally) resembles the free energy response functional for a hyperelastic material. Two remarks are relevant to the assumption in part 2. First of all, in Section 7 it will be shown that $D\psi(f)$ exists for every history f. Secondly, the approximation property in Part 2 of S3 has the following significance. The functions $\psi(\cdot;f)$ and $\nabla\psi(\cdot;f)$, may not be defined in the event that f is not an elastic continuation of itself. It is essential in order to obtain a chain rule property to assume that $D\psi(f)$ satisfies an approximation property analagous to that satisfied by a C^1 function. The assumption that such an approximation property holds is reasonable in view of Lemma 5 (to be proved in this section), which shows that $D\psi(f)$ and $\nabla\psi(f(0);f)$ are equal if f is an elastic continuation of some history. It is convenient to give a name to the pair $D\psi(f)$. Accordingly, $D\psi(f)$ will be referred to as the *elastic gradient of the functional* ψ *at* f. It is clear that $D\psi(f)$ is an element of the set A. (Recall that A is the set of all pairs, positive or not.) Furthermore, $D\psi(f)$ may depend upon all of the values $f(s)$ where $s\in[0,\infty)$.

The assumption that the elastic gradient satisfies the above approximation property implies that another type of gradient of $\hat{\psi}$ exists whenever the elastic gradient itself exists. This gradient is computed from difference quotients

$$\Delta_\sigma\hat{\psi}(f) = \frac{\hat{\psi}(f^{(\sigma)}) - \hat{\psi}(f_{(\sigma)})}{\sigma}$$

where $f^{(\sigma)}$ is an elastic continuation of f to the pair $f(\sigma)$ and where $\|f^{(\sigma)} - f\| \to 0$ as $\sigma \to 0$. Note that A1, Section 5, guarantees that there always exists a sequence of histories $f^{(\sigma)}$ having such properties. If $\lim_{\sigma\to 0} \Delta_\sigma\psi(f)$ exists, the limit is denoted by $\delta\psi(f)$ and is called the *history gradient of* $\hat{\psi}$ *at* f. The history gradient will play the role of the corresponding quantity $\delta\psi$ discussed in the introduction.

Lemma 4. Let f be a history for which the elastic gradient at f exists. Then the history gradient at f exists and is given by

$$\delta\psi(f) = \dot{\hat{\psi}}(f) + D\psi(f)\cdot\dot{f}(0).$$

Note that the last relation is the verstion of the chain rule property appropriate to the present theory.

Proof. The idea of the proof is to write $\Delta_\sigma \hat{\psi}(f)$ as the sum of two difference quotients and to show that each quotient has a limit. Thus

$$\frac{\hat{\psi}(f^{(\sigma)}) - \hat{\psi}(f_{(\sigma)})}{\sigma} = \frac{\hat{\psi}(f) - \hat{\psi}(f_{(\sigma)})}{\sigma} + \frac{\hat{\psi}(f^{(\sigma)}) - \hat{\psi}(f)}{\sigma}.$$

By S2 (Section 4), the first term on the right hand side above tends to $\dot{\hat{\psi}}(f)$ as $\sigma \to 0$. From S3,

$$\frac{\hat{\psi}(f^{(\sigma)}) - \hat{\psi}(f)}{\sigma} = D\psi(f) \cdot \frac{(f(\sigma) - f(0))}{\sigma} + o(1)$$

since $f^{(\sigma)}(0) = f(\sigma)$ and $|\dot{f}(0)|$ is finite. As $\sigma \to 0$, this relation becomes

$$\lim_{\sigma \to 0} \frac{\hat{\psi}(f^{(\sigma)}) - \hat{\psi}(f)}{\sigma} = D\psi(f) \cdot \dot{f}(0).$$

The existence of the history gradient and the validity of the chain rule property follow at once.

Lemma 5. Let g be an elastic continuation of f. Then the elastic gradient and, therefore, the history gradient at g exist. Furthermore,

$$D\psi(g) = \nabla\psi(g(0);g)$$

$$\delta\psi(g) = 0.$$

Proof. To show that $D\psi(g)$ exists let $\{g_n\} \subset \mathcal{E}(g)$ with $\|g_n - g\| \to 0$ as $n \to \infty$. By Lemma 3, $\{g_n\} \subset \mathcal{E}(f)$ and, for n sufficiently large,

$$\psi(g_n(0);g_n) = \hat{\psi}(g_n) = \psi(g_n(0);g).$$

Thus, it follows by similar reasoning that

$$\nabla\psi(g_n(0);g_n) = \nabla\psi(g_n(0);g).$$

Consequently, $D\psi(g)$ exists and has the value

$$\lim_{n\to\infty} \nabla\psi(g_n(0);g) = \nabla\psi(g(0);g).$$

In order to show that $\delta\psi(g) = 0$, observe that if $g \in \mathcal{E}(f)$, then the histories $g_{(\sigma)}$ and $g^{(\sigma)}$ (for σ sufficiently small) can be chosen from a set on which $\hat{\psi}$ is constant. (The validity of this observation rests on Condition 2 in the definition of elastic continuations; the proof is not included here since it is elementary but rather detailed.) Thus,

$$\Delta_\sigma\hat{\psi}(g) = \frac{\hat{\psi}(g^{(\sigma)}) - \hat{\psi}(g_{(\sigma)})}{\sigma} = 0$$

for σ sufficiently small. Clearly, this relation implies $\delta\psi(g) = 0$.

The derivative W' in the theory of PIPKIN and RIVLIN [8] provided the motivation for the introduction of the history gradient in the present theory. Although the two types of "derivatives" are conceptually related, W' and $\delta\psi$ have quite different properties. In fact, the precise analogue of the history gradient in [8] is W' evaluated at the present strain. This quantity is identically zero while, as is shown in the next section, $\delta\psi$ is not.

7. Consequences of the Clausius-Duhem Inequality.

If $D\psi(f)$ exists, then the expression for $\dot{\psi} = \dot{\hat{\psi}}(f)$ obtained in Lemma 4 can be substituted into the Clausius-Duhem inequality to give

$$\{\hat{\Sigma}(f) - D\psi(f)\}\cdot\dot{f}(0) + \delta\psi(f) + \hat{\underline{q}}(f;\underline{g})\cdot\underline{g}/\rho\theta \leq 0.$$

Now, suppose g is an elastic continuation of f. Then, from Lemma 5, $D\psi(g)$ exists and equals $\nabla\psi(g(0);g)$; furthermore, $\delta\psi(g) = 0$. In this case, the Clausius-Duhem inequality takes the form

$$\{\hat{\Sigma}(g) - \nabla\psi(g(0);g)\}\cdot\dot{g}(0) + \hat{\underline{q}}(g;\underline{g})\cdot\underline{g}/\rho\theta \leq 0.$$

The proof of Lemma 3 shows that $\mathcal{E}(f)$ is closed under local linear continuations. From a standard argument (c.f. [3],[5],[7], [12]), one obtains

Lemma 6. Let g be an elastic continuation of f. Then the generalized stress relation and the heat conduction inequality hold, i.e.

$$\hat{\Sigma}(g) = D\psi(g) \qquad (= \nabla\psi(g(0);g))$$

$$\hat{\underline{q}}(\underline{g};g)\cdot\underline{g} \leq 0$$

for every $\underline{g}\in R^3$.

These results correspond to the classical thermodynamic results for hyperelastic materials.

Stronger results can be obtained using the continuity of $\hat{\Sigma}$ and $\hat{\underline{q}}$ (S1, Section 4). These results are included in

Theorem 1. Let S1, S2, S3 and A1 hold. Then necessary and sufficient conditions for the Clausius-Duhem inequality to hold for every admissible thermodynamic process are the existence of the elastic gradient at each history and the relations

$$\hat{\Sigma}(f) = D\psi(f)$$

$$\hat{\underline{q}}(f;\underline{g})\cdot\underline{g} \leq 0$$

$$\delta\psi(f) \leq 0$$

which hold for every history f and for every $\underline{g}\in R^3$.

Proof. Let $f\in\mathcal{P}\mathcal{S}^+$. From A1, there exists $\{g_n\} \subset \mathcal{E}(f)$ such that $\|g_n - f\| \to 0$ as $n \to \infty$. According to Lemma 6,

$$\hat{\Sigma}(g_n) = \nabla\psi(g_n(0);g_n)$$

$$\hat{\underline{q}}(g_n;\underline{g})\cdot\underline{g} \leq 0$$

for $n = 1,2,\ldots$ and for every $\underline{g}\in R^3$. The assumption S1 then implies that $D\psi(f)$ exists, that

$$\hat{\Sigma}(f) = D\psi(f)$$

and that

$$\hat{\underline{q}}(f;\underline{g})\cdot\underline{g} \leq 0$$

for every $\underline{g} \in R^3$. The first relation can be substituted into the Clausius-Duhem inequality to yield

$$\delta\psi(f) + \underline{q}(f;\underline{g})\cdot\underline{g}/\rho\theta \leq 0 .$$

This inequality must hold for every $\underline{g} \in R^3$. Setting $\underline{q} = \underline{0}$, $\delta\psi(f) \leq 0$ is immediate. Clearly, the existence of $D\psi(f)$, the generalized stress relation, and the two inequalities in the statement of Theorem 1 imply that the Clausius-Duhem inequality is satisfied.

The following corollaries are immediate consequences of Theorem 1.

Corollary 1. If g is an elastic continuation of f and g^* is a continuation of g, with g^* sufficiently close to g, then

$$\hat{\Sigma}(g^*) = \nabla\psi(g^*(0);g) .$$

This result shows that the stress and entropy are locally path-independent within the elastic set.

Corollary 2. Let $f \in \wp S^+$. Then

$$\rho\delta\psi(f) = -[\rho\theta\dot{\eta} - (\rho r - \operatorname{div} \underline{q})]$$

where $\theta, \dot{\eta}$ and $\underline{q}$ are computed from the history f.

Corollary 2 follows from Theorem 1 and the energy balance equation

$$\dot{\psi} = -\Sigma\cdot\dot{f}(0) - [\theta\dot{\eta} - (r - \operatorname{div} \underline{q}/\rho)] .$$

The expression for $\delta\psi$ given in this corollary shows that $\delta\psi$ is analogous to the internal dissipation defined by COLEMAN [1]. (It should be noted that the internal dissipation in Coleman's theory is related to a Fréchet derivative whereas $\delta\psi$ is not.) In Section 8 a special theory is given in which $-\rho\delta\psi$ reduces to the rate of performing "plastic work". This special theory along with Corollary 2 indicate that the quantity $\delta\psi$ is a reasonable generalization of the notion of dissipation in elastic-plastic materials.

The fact that $\delta\psi(f) \leq 0$ implies

Corollary 3. Let $f \in \mathcal{P}\mathcal{S}^+$. Then

$$\rho\delta\psi(f) \leq -\hat{\Sigma}(f)\cdot\dot{f}(0).$$

COLEMAN [1] proved a corresponding result for materials with fading memory. It is interesting to note that equality will hold in Corollary 3 if f is an elastic continuation of some history f^*. Hence, roughly speaking, the rate of performing work is accounted for completely by changes in free energy whenever the current history lies within an elastic set. From these thermodynamic considerations, the term "elastic set" is justified in the present context.

8. Infinitesimal Deformations of Elastic-Plastic Materials.

In order to illustrate the results in Theorem 1, a special theory is discussed. This theory uses the definitions of elastic and plastic strain-rate given by HILL [13]; in addition the deformations are assumed to be infinitesimal, i.e. the components of $F-I$ are small in comparision to unity. The assumption of small strains is not essential and is made only for convenience.

We restrict our attention to the class of histories

$$\mathcal{P}\mathcal{S}^+_o = \{\ f\in\mathcal{P}\mathcal{S}^+ \,|\, f(\infty) = (I,\theta),\quad \theta > 0\}.$$

Here, I denotes the unit tensor.

(This restriction also is made for convenience and is not essential to the theory discussed here.) For each $f\in\mathcal{P}\mathcal{S}^+_o$ the Green-St. Venant strain tensor is given by

$$E(f) = \tfrac{1}{2}(F^TF - I)$$

where $f(0) = (F,\theta)$. Of course, $E(f)$ depends upon f only through the pair $f(0)$. The strain-rate $\dot{E}(f)$ is defined by the relation

$$\dot{E}(f) = \lim_{\sigma\downarrow 0} \frac{E(f) - E(f_{(\sigma)})}{\sigma}\ .$$

Clearly, $\dot{E}(f)$ exists for each $f\in\mathcal{P}\mathcal{S}^+_o$. The symmetric part L_s of the velocity gradient tensor satisfies the relation

$$L_s(f) = (F^T)^{-1}\dot{E}(f)F^{-1}$$

for each $f \in \mathcal{P}\mathcal{S}_o^+$. For infinitesimal deformations the last relation takes the form

$$L_s(f) = \dot{E}(f),$$

and it is clear that the relation $\hat{\Sigma} = (\hat{T}(F^{-1})^T/\rho, -\hat{\eta})$ becomes

$$\hat{\Sigma} = (\hat{T}/\rho_o, \quad -\hat{\eta})$$

where ρ_o is the density when $s = +\infty$ and $\hat{T}$ and $\hat{\eta}$ are the stress and entropy response functionals.

The elastic strain-rate for f, $L^e(f)$, is defined through the relation

$$L^e(f) = \mathcal{L}^e(\dot{\hat{T}}(f), \dot{\theta}(f))$$

where $\dot{\theta}(f)$ is the second component of the pair $\dot{f}(0)$ and where $\mathcal{L}^e$ is a bilinear, symmetric tensor-value function. It is assumed that the stress-rate at f,

$$\dot{\hat{T}}(f) \overset{\text{def}}{=} \lim_{\sigma \downarrow 0} \frac{\hat{T}(f) - \hat{T}(f_{(\sigma)})}{\sigma}$$

exists at each $f \in \mathcal{P}\mathcal{S}_o^+$. The elastic strain for f, $E^e(f)$, is defined by

$$E^e(f) = -\int_0^\infty L^e(f_{(\sigma)})\,d\sigma .$$

The *plastic strain-rate for* f, $L^p(f)$, and the *plastic strain for* f, $E^p(f)$, are defined by the relations

$$L^p(f) = L_s(f) - L^e(f)$$

and

$$E^p(f) = -\int_0^{\infty} L^p(f_{(\sigma)})\,d\sigma ,$$

respectively. The definitions of $L^e(f)$ and $L^p(f)$ follow those given by HILL [13]; the definitions of E^e and E^p are natural ones to make for the infinitesimal theory. The relation

$$L_s(f) = \dot{E}(f)$$

and the fact that $f(\infty)$ has the form (I,θ) implies that

$$E(f) = E^e(f) + E^p(f)$$

for each $f \in \mathcal{P}\mathcal{S}_0^+$. It is important to note that $E^e(f)$ and $E^p(f)$ may depend non-trivially upon the entire history f whereas $E(f)$ only depends upon $f(0)$.

Next, two constitutive assumptions are made which embody the basic features of elastic-plastic materials.

C1. <u>Let</u> $A_s = \{(E,\theta) \in A \mid E$ <u>is symmetric and</u> $\theta > 0\}$. <u>Then there exists a</u> C^1 <u>function</u> ψ^* <u>on</u> A_s <u>such that, for every</u> $f \in \mathcal{P}\mathcal{S}_0^+$,

$$\hat{\psi}(f) = \psi^*(E^e(f), \theta).$$

C2. The mapping $f \to E^p(f)$ is continuous on $\wp\mathcal{S}_o^+$ and

$$\dot{E}^p(f) \overset{\text{def}}{=} \lim_{\sigma \downarrow 0} \frac{E^p(f) - E^p(f_{(\sigma)})}{\sigma}$$

exists for each $f \in \wp\mathcal{S}_o^+$. Furthermore, for each $f \in \wp\mathcal{S}_o^+$, there exists a non-empty connected open set $E^*(f) \subset A^+$ such that

1. If $g_{(\sigma)} = f$ and $g(\sigma') \in E^*(f)$ for $o \leq \sigma' < \sigma$, then $E^p(g_{(\sigma)}) = E^p(f)$.
2. There exists $\sigma_o(f) \in (0,\infty)$ such that $f(\sigma') \in E^*(f)$ for $o < \sigma' < \sigma_o$.

The gradient of ψ^* is denoted by $\partial\psi^*(E,\Theta)$ and the first component of the gradient by $\nabla_E \psi^*(E,\Theta)$.

The next lemma shows that the existence of $E^*(f)$ implies that $\mathcal{E}(f)$ is non-empty.

Lemma 7. If S1, C1 and C2 are satisfied, then

1. S2 holds, i.e., $\dot{\hat{\psi}}(f)$ exists at each $f \in \wp\mathcal{S}_o^+$.
2. $\mathcal{E}(f)$ is non-empty and $E^*(f) \subset E(f)$.
3. A1 is satisfied.

Proof. Since ψ^* is smooth and $\dot{E}^p(f)$ exists, $\dot{\hat{\psi}}(f)$ exists for each $f \in \wp\mathcal{S}_o^+$. To show that $\mathcal{E}(f)$ is non-empty, note that any history g satisfying the hypothesis of condition 1 in C2 is an elastic continuation of f. In fact, the special form for $\hat{\psi}$ given in C1 and condition 1 of C2 imply that $\hat{\psi}$ has the appropriate property of path-independence for short extensions of g. The connectedness of $E^*(f)$ and condition 2 of C2 imply that $E^*(f)$ is a subset of $E(f)$. Condition 2 of C2 implies that A1 is satisfied.

Thus, Lemma 7 shows that S1, C1 and C2 define a class of materials to which the theory of Sections 1 through 5 applies.

Denote by $\mathcal{E}^*(f)$ the set of all histories $g \in \mathcal{P}\mathcal{S}_o^+$ such that $g_{(\sigma)} = f$ and $g(\sigma') \in E^*(f)$ for $0 \leq \sigma' < \sigma$; clearly, from Lemma 7, $\mathcal{E}^*(f) \subset \mathcal{E}(f)$. The next lemma shows that the restriction of $\hat{\psi}$ to $\mathcal{E}^*(f)$ determines the quantities $D\psi(f)$ and $\delta\psi(f)$.

Lemma 8. Let $f \in \mathcal{P}\mathcal{S}_o^+$. *Then* $D\psi(f)$ *and* $\delta\psi(f)$ *exist and are given by*

$$D\psi(f) = \nabla\psi^*(E^e(f), \Theta)$$

and

$$\delta\psi(f) = -\operatorname{tr}[\nabla_E\psi^*(E^e(f), \Theta)\dot{E}^p(f)],$$

where the first relation is valid to lowest order in the quantity F-I *and the second relation is exact.*

Proof. Let $f \in \mathcal{P}\mathcal{S}_o^+$ with $f(0) = (F, \Theta)$, let $a' = (F', \Theta') \in E^*(f)$, and let $g, g' \in \mathcal{E}^*(f)$ with $g' \in \mathcal{C}(g)$ and $g'(0) = a'$. Recall that $\psi(a';g)$ is defined by the relation

$$\psi(a';g) = \hat{\psi}(g')$$

whenever g' is sufficiently close to g. It follows that

$$\begin{aligned} \psi(a';g) &= \hat{\psi}(g') \\ &= \psi^*(E^e(g'),\Theta') \\ &= \psi^*(E(g') - E^p(g'),\Theta') \\ &= \psi^*(E' - E^p(f),\Theta') \end{aligned}$$

where $E' = \frac{1}{2}(F'^TF' - I)$. Thus, $\psi(a';g)$ only depends upon f and a'. Consequently, $\nabla\psi(a';g)$ is given by

$$\nabla\psi(a';g) = \nabla\psi^*(E' - E^p(f),\Theta'),$$

to lowest order in $F' - I$. Since ψ^* is C^1 it follows that $D\psi(f)$ exists and is given by the relation

$$D\psi(f) = \nabla\psi^*(E^e(f),\Theta).$$

The proof that $\delta\psi(f)$ exists is accomplished by means of a direct calculation. In fact, if $\tilde{f}^\sigma$ is any element of $\mathcal{E}^*(f)$ such that $\tilde{f}^\sigma(0) = f(\sigma)$,

$$\begin{aligned} \delta\psi(f) &= \lim_{\sigma\downarrow 0} \frac{\hat{\psi}(\tilde{f}^\sigma) - \hat{\psi}(f_{(\sigma)})}{\sigma} \\ &= \lim_{\sigma\downarrow 0} \frac{\psi^*(E(\tilde{f}^\sigma) - E^p(\tilde{f}),\Theta(\sigma)) - \psi^*(E(f_{(\sigma)}) - E^p(f_{(\sigma)}),\Theta(\sigma))}{\sigma} \\ &= \lim_{\sigma\downarrow 0} \frac{\psi^*(E(f_{(\sigma)}) - E^p(f),\Theta(\sigma)) - \psi^*(E(f_{(\sigma)}) - E^p(f_{(\sigma)}),\Theta(\sigma))}{\sigma} \\ &= -\operatorname{tr}[\nabla_E\psi^*(E^e(f),\Theta)\dot{E}^p(f)]. \end{aligned}$$

Now, define a function W^p by the relation

$$W^p(f) = \operatorname{tr}\{\nabla_E \psi^*(E^e(f),\Theta)\dot{E}^p(f)\}.$$

Let us apply Theorem 1 in the special context outlined above. The most interesting results obtained are

$$\hat{T}(f) = \rho_o \nabla_E \psi^*(E^e(f),\Theta)$$

and

$$-\delta\psi(f) = W^p(f) \geq 0.$$

Using the first result in the definition of $W^p(f)$, it follows that

$$\rho_o W^p(f) = \operatorname{tr}\{\hat{T}(f)\dot{E}^p(f)\};$$

hence we can refer to W^p as the "*plastic power production*."

Theorem 2. In the infinitesimal theory $\delta\psi(f)$ equals $-W^p(f)$, the negative of the plastic power production; hence $W^p(f) \geq 0$.

The fact that the result $W^p(f) \geq 0$ follows as a special case of Theorem 1 indicates that the theory of Sections 2-7 is a proper generalization of the classical theories. It should be noted that the condition $W^p(f) < 0$ does not violate any thermodynamic restrictions if one takes, for example, a more general constitutive relation of the form

$$\hat{\psi}(f) = \tilde{\psi}(E(f), E^p(f), \theta)$$

for the free energy. Hence, if in a particular deformation the condition $W^p(f) \geq 0$ is violated, a form for $\hat{\psi}$ more complicated than the expression in terms of ψ^* must be chosen. Thus, the Clausius-Duhem inequality can be used to indicate when constitutive relations are too restrictive in form.

Part II: Rate-Independent Materials

1. Introduction.

The important role that rate-independent materials play in theories of materials with elastic range is seen in the theories of plasticity. Having established in Part I thermodynamic results for materials with elastic range, I devote Part II of this paper to a systematic investigation of the consequences of the assumption that $\hat{\psi}$ is a rate-independent functional.

The definition of rate-independence used here is based on that given by TRUESDELL and NOLL[14]; the specific form used here was introduced by OWEN and WILLIAMS [15]. This definition requires that $\hat{\psi}$ be invariant under time rescalings of each fixed history in its domain and has an advantage over the equivalent [15] definition of PIPKIN and RIVLIN [8]. The advantage lies in the fact that the former definition does not distinguish between different rescalings of a given history, while the latter definition singles out a particular rescaling function, the arc-length rescaling function, and replaces each history by its arc-length representation. The use of the arc-length representation complicates the analysis because there appears to be no simple way of comparing different histories in terms of their arc-length representations.

The rate-independence of $\hat{\psi}$ is the central hypothesis in each of the results in Sections 3 and 4. In particular, I show in

Section 3 that this hypothesis implies that the elastic range is invariant under rescalings of a given history; furthermore, the elastic set changes only in a trivial way under time rescalings. In Section 4, similar results specify the restrictions which the rate-independence of $\hat{\psi}$ imposes on the stress and on the history gradient of the free energy. Specifically, the stress functional is a rate-independent functional and the history gradient depends upon the history of the magnitude of strain and temperature rates only through the present value. An important corollary of the last result is the fact that there can be no internal dissipation during static continuations for a rate-independent material. Thus, certain internal processes must cease when a material point is subject to conditions of constant strain and temperature. In the final section, a result is presented which shows that the rate-independence of $\hat{\Sigma}$ and the special property of $\delta\psi$ deduced in Section 4 suffice to establish the rate-independence of a functional closely related to $\hat{\psi}$. This final result, when considered in the context of the theory presented in Section 8 of Part I, gives testable sufficient conditions for the rate-independence of $\hat{\psi}$. In fact, if during isothermal processes the stress is a rate-independent functional of strain-history and if the rate of plastic work is linear in the magnitude of the current strain rate, then $\hat{\psi}$ is a rate-independent functional of the strain-history.

The method of proof for the results in Part II relies on many concepts and techniques used in Part I. The main new concept introduced in Part II and used throughout is that of *a σ-rescaling function*. This function, denoted by ϕ^{σ}, rescales only the segment of a history f which was traversed in the time interval $(-\infty, -\sigma]$. The importance of such rescalings is demonstrated in Lemma 2 where I show that every elastic continuation of a rescaling of a history f is characterized by the property of being a σ-rescaling of an elastic continuation of f.

2. Rate-Independence and Related Concepts.

A functional $\gamma: \mathcal{PS}^+ \to R^n$ is said to be rate-independent if, for each history f,

$$\gamma(f \circ \phi) = \gamma(f)$$

for every monotone non-decreasing continuous function ϕ mapping $[0,\infty)$ onto a set containing the support of $\dot{f}$. To insure that the composition $f \circ \phi$ is an element of $\mathcal{PS}^+$, it suffices to assume that ϕ also is piecewise smooth and has a bounded derivative. The symbol ϕ is used exclusively to denote a function with these smoothness properties. If the range of ϕ contains the support of $\dot{f}$, then ϕ is called a *rescaling function for* f, and the set of all rescaling functions for f is denoted by Φ_f. The condition on the range of ϕ guarantees that $f \circ \phi$ attains the same values as does f. Further technical points related to the definition of rate-independence are discussed in detail by OWEN and WILLIAMS [15]. In particular, these authors show that the definition of rate-independence given here, which is a refinement of that given by TRUESDELL and NOLL [14], is equivalent to the definition given by PIPKIN and RIVLIN [8].

Henceforth, the free energy functional $\hat{\psi}$, introduced in Part I, is assumed to be a rate-independent functional. The remainder of Part II is devoted to obtaining the consequences of this assumption.

Before proceeding, it is convenient to introduce the concept of a σ-rescaling function. Let $\phi \in \Phi_f$ and $\sigma \in [0,\infty)$. Then the *σ-rescaling function* ϕ^σ is defined by the relation

$$\phi^\sigma(s) = \begin{cases} s & 0 \leq s < \sigma \\ \sigma + \phi(s-\sigma), & \sigma \leq s. \end{cases}$$

Note that the composition $f \circ \phi^\sigma$ agrees with f on the interval $[0,\infty)$, while on the semi-infinite interval $[\sigma,\infty)$,

$$\begin{aligned} (f \circ \phi^\sigma)(s) &= f(\sigma + \phi(s-\sigma)) \\ &= (f_{(\sigma)} \circ \phi)(s-\sigma). \end{aligned}$$

It follows that

$$(f \circ \phi^\sigma)_{(\sigma)} = f_{(\sigma)} \circ \phi.$$

Clearly, if $\phi \in \Phi_f$, then $\phi^\sigma \in \Phi_f$ and ϕ is a rescaling function for $f_{(\sigma)}$.

The functions ϕ^σ are important in determining the restrictions imposed on the elastic range by the rate-independence of $\hat{\psi}$. The relevant properties of these functions are given in

Lemma 1. Let g be a history, σ^*, σ' and δ non-negative numbers with $\sigma^* < \sigma'$, and ϕ a rescaling function for $g_{(\sigma')}$. The following conditions hold:

1. If h is a δ-continuation of g with $h_{(\sigma^*)} = g$, then $h \circ \phi^{\sigma^* + \sigma'}$ is a δ^*-continuation of $g \circ \phi^{\sigma'}$, where

$$\delta^* = \{3 + M_\phi\}\delta$$

and $M_\phi = \sup_{s \in [0,\infty)} \dot{\phi}(s)$.

2. If $h \circ \phi^{\sigma^* + \sigma'}$ is a δ-continuation of $g \circ \phi^{\sigma'}$, then h is a δ^*-continuation of g.

Proof. The proof of 1 is given here; the proof of 2 is similar. Thus, let $g, h, \phi, \delta, \sigma^*$, and σ' be given as in 1. The proof rests on the following observation. Let g^* be any history and $\sigma^* \in [0,\infty)$. Then for each $\phi_o \in \Phi_{g^*}$

$$\begin{aligned} \sup_{s \geq \sigma^*} |g^*(s)| &= \sup_{s \geq \sigma^*} |g^*(\phi_o(s-\sigma^*) + \sigma^*)| \\ &= \sup_{s \geq \sigma^*} |(g^* \circ \phi_o^{\sigma^*})(s)|, \end{aligned}$$

Thus, if $\|g^*\|_{[\sigma^*,\infty)} \overset{\text{def}}{=} \sup_{\sigma^* \leq s < \infty} |g^*(s)|$, the last relation becomes

$$\|g^*\|_{[\sigma^*,\infty)} = \|g^* \circ \phi_o^{\sigma^*}\|_{[\sigma^*,\infty)}$$

Consider now the quantity

$$\begin{aligned} &\|h \circ \phi^{\sigma'+\sigma^*} - g \circ \phi^{\sigma'}\|_{[\sigma'+\sigma^*,\infty)} \leq \\ &\quad \|h \circ \phi^{\sigma'+\sigma^*} - g \circ \phi^{\sigma'+\sigma^*}\|_{[\sigma'+\sigma^*,\infty)} + \\ &\quad \|g \circ \phi^{\sigma'+\sigma^*} - g \circ \phi^{\sigma'}\|_{[\sigma'+\sigma^*,\infty)}. \end{aligned}$$

On the right hand side of this inequality, the first term equals $\|h - g\|_{[\sigma'+\sigma^*,\infty)}$ which, by hypothesis, is less than δ. The second term can be estimated as follows:

$$\|g\circ\phi^{\sigma'+\sigma^*} - g\circ\phi^{\sigma'}\|_{[\sigma'+\sigma^*,\infty)} =$$

$$= \sup_{s\geq\sigma'+\sigma^*} |g_{(\sigma'+\sigma^*)}(\phi(s-\sigma'-\sigma^*)) - g_{(\sigma')}(\phi(s-\sigma'))|$$

$$= \sup_{s\geq\sigma'+\sigma^*} |g_{(\sigma')}(\phi(s-\sigma'-\sigma^*)+\sigma^*) - g_{(\sigma')}(\phi(s-\sigma'))|$$

$$\leq M_g \sup_{s\geq\sigma'+\sigma^*} \{|\phi(s - \sigma' - \sigma^*) - \phi(s - \sigma') + \sigma^*|\}$$

$$\leq M_g (1 + M_\phi)\sigma^*,$$

where $M_g \overset{\text{def}}{=} \sqrt{10} \sup_{s\geq 0} |\dot{g}(s)|$. In this estimate, the mean-value theorem has been applied to ϕ and to each component function of g. (Note that the mean-value theorem applies here since both $\dot{\phi}$ and $\dot{g}$ have only isolated jumps, c.f. Lemma 1, Part I.) Furthermore, the inequality

$$\|h - g\| < \delta$$

and the mean-value theorem imply that

$$M_g\sigma^* \leq \delta .$$

It follows that $\|g\circ\phi^{\sigma'+\sigma^*} - g\circ\phi^{\sigma'}\|_{[\sigma'+\sigma^*,\infty)} < (3 + M_\phi)\delta$.

Consider next the estimate

$$\|h\circ\phi^{\sigma'+\sigma^*} - g\circ\phi^{\sigma'}\|_{[\sigma',\sigma'+\sigma^*]} \leq$$

$$\|h\circ\phi^{\sigma'+\sigma^*} - h\circ\phi^{\sigma'}\|_{[\sigma',\sigma'+\sigma^*]} +$$

$$\|h\circ\phi^{\sigma'} - g\circ\phi^{\sigma'}\|_{[\sigma',\sigma'+\sigma^*]}.$$

Using the method given above, one can show that the first term is dominated by $\delta\{1 + M_\phi\}$; the second term is less than δ, since

$$\|h\circ\phi^{\sigma'} - g\circ\phi^{\sigma'}\|_{[\sigma',\infty)} = \|h - g\|_{[\sigma',\infty)}.$$

Thus,

$$\|h\circ\phi^{\sigma'+\sigma^*} - g\circ\phi^{\sigma'}\|_{[\sigma',\sigma'+\sigma^*]} < (3 + M_\phi)\delta.$$

Finally, it is clear that

$$\|h\circ\phi^{\sigma'+\sigma^*} - g\circ\phi^{\sigma'}\|_{[0,\sigma']} = \|h - g\|_{[0,\sigma']}$$

$$< \delta < (3 + M_\phi)\delta$$

since $h\circ\phi^{\sigma'+\sigma^*}$ and $g\circ\phi^{\sigma'}$ equal h and g, respectively, on $[0,\sigma']$. Combining the estimates obtained separately for the intervals $[0,\sigma'], [\sigma',\sigma'+\sigma^*]$ and $[\sigma'+\sigma^*,\infty)$, it follows that

$$\|h\circ\phi^{\sigma'+\sigma^*} - g\circ\phi^{\sigma'}\| < \delta^* = (3 + M_\phi)\delta.$$

3. Restrictions Imposed on $\mathcal{E}(f)$ and $\mathbb{E}(f)$.

The results of this section will show that the rate-independence of $\hat{\psi}$ implies that both the elastic set and the elastic range are rate-independent, each in a sense to be specified below.

The substance of these results is contained in

Lemma 2. *Let f be any history and ϕ a rescaling function for f. Then the following conditions hold:*

1. *If g is an elastic continuation of f with $g_{(\sigma)} = f$, then $g \circ \phi^{\sigma}$ is an elastic continuation of $f \circ \phi$.*
2. *If $\tilde{g}$ is an elastic continuation of $f \circ \phi$ and $g_{(\tilde{\sigma})} = f \circ \phi$, then the history $\bar{g}$ given by*

$$\bar{g}(s) = \begin{cases} \tilde{g}(s) & 0 \leq s < \tilde{\sigma} \\ f(s-\tilde{\sigma}) & \tilde{\sigma} \leq s \end{cases}$$

is an elastic continuation of f.

This lemma makes precise the sense in which the elastic set is rate-independent. In fact, the results of the lemma are equivalent to the assertion that there is a one-to-one mapping of the set of elastic continuations of f onto the set of elastic continuations of $f \circ \phi$.

Proof. The proof of 2 is given here; the proof of 1 is similar. Let $\tilde{g} \in \mathcal{E}(f \circ \phi)$ with $\tilde{g}_{(\tilde{\sigma})} = f \circ \phi$. Let $0 \leq \sigma' \leq \tilde{\sigma}$ and consider the history $\bar{g}_{(\sigma')}$, where $\bar{g}$ is defined in the statement of part 2 of this lemma. Note that $\tilde{g}_{(\sigma')} = \bar{g}_{(\sigma')} \circ \phi^{\tilde{\sigma}-\sigma'}$. Now, take $\delta \epsilon\ (0,\infty)$ such that

$$\delta^* = \delta(3 + M_{\phi}) < \delta(\sigma',\tilde{g}),$$

where $\delta(\sigma',\tilde{g})$ is the positive number which determines which continuations of $\tilde{g}_{(\sigma')}$ give a path-independent response. With δ so chosen, let $\bar{h} \epsilon C_{\delta}(\bar{g}_{(\sigma')})$ with $\bar{h}_{(\sigma^*)} = \bar{g}_{(\sigma')}$. Then, by Lemma 1, $\bar{h} \circ \phi^{\tilde{\sigma}-\sigma'+\sigma^*}$ is a δ^*, and hence, a $\delta(\sigma',\tilde{g})$ continuation of $\bar{g}_{(\sigma')} \circ \phi^{\tilde{\sigma}-\sigma'} = \tilde{g}_{(\sigma')}$. Consequently, if $\bar{h}_1, \bar{h}_2 \epsilon C_{\delta}(\bar{g}_{(\sigma')})$ with $\bar{h}_1(0) = \bar{h}_2(0)$ and $(\bar{h}_1)_{(\sigma_1)} = (\bar{h}_2)_{(\sigma_2)} = \bar{g}_{(\sigma')}$, then it follows that $\tilde{h}_1 \stackrel{def}{=} h_1 \circ \phi^{\tilde{\sigma}-\sigma'+\sigma_1}$ and $\tilde{h}_2 \stackrel{def}{=} h_2 \circ \phi^{\tilde{\sigma}-\sigma'+\sigma_2}$ are $\delta(\sigma',\tilde{g})$-continuous of $\tilde{g}_{(\sigma')}$. Furthermore, $\tilde{h}_1(0) = \tilde{h}_2(0)$. Therefore, since $\hat{\mathfrak{P}}$ is path-independent on the $\delta(\sigma',\tilde{g})$ continuations of $\tilde{g}_{(\sigma')}$,

$$\begin{aligned} \hat{\mathfrak{P}}(\bar{h}_1) &= \hat{\mathfrak{P}}(\bar{h}_1 \circ \phi^{\tilde{\sigma}-\sigma'+\sigma_1}) \\ &= \hat{\mathfrak{P}}(\bar{h}_2 \circ \phi^{\tilde{\sigma}-\sigma'+\sigma_2}) \\ &= \hat{\mathfrak{P}}(\bar{h}_2). \end{aligned}$$

Thus, $\hat{\mathfrak{P}}$ is path-independent on $C_{\delta}(\bar{g}_{(\sigma')})$. This argument shows that part 1 of the definition of elastic continuation is satisfied by g(c.f. Part I, Section 5). The results in Lemma 1 readily show that part 2 of the definition also is satisfied.

An immediate consequence of Lemma 2 is the main result of this section:

Theorem 1. *Let* f *be any history and let* ϕ *be a rescaling function for* f. *Then the elastic ranges for* f *and for* $f \circ \phi$ *are identical*.

Thus, Lemma 2 and Theorem 1 show that the rate-independence of $\hat{\psi}$ places strong restrictions on the dependence of the elastic set and elastic range upon rescalings of a given history. In the next section, the thermodynamic results obtained in Part I will be reexamined in light of the rate-independence of $\hat{\psi}$.

4. Restrictions Imposed on $\hat{\Sigma}$ and $\delta\psi$.

In this section, the assumptions S1-S3 and A1 (Part I) and the assumption that $\hat{\psi}$ is rate-independent are taken as a starting point. (The assumptions from Part I were those used in establishing Theorem 1, Section 7.) With these assumptions taken as hypotheses, the main results of this section are given in

Theorem 2. *Let f be any history. Then*

1. $\hat{\Sigma}$ *is rate-independent.*
2. $\delta\psi(f\circ\phi) = \delta\psi(f)\dot{\phi}(0)$ *for each rescaling function* ϕ.

The proof of Theorem 2 requires two lemmas.

Lemma 3. *Under the hypotheses of Theorem 2,* $D\psi$ *is a rate-independent functional.*

Proof. Let $g\in\mathcal{E}(f)$ with $g_{(\sigma)} = f$. Let $\{\sigma_n\}$ be a monotone increasing sequence of positive numbers with limit σ. For each $\phi\in\Phi_f$ and for $n = 1,2,\ldots,$ $g_{(\sigma_n)}\circ\phi^{\sigma-\sigma_n}$ is an elastic continuation of $f\circ\phi$ (Lemma 2, Part II). An argument similar to the proof of Lemma 1, Part I implies that $\|g_{(\sigma_n)} - f\| \to 0$ as $n \to \infty$ and that $\|g_{(\sigma_n)}\circ\phi^{\sigma-\sigma_n} - f\circ\phi\| \to 0$ as $n \to \infty$. It follows from the rate-independence of $\hat{\psi}$ that, for each n,

$$\psi(a;g_{(\sigma_n)}\circ\phi^{\sigma-\sigma_n}) = \psi(a;g_{(\sigma_n)})$$

for every $a\in A^+$ sufficiently close to $g(\sigma_n)$. This relation implies that

$$\nabla\psi(g(\sigma_n);\ g_{(\sigma_n)}\circ \emptyset^{\sigma-\sigma_n}) = \nabla\psi(g(\sigma_n);\ g_{(\sigma_n)}).$$

Consequently, one can write

$$D\psi(f\circ\emptyset) = \lim_{n\to\infty} \nabla\psi(g(\sigma_n);\ g_{(\sigma_n)}\circ \emptyset^{\sigma-\sigma_n})$$

$$= \lim_{n\to\infty} \nabla\psi(g(\sigma_n);\ g_{(\sigma_n)}) = D\psi(f),$$

which is the desired result.

Lemma 4. *Let* f *be any history,* $\emptyset$ *a rescaling function for* f, *and* $\sigma\in[0,\infty)$. *Then*

$$(f\circ\emptyset)_{(\sigma)} = f_{(\emptyset(\sigma))}\circ\mu\{\emptyset,\sigma\}$$

where $\mu\{\emptyset,\sigma\}$ *is the rescaling function of* $f_{(\emptyset(\sigma))}$ *given by*

$$\mu\{\emptyset,\sigma\}(s) = \emptyset(s+\sigma) - \emptyset(\sigma),\quad s\in[0,\infty).$$

Proof. For each $s\in[0,\infty)$,

$$\begin{aligned}(f\circ\emptyset)_{(\sigma)}(s) &= (f\circ\emptyset)(s+\sigma)\\ &= f(\emptyset(s+\sigma))\\ &= f(\emptyset(s+\sigma) - \emptyset(\sigma) + \emptyset(\sigma))\\ &= f_{(\emptyset(\sigma))}(\emptyset(s+\sigma) - \emptyset(\sigma))\\ &= [f_{(\emptyset(\sigma))}\circ\mu\{\emptyset,\sigma\}](s).\end{aligned}$$

Proof of Theorem 2. The fact that $\hat{\Sigma}$ is rate-independent is an immediate consequence of Lemma 3 and the generalized stress relation.

In order to prove 2, let $f \in \mathcal{P}\mathcal{S}^+$ and $\phi \in \Phi_f$. Using S3, two approximations for the difference $\hat{\psi}(f) - \hat{\psi}(f_{(\phi(\sigma))})$ will be obtained. First, the fact that $\hat{\psi}$ is rate-independent gives

$$\begin{aligned}
\hat{\psi}(f) - \hat{\psi}(f_{(\phi(\sigma))}) = \\
&= \hat{\psi}(f \circ \phi) - \hat{\psi}(f_{(\phi(\sigma))} \circ \mu\{\phi, \sigma\}) \\
&= \hat{\psi}(f \circ \phi) - \hat{\psi}((f \circ \phi)_{(\sigma)}) \\
&= \hat{\psi}((f \circ \phi)^{\sigma}) - \hat{\psi}((f \circ \phi)_{(\sigma)}) + \hat{\psi}(f \circ \phi) - \hat{\psi}((f \circ \phi)^{\sigma}) \\
&= \hat{\psi}((f \circ \phi)^{\sigma}) - \hat{\psi}((f \circ \phi)_{(\sigma)}) \\
&\quad - D\psi(f \circ \phi) \cdot [(f \circ \phi)(\sigma) - f(0)] \\
&\quad + o(|f(0) - f(\phi(\sigma))|),
\end{aligned}$$

where $(f \circ \phi)^{\sigma}$ is any elastic continuation of $f \circ \phi$ to $f(\phi(\sigma))$. Hence

$$\begin{aligned}
\hat{\psi}(f) - \hat{\psi}(f_{(\phi(\sigma))}) = \sigma\left[\frac{\hat{\psi}((f \circ \phi)^{\sigma}) - \hat{\psi}((f \circ \phi)_{(\sigma)})}{\sigma}\right] \\
- D\psi(f \circ \phi) \cdot [f(\phi(\sigma)) - f(0)] \\
+ o(|f(0) - f(\phi(\sigma)|).
\end{aligned}$$

On the other hand, for σ sufficiently small, let $f^{\phi(\sigma)}$ be an elastic continuation of f to $f(\phi(\sigma))$ and write

$$\hat{\psi}(f) - \hat{\psi}(f_{(\phi(\sigma))}) = \hat{\psi}(f) - \hat{\psi}(f^{\phi(\sigma)})$$

$$+ \hat{\psi}(f^{\phi(\sigma)}) - \hat{\psi}(f_{(\phi(\sigma))})$$

$$= -D\psi(f) \cdot [(f \circ \phi)(\sigma) - f(0)]$$

$$+ o(|f(\phi(\sigma)) - f(0)|)$$

$$+ \phi(\sigma) \left[\frac{\hat{\psi}(f^{\phi(\sigma)}) - \hat{\psi}(f_{(\phi(\sigma))})}{\phi(\sigma)} \right] .$$

(In this argument, it is assumed that $\phi(\sigma') > 0$ for all σ' near 0. The case where $\phi(\sigma') = 0$ for one and hence for all σ' near 0 can be treated easily by looking at the first estimate for $\hat{\psi}(f) - \hat{\psi}(f_{(\phi(\sigma))})$.) Equating the two expressions for $\hat{\psi}(f) - \hat{\psi}(f_{(\phi(\sigma))})$, dividing by σ, letting σ tend to zero, and using Lemma 3, it follows that

$$\delta\psi(f \circ \phi) = \delta\psi(f)\dot{\phi}(0).$$

Part 2 of Theorem 2 can be restated as follows: $\delta\psi(f)$ depends upon $|\dot{f}|$, the history of the magnitude of the strain-temperature rates, only through the present value $|\dot{f}(0)|$; furthermore, the dependence is linear. In particular, this observation implies that <u>the history gradient of $\hat{\psi}$ (for a rate-independent material) vanishes during static continuations</u>. Hence, for rate-independent materials, **static continuations** produce no internal dissipation.

5. A Sufficient Condition for the Rate-Independence of $\hat{\psi}$.

This section contains a theorem which is a partial converse to Theorem 2 of Section 4. This theorem provides sufficient conditions for the rate-independence of $\hat{\psi}$ when $\hat{\psi}$ is restricted to a special class of histories.

Theorem 3. *Suppose that* $D\psi$ *is a rate-independent functional and that*

$$\delta\psi(f\circ\phi) = \delta\psi(f)\dot{\phi}(0)$$

for every history f *and every rescaling function* ϕ *for* f. *Then the relation*

$$\tilde{\psi}(f) = \int_0^\infty \{-\hat{\Sigma}(f_{(\sigma)})\cdot\dot{f}(0) + \delta\psi(f_{(\sigma)})\}\, d\sigma$$

defines a rate-independent functional $\tilde{\psi}$ *whose domain includes those histories for which the integral exists*.

Proof. Assume that $\tilde{\psi}(f)$ exists and write

$$\begin{aligned}
\tilde{\psi}(f) &= \int_0^\infty \{-\hat{\Sigma}(f_{(\sigma)})\cdot\dot{f}(0) + \delta\psi(f_{(\sigma)})\}d\sigma \\
&= \int_0^\infty \{-\hat{\Sigma}(f_{(\phi(\sigma))})\cdot\dot{f}(\phi(\sigma)) + \delta\psi(f_{(\phi(\sigma))})\}\dot{\phi}(\sigma)\,d\sigma \\
&= \int_0^\infty \{-\hat{\Sigma}(f_{(\phi(\sigma))}\circ\mu\{\phi,\sigma\})\cdot\dot{\overline{(f\circ\phi)}}(\sigma) \\
&\qquad + \delta\psi(f_{(\phi(\sigma))}\circ\mu\{\phi,\sigma\})\}d\sigma \\
&= \int_0^\infty \{-\hat{\Sigma}((f\circ\phi)_{(\sigma)})\cdot\dot{\overline{(f\circ\phi)}}(\sigma) + \delta\psi((f\circ\phi)_{(\sigma)})\}d\sigma = \tilde{\psi}(f\circ\phi).
\end{aligned}$$

Thus, $\tilde{\psi}$ is defined on $f \circ \phi$ and has the value $\tilde{\psi}(f)$.

In Section 8, Part I, I showed that, in a special theory, $\delta\psi(f)$ represents negative the rate of performing "plastic work." Thus, for the theory of Section 8, the condition

$$\delta\psi(f \circ \phi) = \delta\psi(f)\dot{\phi}(0)$$

can be verified, in principle. Moreover, for isothernal deformations, $\tilde{\psi}$ is independent of the entropy functional $\hat{\eta}$. Hence, the verification that $\hat{\Sigma}$ is rate-independent reduces, in the isothermal case, to the verification that $\hat{T}$ is rate-independent. Since $\hat{\psi}$ and $-\tilde{\psi}$ agree on a class of histories for which the free energy at $s = +\infty$ is zero, Theorem 3 gives a testable set of sufficient conditions for the rate-independence of $\hat{\psi}$ over a large class of histories.

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Corrections to: THERMODYNAMICS OF MATERIALS WITH ELASTIC RANGE

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1.	Contents	Part II no. 3	$E(f) \to \mathbf{E}(f)$
2.	Page 14	S1	$\hat{q} \to \hat{\underline{g}}$
3.	Page 23	Line 1	verstien → version
4.	Page 28	Line 3	$\underline{q}(\underline{f};\underline{g}) \to \hat{\underline{q}}(\underline{f};\underline{g})$
		Line 4	$q = 0 \to g = 0$
5.	Page 35	Third line from bottom	$\bar{E}^p(\tilde{f}) \to \bar{E}^p(\tilde{f}^\sigma)$
6.	Page 42	Line 9	$[0,\infty)$, while → $[0,\sigma)$, while
7.	Page 46	Lemma 2, No. 2	$g_{(\tilde{\sigma})} \to \tilde{g}_{(\tilde{\sigma})}$
8.	Page 53	Second Equation	$-\hat{\Upsilon}(f_{(\sigma)})\cdot\dot{f}(0) \to -\hat{\Upsilon}(f_{(\sigma)})\circ\mathring{f}(\sigma)$
		Third Equation	$\dot{f}(0) \to \dot{f}(\sigma)$