SERIES AND PARALLEL ADDITION OF MATRICES

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ABSTRACT

Let A and B be Hermitian semi-definite matrices and let A^+ denote the Moore-Penrose generalized inverse. Then we define the parallel sum of A and B by the formula $A(A + B)^+B$ and denote it by $A : B$. If A and B are nonsingular this reduces to $A : B = (A^{-1} + B^{-1})^{-1}$ which is the well known electrical formula for addition of resistors in parallel. Then it is shown that the Hermitian semi-definite matrices form a commutative partially ordered semigroup under the parallel sum operation. Here the ordering $A \geq B$ means $A - B$ is semidefinite and the following inequality holds: $(A + B) : (C + D) \geq A : C + B : D$. If $R(A)$ denotes the range of A then it is found that $R(A : B) = R(A) \cap R(B)$. Moreover if A and B are orthogonal projection operators then $2\, A : B$ is the orthogonal projection on $R(A) \cap R(B)$. The norms are found to satisfy the inequality $\|A:B\| \leq \|A\| : \|B\|$. Generalization to non-Hermitian operators are also developed.

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I. Introduction

In this paper a new operation, called parallel addition, is defined for pairs of linear operators. Parallel addition originally arose in an attempt to generalize a network synthesis procedure of Duffin [3] and has already been studied in the scalar case by Erickson [5].

The connection of resistors in series and parallel is a familiar concept from elementary network theory. If two resistors having resistances A and E are connected in series the joint resistance is S » A + B, and if they are in parallel, the joint resistance is P «. (A'''* •}• f)" « AB/(A + B). These two methods of combining resistance are then called series and parallel addition.

It has been tacitly assumed in the above that A and B are positive numbers, since the formula for P is not necessarily defined otherwise* In the physical context the normal situation is for A and B to be positive, however, the case A and B » 0 - a short circuit - can be handled by lotting P » 0 if A » 0 and B is non-negative. Thus the concept of parallel addition of non-negative scalars is, a well defined mathematical operation; it will be denoted here by A:B.

As is well known, positive semidefinite matrices are a generalization of non-negative scalars. This suggests that the parallel addition of such matrices be defined by A:B « A(A + B)~1 £. If A and B are both nonsingular this is equivalent to A:B » (A~1 + B~1)~1. The latter formula is not defined for singular A and B, however, the former will be defined if A + B has an inverse. Even when A + B is singular, the parallel addition may be defined by replacing $(A + B)"^{1}$ by $(A + B)^{+}$, the Moore-Penrose generalized inverse. The generalized inverse has a particularly simple form when A + B is symmetric semidefinite. In fact, A + B when restricted to its range is one to one, so that a genuine inverse may be defined there.

It proves just as simple to consider Hermitian semidefinite matrices rather than matrices which are necessarily real. However, in this paper, the operators are restricted to a finite dimensional vector space V.

Suppose that A and B are Hermitian semidefinite matrices. If they are both nonsingular it is obvious that parallel addition is commutative and associative. Moreover, the resulting matrix A:B is also Hermitian semidefinite. However, if A and B are singular then these properties are not clear consequences of the definition. Nevertheless indirect arguments show thfct these properties hold in all situations.

Let R(A) denote the range subspace of the matrix A. Then it is well known that R(A + B) = R(A) + R(B) if A and B are Hermitian semidefinite matrices. It is found here that R(A:B) « R(A) D R(B). The above two operations are just the operations used in defining the lattice of subspaces of the vector space V. As a consequence a sequence of series and parallel additions of matrices induces corresponding lattice operations. As is well known the lattice of subspaces is modular. This property is found to give identities relating certain series-parallel connections.

Orthogonal projection operators are, of course, Hermitian semidefinite and the theorem just mentioned has the following interesting application.

If A and B are orthogonal projection operators then 2A:B is the orthogonal projection into the subspace $R(A) \cap R(B)$.

Semidefinite matrices form a partially ordered system if $A \geq B$ is taken to mean A - B is semidefinite. In terms of this system the following general inequality holds

$$(A + B) : (C + D) \geq A:C + B:D \quad .$$

We term this the series-parallel inequality and it is a generalization of a scalar inequality of A. Lehman [11]. In particular if $A \geq B$ then it results that $A:C \geq B:C$. Thus we can say that the Hermitian semidefinite operators on a finite dimensional space form a partially ordered commutative semigroup with parallel addition as the group operation.

It is of interest to note that the boundary inequalities for parallel additions of matrices seem to be best expressed in terms of parallel addition of scalars. For example the following inequalities hold for the norm, trace, and determinant:

$$\|A:B\| \leq \|A\| : \|B\| \quad ,$$

$$tr(A:B) \leq (trA) : (trB) \quad ,$$

$$|A:B| \leq |A| : |B| \quad .$$

These inequalities are best possible.

II. Preliminaries

In this paper we will consider operators on a finite dimensional complex inner product space. The range and null space of an operator A will be denoted by R(A) and N(A) respectively. The orthogonal projection onto R(A) will be denoted by P_A. A Hemitian operator A will be said to be positive semideflnite if $(Ax,x) \geq 0$ for all x; the abbreviation HSD will be used. For an HSD operator A it is easy to prove that $(Ax,x) = 0$ iff $Ax * 0$, and that $R(A + B) = R(A) + R(B)$.

The Moore-Penrose generalized inverse [l], [13] of an operator A will be denoted by A^+. In the cases considered here A will be HSD. Then A when restricted to its range is one to one, and therefore invertible ; A^+ is this inverse. It then follows that $AA^+ « A^+A « P_A$, and that A^{-f} is HSD iff A is HSD.

III. Algebraic Properties

Definition: Let A and B be Hermitian semidefinite operators on the finite dimensional complex vector space V. The se£ie£ sum of A and B is defined to be the ordinary sum A + B. The parallel sum of A and B is defined by

(l) $$A:B « A(A + B)^{+}B.$$

Since R (B) C R(A + B)$_f$ for any x we have Bx e H(A + B) . Then, since A + B is invertible on its range, the complete definition of the generalized inverse is never used. In the later development of-the theory it v/ill turn out that reasonable results are obtained only when we can guarantee that R (B) C^R (^{A} +^{B})t as is the case here.

For scalars a,b we define $a:b « a(a+b)^{+}b$, where $0^{+} = 0$. This is the case studied by Erickson [5]•

Lemma 1; If A and B are Hermitian semidefinite, then A:B = B:A.

Proof: $A:B - A(A + B)^{+}B$

$$- (A + B - B)(A + B)^{+}(A + B - A)$$

$$- (A + B)(A + B)^{+}(A + B) - B(A + B)^{+}(A + B) - (A + B)(A + B)^{+}A + B:A$$

$$" - A + B - B P_{M)} > - P_{A+B} A + B:A$$

$$- B:A$$

QEO)

If the definition of A:B is extended to general operators A and B, this proof would work if A + B is nonsingular; if not, the lemma need not hold.

Lemma 2: Lf A and B e.re Hermitian semidefinite, then A:B is Hermitian.

Proof: $(A:B)^* - (A(A + B)^+B)^*$

$- B^* (A + B)^{*+}A^*$ «$B(A + B)^*A$

« B:A

« A:B by lemma $1_\#$ QED

Lemma }: If A and B are Hermitlan semidefinite, then R(A:B) « R(A)HR(B).

Proof: Consider x e R(A)AR(B). Then

$A:B(A^+$ ÷ $B^+)x$ « $A(A + B)^+BB^+x + B(A + B)^+AA^+x$

$\sim A(A+B)^+x+B(A+B)^+x$

» P_{A+B} • x « x

Therefore R(A:B) 3R(A)0R(B). Since R(A(A + B)$^+$B)C R(A)$_f$ the lemma then follows from leraiaa *1.* QJID

Lgmm ***k;*** **If A** and B are Herxnitian senidefinite, then A:B jLs semidefinite.

Proof: For any K, let A:Bz=x. Then, as in lcninia]Af A:B(A $^+$ -I- B^{-1})x=x and

(A:Bz$_f$z) » (x,z) - (A:B(A$^+$ 4- B$^+$)x,z)

« ((A$^+$ + B$^+$)x, A:Bz) by lemma 2

= ((A$^+$ + B$^+$)x,x)

o (A$^+$x,x) + (B$^+$x,x) $\geq$ 0

QED

Although we cannot in general write A:B « (A~ ^{1}+ B~"$^{?}$)~^{1}, and by letting

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} f\,0 & 0 \\ « & 1 \end{bmatrix}$$

we see that the "obvious" extension A:B * (A* + B*)*^{1}" will not work either, we do have an alternative definition of this type.

Theorem 5; If A and B are Hermitian semide.finite, then A:B = (P(A^+ + B^+)P)$^+$, where P jis the projection onto R(A)AR(B).

Proof; Consider any x e V. Since R(A:B) » R(A)f|R(B) and A:B is Hermitian, (A:B)$^+$x » (A:B)$^+$Px. But, as in lemma 3, A:B(A^+ + B^+)PJC « Px. Then, by lemma II.6, (A:B)$^+$Px « p(A^+ + B*)Px. But A$^{4"1"}$" * A, therefore A:B *m* (p(A^+ +,B^+)P)$^+$. QED

Lemma6; IT A, B, and C are Hermitian semidefinite, then (A:B):C « A:(E:C).

Proof: By *lemma.* J, the range of both sides is R(A)fiR(3)f|R(c). How, for x e R(A)flR(B)riR(C)$_f$

A:(B:G)(A^+ + B^+ + C^+)x « A(A + B:C)$^+$B:C(A^+ + B^+ •+ C^+)x

« A(A + B:C)$^+$B(B + C)$^+$CC^+ x 4- A(A + B:C)$^+$C(B + C)$^+$BB"^{I}x + 7J:C(A +B:C)"!AA^+X

« A(A + B:C)$^+$(B + C)(B + C)$^+$x + B:C(A + B:C)$^+$x

« (A + B:C)(A + B:C)$^+$x - *x.*

A si-nilar computation holds for (A:B):C. Then, as in the proof of theorem 6, if P is the projection onto R(A)nR(B)f)R(C), A:(B:C) - (A:B):C - (P(A^+ + B^+ + C^+)P)$^+$. QED

Theorem 7: The Hermitian semidefinite operators on V form a partially ordered commutative semigroup with the semi^rroup or»ex^ation paralH.el addition.

Proof; The semigroup property follows from leninas 1, 2, 3> Q-nd "[. The partial order property is proved in corollary 21, to follow.

It has been proved that if A and B are HSD, then R(A + B) = R(A) + R(B) and R(A:B) = R(A) 0 R(B). These are just the operations used in defining the lattice of subspaces; it seems natural, therefore, to consider that lattice* One question in this regard may be easily answered.

Theorem 8: If P and Q are projections, then the projection onto R(P) 0 R(Q) i£ 2P:Q.

Proof: By lemmas 2 and 3> 2P:Q is an Hermitian operator with the correct range. For x *e* R(P) D R(Q) we have x = P:Q(P^+ + Q^+)x = P:Q(2x) = 2P:Qx. QED.

The formula of theorem 8 answers problem 96 of Halmos [8], for the finite dimensional case. In the published solution it is stated that the familiar algebraic operations are not likely to furnish a solution; theorem 8 appears to meet the requirement.

Because the lattice of subspaces is modular, equalities may be between the range of various series-parallel combinations of operators. In any modular lattice (aA(b v c)) A (bA(c va)) = (a v b) A (b v c) A (c v a) [2]; it follows that for HSD operators A,B,C, the operators A:(B+C) + B:(C+A) and (A+B) : (B+C) : (C+A) have the same range. It appears that these facts about range spaces could be used to study switching circuits, possibly by using projections to represent their range spaces.

Theorem 9s JJL A IS1 B a re Hermitian seffiiclefinite, and Ax » ax, Ex «= bx, then A:Bx = a:bx.

Proof: $(A:B)^{+}x$ - $P_{A:B}(A^{+} + B^{+})P_{AjB}x$

$$= P_{A:B}(A^{+} + B^{+})x$$

$$= (a^{+} + b^{+})x$$

Then A:Bx » $(a^{+} + b^{+})^{+}x$ » a:bx.

The parallel addition is not distributive with respect to series addition. There is, however, a weak distributive law with respect to multiplication.

Theorem 10: If A, B, and C are Hermitian semidefinite, and AC = CA, BC = CB, then (AC):(BC) « (A:B)C* In particular, C night be CI, for some non-negative scalar C.

Proof: (A0):(BC) = $AC(AC + BC)^{+}BC$

$= ACC^{+}(A + B)^{+}BG$

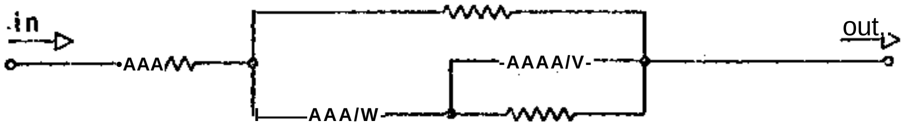
$= A(A + B)^{+} BCC^{+}C$ - (A:B)C QSD

This is the most general distributive lav; that could be expected, since if AC / CA, then AC is not Hermitian.

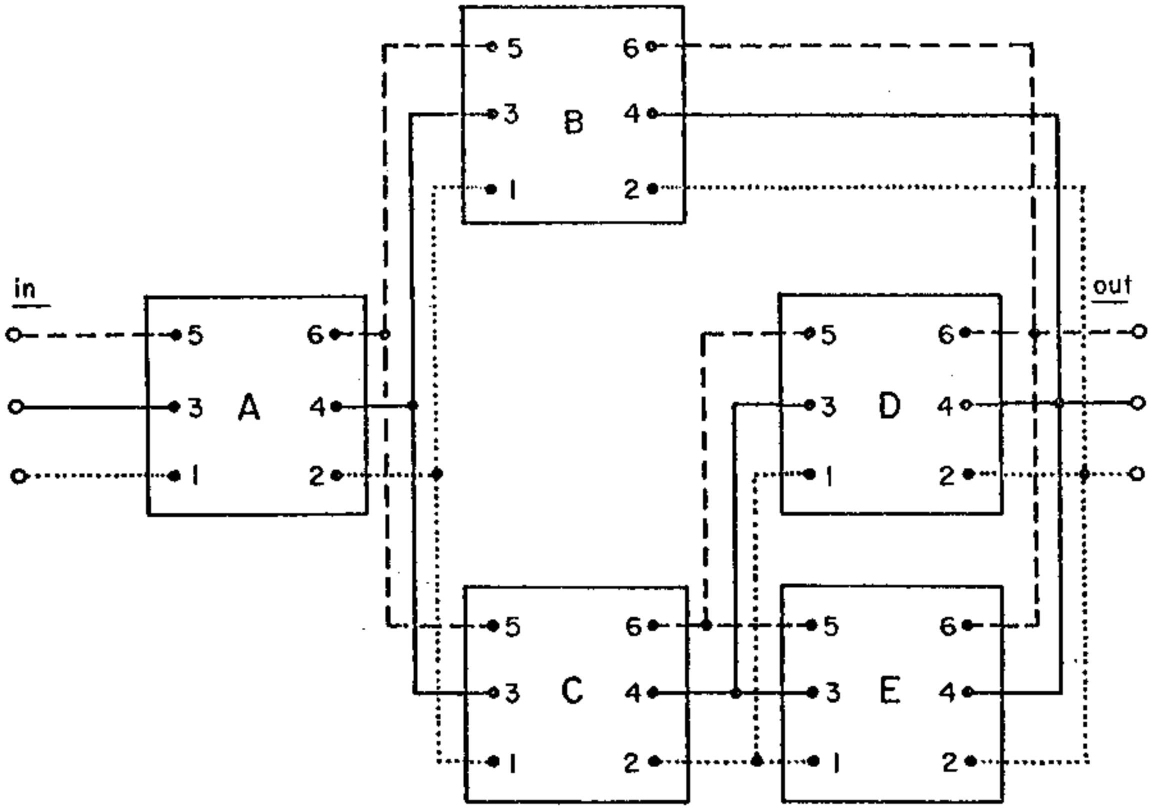
Lemma 11: IT A and B are Hermitian semidefinite, and Aw « Bx * u, •Ay = Bz « v, and w+x = y+z, then u « v.

Proof: By direct computation, A:B(w+x) - u, A:B(y+z) = v, and since w+x « y+z, it follows that u « v. QED

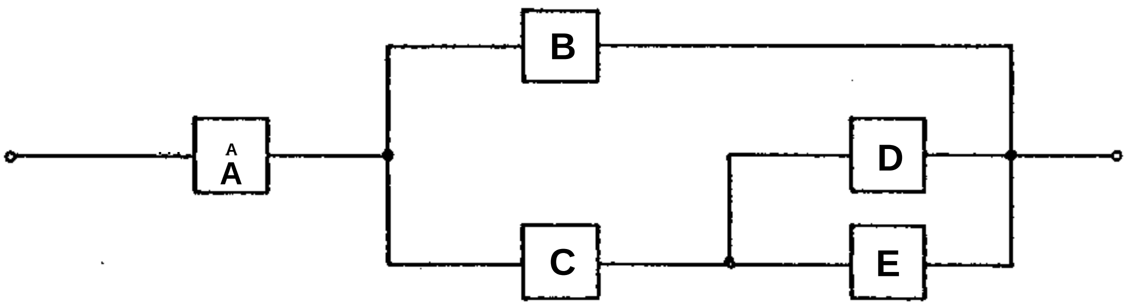
The impedance matrix of a resistive n-port network is semidefinite [10]. A series-parallel connection of 3-port networks is shown in figure 1. If two n-port networks with nonsingular impedance matrices A and B are connected in parallel, then the impedance matrix C of the parallel connection is given by $C = (A^{-1} + B^{-1})^{-1} = A:B$. If A or B is singular, then our formula for A:B still gives the correct impedance. Lemma 11 is the crucial step in the proof, which will not be further discussed here.



a. A SERIES PARALLEL NETWORK (I-PORT)



b. SERIES PARALLEL CONNECTION OF 3- PORTS



C. SYMBOLIC REPRESENTATION OF SERIES-PARALLEL CONNECTION

FIGURE 1

IV. The Matrix of a Parallel Sum

If A and B are Herniitian, an orthononnal basis may be chosen such that A +B is diagonal. Let A + B = C, A:B « *D.* Then a. ~ -b. for i ^ j, and C^+ has the matrix diag(c_{ii}^+) » diag((a_{ii} + b_{ii})$^+$)• Then

IJ ij

$$d_{ij} = \sum_k a_{ik} \sum_m C^+ krAg$$

$$= \sum_k a_{ik}Kk - b_{kk}>^{+b}{}_{tj}$$

since C is diagonal* If a_{kk} = 0, then since A is HSD, a_{ik} « 0 for all i, and we may then write with the convention ^0_j « 0.

(2) $$d_{ij} = \sum_k \frac{a_{ik} \quad b_{kj}}{a_{kk} + b_{kk}}$$

and, in particular,

(3) $$d_{11} = \frac{a_{ii} \quad b_{ii}}{a_{ii} + b_{ii}} - \sum_{i \neq j} \frac{a_{ij}^2}{a_{jj} + b_{jj}} .$$

Lemma 12: Let a.,•••,a , b_n,•••,b be non-negative real numbers. Then

$$\left(\sum_{i=1}^n a_i\right) : \left(\sum_{i=1}^n b_i\right) \geq \sum_{i=1}^n a_i : b_i .$$

Proof: This is merely Hinkowski's inequality [19] . This inequality is more extensively discussed in section V.

Theorem 13: If A and B are Hermitj.an seinidefinite, then tr(A:B) ≤ tr(A): tr(B), with equality iff A « cB, for some scalar c (necessarily real)•

Proof: Since trace is invariant, we may choose a basis such that A + B is diagonal. Then by (2),

$$tr(A:B) = \sum_i \left(a_{ii}:b_{ii} - \sum_j \frac{|a_{ij}|^2}{a_{ii}+b_{ii}} \right)$$

$$\leq \sum_i a_{ii}:b_{ii}$$

$$\leq \left(\sum a_{ii}\right) : \left(\sum b_{ii}\right) = tr(A):tr(B).$$

In the first inequality, equality holds iff all $a_{ij} = 0$, that is, iff AB = BA. In the second inequality, equality holds iff the a_{ii} and b_{ii} are proportional. Thus equality will hold in both iff A = cB. QED

Theorem 14: If A and B are Hermitian semidefinite, then $|A:B| \leq |A|:|B|$.

Proof: If either $|A| = 0$ or $|B| = 0$, then $|A:B| = 0$. If not, then A and B are invertible, and

$$|A^{-1} + B^{-1}| \geq |A^{-1}| + |B^{-1}| = |A|^{-1} + |B|^{-1}.$$

Then

$$|A:B| = |A^{-1} + B^{-1}|^{-1} \leq (|A|^{-1} + |B|^{-1})^{-1} = |A|:|B| \qquad \text{QED}$$

Theorem 15: If V is 2-dimensional, and A + B is nonsingular, then

. – A|B[+ BUI
+ B| .

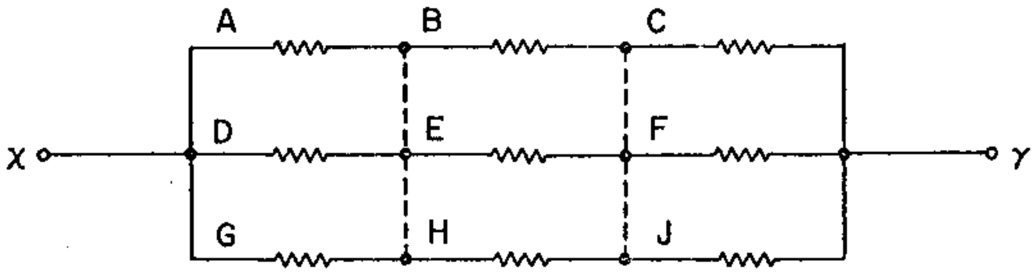
Proof: The result follows from direct computation with matrix elements. QED

No generalization to higher dimensions has been found.

V* Partial Ordering and Parallel .Addition

Definition: If A and B are linear operators, then A $\geq$ B is defined to mean that A - B is positive semidefinite.

The results of this section will be motivated by electrical networks. The proofs, however, will be purely algebraic. Consider the diagram below, which shows nine resistors in a series-parallel network*



FIGUEE 2

If the dotted connections are not present, the joint impedance between terminals x and y is

(4) B -(A + B + C):(D + E + P):(G + H + J),

and if the dotted connections are present

(5) Z^1 « A:D:G + B:E:H + C:P:J .

It v/as observed by Lehman [llj that Z $\geq$ Z' because the current takes the path of least resistance, and in the second case more paths are available. The general case v/ould then be

Theorem 16: If $r_{ij} > 0$, $i = 1, \dots, m$, $j = 1, \dots, n$, then

$$(6) \qquad \sum_{i=1}^{m} \left(\sum_{j=1}^{n} r_{ij}^{-1} \right)^{-1} \le \left(\sum_{j=1}^{n} \left(\sum_{i=1}^{m} r_{ij} \right)^{-1} \right)^{-1} .$$

Proof; This is Minkowski's inequality [9]. In the present notation we may rewrite this.

Corollary 17: If $r_{ij} \ge 0$, $i = 1, \dots, m$, $j = 1, \dots, n$, then

$$(7) \qquad \sum_{i=1}^{m} \mathop{:}_{j=1}^{n} r_{ij} \le \mathop{:}_{j=1}^{n} \sum_{i=1}^{m} r_{ij}$$

where $\mathop{:}_{j=1}^{n} r_{ij} = r_{i1} : r_{i2} : \cdots : r_{in}$.

Proof; For $r_{ij} > 0$ this is theorem 16 above; the case $r_{ij} = 0$ is easily obtained by continuity.

It seems that the same physical reasoning must apply to n-port networks, so that corollary 17 will hold for the joint impedance matrices. The proof below follows a proof of theorem 16 given by Reza [11].

Lemma 18: If A and B are Hermitian semidefinite, then for any x, y, z such that $x + y = z$

$$(8) \qquad (A{:}Bz, z) \le (Ax, x) + (By, y).$$

Moreover, if $z \in R(A) + R(B)$, then if $x_0 = (A + B)^{+}Bz$ and $y_0 = (A + B)^{+}Az$, $x_0 + y_0 = z$ and equality holds in (8).

We first give a heuristic argument. Consider the network of Figure 3

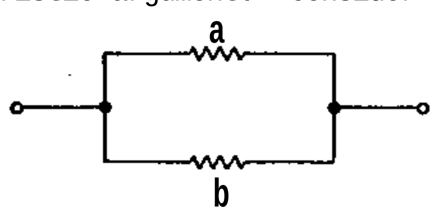


FIGURE 3

For a given current input z, the current will divide $z = x + y$ in such a fashion that the power dissipated $ax^2 + by^2$ is minimum. That is $a{:}bz^2 \le ax^2 + by^2$ which is lemma 18 in the scalar case. The same argument may be used for n-ports with impedance matrices.

Proof: For x_o, y_o as above, $Ax_o = By_o = A{:}Bz$, and $x_o + y_o = (A + B)^+(A + B)z = z$. Then

$$(A{:}Bz,z) = (A{:}Bz,x_o) + (A{:}Bz,y_o) = (Ax_o,x_o) + (By_o,y_o)$$

so that equality holds.

For any z, and $x + y = z$, let $x^1 = P_{A+B}x$, $y^1 = P_{A+B}y$ and $z^1 = P_{A+B}z$. Then let $x_o^1 = (A + B)^+Bz^1$, $y_o^1 = (A + B)^+Az^1$. Since $x^1 + y^1 = z^1 = x_o^1 + y_o^1$, we may write $x^1 = x_o^1 + t$, $y^1 = y_o^1 - t$. Then

$$(Ax^1,x^1) = (Ax_o^1,x_o^1) + 2\,\mathrm{Re}(Ax_o^1,t) + (At,t)$$

$$(By^1,y^1) = (By_o^1,y_o^1) - 2\,\mathrm{Re}(By_o^1,t) + (Bt,t).$$

Since $Ax_o^1 = By_o^1$ it follows that

$$(9)\quad (Ax^1,x^1) + (By^1,y^1) = (Ax_o^1,x_o^1) + (By_o^1,y_o^1) + (At,t) + (Bt,t) \ge (A{:}Bz^1,z^1).$$

But $(Ax^1,x^1) = (AP_{A+B}x, P_{A+B}x) = (Ax,x)$ and similarly for y^1, z^1. The lemma then follows from (9). QED

Corollary 19: *If* A *and* B *are Hermitian semidefinite, then* $(A{:}Bz,z) \le (Az,z) : (Bz,z)$.

Proof: If (Azfz) + (Bz,z) « 0 then the result is clear. If not let

(Bzjjz) , JL^AZ IJ?L)

X ~ ((A"+ B)z,z) Z anQ y " ((A + B)z,z) Z,

Then x + y « z and by lemma 17

(A:Bz,z) ≤ (Ax,x) + (By,y) = (A:Bz,z) QED

Lemma 20: Let A, B, C, and D be Hermitian semidefinite, then (A + B):(C + D) ≥ A:C + BiD.

Proof: It suffices to consider z e R(A + B + C + D), then by lemma 18, for suitable x^o, y^o

((A + B) x (C + D)z,z) = ((A + B)x_Q,X_o) + ((C +'D)y_o,7_o)

« (Ax_o,x_o) + (Bx_o,x_o) + (Cy_o,y_o) + (l^$_o$,y_o)

≥ (A:Cz,z) + (B:Dz,z) QED

Corollary 21: II A, B, and C are Hermitian semidefinite, then A>_B implies A:C ≥ B:C.

Proof: Let A - B « D. Then D is ESD and

A:C « (B + D) : (C + 0) ≥ B:C + D:0 = B:C. QED

This completes the proof of theorem 7. The property of corollary 21 is the necessary relation between the semigroup operation and the partial order \ 6j#

Ve may now prove the series-parallel inequality for operators.

Theorem 22: let A_{ij} i * l,***,m j »'l,«««,n be Hermitian semidefinite. Then

(10) $$\prod_{i=1}^{m} \sum_{j=1}^{n} A_{ij} \geq \sum_{j=1}^{n} \prod_{i=1}^{m} A_{ij}$$

<u>Proof</u>: The proof is by induction. Lemma 20 is the case $m = 2$, $n = 2$. Now assume that (10) is true for m,n and all smaller, then for $m,n+1$

$$\mathop{\prod:}_{i=1}^{m} \sum_{j=1}^{n+1} A_{ij} = \mathop{\prod:}_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij} + A_{i,n+1} \right)$$

$$\geq \mathop{\prod:}_{i=1}^{m} \sum_{j=1}^{n} A_{ij} + \mathop{\prod:}_{i=1}^{m} A_{i,n+1} \qquad \text{by the case } m,2$$

$$\geq \sum_{j=1}^{n} \mathop{\prod:}_{i=1}^{m} A_{ij} + \mathop{\prod:}_{i=1}^{m} A_{i,n+1} \qquad \text{by the case } m,n$$

$$= \sum_{j=1}^{n+1} \mathop{\prod:}_{i=1}^{m} A_{ij}$$

And for $m+1,n$

$$\mathop{\prod:}_{i=1}^{m+1} \sum_{j=1}^{n} A_{ij} = \left(\mathop{\prod:}_{i=1}^{m} \sum_{j=1}^{n} A_{ij} \right) : \sum_{j=1}^{n} A_{m+1,j}$$

$$\geq \left(\sum_{j=1}^{n} \mathop{\prod:}_{i=1}^{m} A_{ij} \right) : \sum_{j=1}^{n} A_{m+1,j} \qquad \text{by the case } m,n$$

$$\geq \sum_{j=1}^{n} \left(\mathop{\prod:}_{i=1}^{m} A_{ij} \right) : A_{m+1,j} \qquad \text{by the case } 2,n$$

$$= \sum_{j=1}^{n} \mathop{\prod:}_{i=1}^{m+1} A_{ij} \qquad \text{QED}$$

<u>Corollary 23</u>; <u>Let</u> $A_1,\ldots,A_n$ <u>be Hermitian semidefinite</u>. <u>Then</u>

$$\sum_{i=1}^{n} A_i \geq n^2 \mathop{\prod:}_{i=1}^{n} A_i .$$

Proof: This follows from theorem 22 with m « n. For example, if n « 3

$$(A + B + C) : (B + C + A) : (C + A + B) \geq A:B:C + B:C:A + C:A:B$$

$$\frac{1}{3} (A + B + C) \geq 5 A:B:C \qquad \text{Q,ED}$$

Corollary 23 is a generalization of the classical inequality betv/een the arithmetic and harmonic means [9 J* This inequality is sometimes proved by a convexity argument* In the present case, we have

Theorem 24: Parallel addition ijs a concave operation for Hemnitian semidefinite operators'.

Proof: Let 8(AfB) = A:B. Then, if a is a scalar, θ(aA,aB) « aO(A,B). Then if a_1, $a_? \geq 0$ and $a_1 + a$^ » 1, using theorem 22

$$\theta(a_1A_1 + a_2A_2, a_1B_1 + a_2B_2) \geq a_1\theta(A_1,B_1) + a_2\theta(A_2,B_2) \qquad \text{QKD'}$$

VI. Continuity

Theorem 25: If A and B are Hermitian semidefinite, then $\|A{:}B\| \le \|A\| : \|B\|$.

Proof: By lemma II.2, for any x

$$(11) \qquad (Ax, Ax) \le \|A\| (Ax, x)$$

and in fact, if $A \ne 0$, for any ϵ there is an x_o such that

$$(12) \qquad \frac{(Ax_o, Ax_o)}{(Ax_o, x_o)} \ge \|A\| - \epsilon .$$

Now let $y = A{:}Bx$, where $\frac{(A{:}Bx, A{:}Bx)}{(A{:}Bx, x)} \ge \|A{:}B\| - \epsilon$. Then

$A{:}B(A^+ + B^+)y = y$ as in lemma 3, and

$$(13) \qquad (A{:}Bx, x) = (y,x) = (A{:}B(A^+ + B^+)y, x) = ((A^+ + B^+)y, A{:}Bx) = (A^+y,y) + (B^+y,y).$$

Since $y \in R(A{:}B)$, let $y = Au = Bv$. Then $(A^+y,y) = (u,Au)$ and $(B^+y,y) = (v,Bv)$. Then from (11) and (12)

$$\frac{(u,Au)}{(Au,Au)} + \frac{(v,Bv)}{(Bv,Bv)} = \frac{(A{:}Bx,x)}{(A{:}Bx, A{:}Bx)}$$

and then from (11)

$$\frac{1}{\|A\|} + \frac{1}{\|B\|} \le \frac{1}{\|A{:}B\| - \epsilon}$$

$$\text{or} \quad \|A\| : \|B\| \ge \|A{:}B\| - \epsilon ,$$

but ϵ was arbitrary. QED

By theorem 9, equality will hold if there is a y such that $Ay = \|A\| y$ and $By = \|B\| y$.

Theorem 25 expresses continuity of parallel addition about 0; the next theorem applies at other points. We thus consider $(A + X) : (B + Y) - A{:}B$

and obtain bounds for the error in terms of the partial order of operators.

Lemma 26: Lf A and B are Hermitian semidefinite operators such that $B \geq A$, then

(14) $$(B^+ - A^+)P_A - B^+(B - A)A^+$$

Proof: $B^+(B - A)A^+$ « B^+BA^* - B^+AA^+ « $PJl^+ - B^+P_A$ « $A^+ - B^+P$ since $R(B) \supset R(A)$. QED

Lemma 27: I£ A and B are Hermitian seroidefinite operators and $C - A + B$ then $P_C(I - C^+B)$ •« C^+A.

Proof: $C^+A + C^+B - C^+C - P_p$» Then $P(l - C^+B) - P_n - C^+B = C^+A$. QED

Theorem 28: Let A, B and X b£ Hermitian semidefinite and G » A:(B + X) - A:B. Then- G i^s HSD and if C = A + B,

(15) $$G - AC^+(C:X)C^+A \quad \text{and}$$

(16) $$||G||| l ||C^+A||^2 | I X | 1$$

Rroof:

$$G - A(C + X)^+ (B + X) - AC^+B$$

$$= A[(C + X)^+ - C^+]B + A(C + X)^+X$$

Since $P_C B$ « B lemma 24 applies and then

(17) $$G - - A(C + X)^+XC^+B + A(C + X)^+X$$

By the definition of G, $P_nG = G$, and since G is Hermitian GP« « G. Therefore

$$G = GP_n = - A(C + X)^+XC^+BP_0 + A(C + X)^+XP_{rt}$$

$$- A(C + X)^+XP_n(l - C^+B).$$

Then lemma 27 applies and we have

$$G - A(C + X)^{+}XC^{+}A = AC^{+}C(C + X)^{+}XC^{+}A$$
(18)
$$- AC^{+}(C:X)C^{+}A.$$

Since C:X is HSD and (l8) is congruent to CJX, G is HSD. Then using theorem 25

IMUIKII Ml H^{C+A}ll
$\leq$ ||A0^{+}||^{2} ||X|| QED

Lemma 29; If A, B, and X are HSD, then

2(A + X):(B + X) + (A + B):(2X) - 2(A + B + X):X + A:(B + 2X) + (A + 2X):B

Proof: Let A + B'+ 2X » D, then by computation both sides equal

$$2AD^{+}B + 4AD^{+}X + 2XD^{+}B + 2BD^{+}X + 2XD^{+}X$$

Lenim $0: Jf A, B and X are HSD, and

H - (A + X):(B + X)'- A:B - X:X

then H JLS HSD and for C » A + B

(19) . 2H « A0^{+}(C:2X)C^{+}A + BC^{+}(C:2X)C^{+}B - | C:2X

and

(20) $$||H|| \leq 2(||C^{+}A||^{2} + ||C^{+}B||^{2})||X||.$$

Proof; By lemma 29,

2H = A:(B + 2X) - A:B + B?(A + 2X) - B:A + 2X:(A + B + X) - 2X:X - (A + B);(2X)

then using theorem 28

2H = AC^{+}(C;2X)C^{+}A + BC^{+}(C:2X)C^{+}B + 2X(2X^$^{+}$(C:2X)(2X)$^{+}$X - C:(2X)

which simplifies to (19). By theorem 22 H is HSD. Therefore

2H$\leq$ AC^{+}(C:2X)C^{+}A + BC^{+}(C:2X)C^{+}B and, as in theorem 28

|| H || $\leq$ || A0^{+}||^{2}|| 2X || + ||BC^{+} ||^{2} j| 2X|| which is (20). QED

Theorem 51: If A, B, X and Y are Herctitian semi-definite, then

(21) $\|(A+X):(B+Y) - A:B\| \le 2(\|(A+B)^+A\|^2 + \|(A+B)^+B\|^2 + \frac{1}{4})\|X+Y\|$

Proof; It follows from corollary 21 that

$$(A+X):(B+Y) - A:B \le (A+X+Y):(B+X+Y) - A:B,$$

and using lemma 30

$$\begin{aligned}
&\|(A+X+Y):(B+X+Y) - A:B\| \\
&\le \|(A+X+Y):(B+X+Y) - A:B - (X+Y):(X+Y)\| \\
&\quad + \|(X+Y):(X+Y)\| \\
&\le 2(\|(A+B)^+A\|^2 + \|(A+B)^+B\|^2)\,\|(X+Y)\| + \|\tfrac{1}{2}(X+Y)\| \\
&= 2(\|(A+B)^+A\|^2 + \|(A+B)^+B\|^2 + \tfrac{1}{4})\|X+Y\|
\end{aligned}$$

QED

VII. Generalization

The definition of parallel addition A:B « A(A + B)$^{+}$B may be applied to any pair of linear operators, since the generalized inverse is always defined. However, without suitable restrictions, very little of the preceding theory will hold. Four rather natural extensions have been considered, and will be developed in another paper. These extensions are the following:

(I) The networks which motivated the theory here were resistive. A natural extension is to consider networks with reactive elements. In that case the impedance matrices will be "positive real" [4], and will in general be non-Hermitian. Certain parts of the series-parallel algebra will extend.

(II) A lineax operator A on a real vector space is said to be almost positive definite if (Ax,x) $\geq$ 0 for all x and (Ax,x) = 0 only if Ax = 0 [12]; alternatively, if A = H + S with H Hermitian semidefinite, S skew, and E(S) c R(H). The latter definition is related to positive real matrices, where a similar range relation holds between the real and imaginary parts. The algebra of section III holds for almost positive definite matrices; however, lemma 18 is not true in this case, and the remainder of the theory will not follow.

(III) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a partitioned matrix with a square, then the gyration of A, r (A), is defined by

$$r(A) = \begin{bmatrix} a^{-1} & -a^{-1}b \\ ca^{-1} & d-ca^{-1}D \end{bmatrix} .$$

This operation has been studied by Tucker [l^], and is the basis of the network synthesis method Duffin, Hazony, and Morrison [^J. A hybrid addition of operators A and B may then be defined by A:B = r(r(A) + r(B)). For nonsingular A and B this includes series and parallel addition as special cases, depending on the partition. Even in the singular case hybrid addition may be made the basis for an algebra similar to that developed here.

(IV) The question of extending the definition of parallel sum to operators in Hilbert space suggests itself. Although $(A + B)^+$ is defined [ljj it need not be bounded, and thus $A(A + B)^+B$ may not be defined on the whole space. It would appear that a different definition of parallel addition is necessary.

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