REPRESENTATION OF FAITHFUL NORMAL EXPECTATIONS IN von NEUMANN ALGEBRAS

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Introduction.

Let G and IB be two C* algebras with identity. . Suppose 8 c G. Let <p be a positive linear map of G on IB such that <p preserves the identity and such that <p(BX) = B<p (X) for all B in IB and all X in. G . <p is then defined to be an expectation of G on IB. The extension of the notion of an expectation in the probability theory sense, to expectations on finite von Neumann algebra is largely due to J. Dixmier and H. Umegaki [1]. In [4] Tomiyama considers an expectation on von Neumann algrbras to be a projection of norm one. If <p is an expectation in the sense cp(BX) = *Bcp(X)* , <p positive and <p preserves identities, then <p(XB) = <p (X) B for all X in G , B in B. IB is the set of fixed points of <p. By writing <p[(X - (p(X))*(X - (p(X))] >_ 0 we have <p(X*X) >_ <p (X) *<p (X) . In particular <p is a bounded map.

Let h and k be two Hilbert spaces,- h ©k will denote the tensor product of h and k . Let G be a von Neumann algebra acting on h , by an ampliation of G in h © k one means a map *ij)* of G in L(h {^k) such that $(A) = A © I_v where I_k denotes the identity operator on k. The image of G by an ampliation is then a von Neumann subalgebra of L (h © k) . In what follows CT will designated the image of G by an ampliation 0 and $\tilde{A}$ will stand for i/)(A) .

In this paper expectations of a particular type are considered. If IB is a subalgebra of G and if IB is the range of a faithful, normal expectation <p defined on G , then it will be shown that

there exists an ampliation of G in h (D }L , independent of B and of cp , such that <p θ I_K is a spatial isomorphism of S\ This result extends a result by Nakamera, Takesaki, and Umegaki [2], which consider the case when G is a finite von Neumann algebra.

Definitions.

Let M and N be C* algebras and <p a positive linear map of M on N. Let M_n be the set of all nXn matrices whose entries are elements of M , call those entries $A_{i,j}$. Define for each n, $<p^{(n)}$ $(A_{i,j})$.* = $((p(A_{ij}))$ • $<p^n$ is then a map of M_n on N_n .

<p is called completely positive if each <p is.

Let G and 6 be two von Neumann algebras, with G c B. Let <p be an expectation of R~ on G. (p is called faithful if for any T in G $_y$ (p(TT^) = 0 implies T = 0, Let A_α be a net of uniformly bounded self adjoint operators in G. <p is called normal if sup <p(Aj = <p(sup A_α)

The ultra-weak topology on G will be the weakest which will make all E w_{x_i,y_i} (A) = L(Ax.,y.) continuous where $S||x_i||^2$ < oo and $S||y_i||^2$ < co . In what follows if N is arbitrary von Neumann algebra, N^T will denote the commutant of N. If h is any Hilbert space, dim h will denote the cardinality of the dimension of h.

Propo ition 1.

Let M and N be two von Neumann algebras acting on h^M and h^. Let <p be a * isomorphism of M on B. Let k be a Hilbert space such that dim k >_ Max (^, dim h^M, dim h^) , then (p ® I_K is a spatial isomorphism.

This theorem says that there exists a isometry V of h^M θk on h_N ⊗ k such that <p ® I_k (A®C_k) = (p(A)⊗ I_k = v (A @ I_k) V^ (= VA V*).

Tomiyama has shown this result in [5].

Proposition 2.

Let M and N be two C^* algebras with identities. Let φ be an expectation of M on N, then φ is completely positive. This result was shown by Nakamura, Takesaki, and Umegaki in [2].

One of the tools for the proof of the theorem will be the Stinespring construction which is given in [3] and which will be sketched here for completeness sake.

Let M be any von Neumann algebra acting on h. Let $M \otimes h$ denote the tensor product of M and h as linear spaces. Let N be von Neumann algebra of M which is the range of a faithful, normal expectation φ. On $M \otimes h$ define an inner product by:

$$\langle \sum_{i=1}^{n} a_i \otimes x_i , \sum_{j=1}^{n} b_j \otimes y_j \rangle = \sum_{i,j} (\varphi(b_j^* a_i) x_i , y_j)$$

where a_i, b_j are in M, x_i, y_i are in h and where $(\;)$ denotes the inner product in h. Now:

$$\sum_{i,j} (a_j^* a_i x_i , x_j) = \left(\sum_{i=1}^{n} a_i x_i , \sum_{i=1}^{n} a_i x_i \right) \geq 0 .$$

Let A be in M_n with $A_{ij} = a_j^* a_i$ then if $x = (x_1, x_2, \ldots, x_n)$

$$(Ax, x) = \sum_{i,j} (a_j^* a_i x_i , x_j) \geq 0$$

By proposition 2,

$$\sum (\varphi(a_j^* a_i) x_i , x_j) \geq 0 .$$

Hence the product defined on $M \otimes h$ is bilinear and positive. However it is possible to have $\langle \xi , \xi \rangle = 0$ with $\xi \neq 0$. Divide out the space $M \otimes h$ by all vectors of norm zero. Then taking the completion of that space, one obtains a Hilbert space which will be denoted $M \bar{\otimes} h$.

<u>Lemma</u> 3. h is imbedded as a Hilbert space in $M \otimes h$.
Proof: In fact we shall show that h is isomorphic to $N \otimes h$. Let a_i $i = 1,2,\ldots,n$ be operators in B, consider the map

$$S\left(\sum_{i=1}^{n} a_i \otimes x_i\right) = \sum_{i=1}^{n} a_i x_i$$

then

$$\left\langle \sum_{i}^{n} a_i \otimes x_i , \sum_{j}^{n} a_j \otimes x_j \right\rangle$$

$$= \sum_{i,j} (\varphi(a_j^* a_i) x_i , x_j)$$

$$= \sum_{i,j} (a_j^* a_i x_i , x_j)$$

$$= \left(\sum_{i=1}^{n} a_i x_i\right)$$

Hence S is an isometry of $N \otimes h$ on h. In particular then, one can view h as a subspace of $M \otimes h$.

<u>Lemma 4</u>.

φ defines a self adjoint projection E of $M \otimes h$ on $N \otimes h$.

Proof: Let a_i, $i = 1,2,\ldots,n$ be operators of M. Define

$$E\left(\sum_{i=1}^{n} a_i \otimes x_i\right) = \sum_{i=1}^{n} \varphi(a_i) \otimes x_i$$

the proof in [2] shows that E is a well defined self adjoint projection of $M \otimes h$ on $N \otimes h$. Recall for example how self adjointness is checked out.

$$\langle E(\sum_i a_i \otimes x_i), \sum_j b_j \otimes y_j \rangle$$

$$= \langle \sum_i \varphi(a_i) \otimes x_i, \sum_j b_j \otimes y_j \rangle = \sum_{i,j} (\varphi(b_j^* \varphi(a_i)) x_i, y_j)$$

$$= \sum_{i,j} (\varphi(\varphi(b_j^*) a_i) x_i, y_j)$$

$$= \langle \sum_i a_i \otimes x_i, \sum_j \varphi(b_j) \otimes y_j \rangle = \langle \sum_i a_i \otimes x_i, E(\sum_j b_j \otimes y_j) \rangle$$

Lemma 5.

There exists an ultra-weakly continuous representation ℓ of M in $L(M \otimes h)$ such that $\ell(b)E = E\ell(b)$ for all b in N. Moreover if h and $N \otimes h$ are identified by the isometry S of lemma 3, then $\varphi(a) = E\ell(a)E$ for all a in M.

Proof: For each a in M define

$$\ell(a)(\sum_i a_i \otimes x_i) = \sum_i a a_i \otimes x_i$$

ℓ is then a representation of M in $L(M \otimes h)$. Let b_i, $i=1,2,\ldots,n$ be operators in N then:

$$E\ell(a)(\sum b_j \otimes x_j) = E(\sum a b_j \otimes x_j)$$

$$= \sum_j \varphi(a) b_j \otimes x_j = \varphi(a)(\sum_j b_j x_j)$$

identifying $\sum_j b_j \otimes x_j$ with $\sum_j b_j x_j$. this shows that $E\ell(a)E = \varphi(a)$. Let b be in N then

$$\ell(b)E(\sum a_i \times x_i) = \ell(b)(\sum \varphi(a_i) \otimes x_i)$$

$$= \sum b\varphi(a_i) \otimes x_i = E\ell(b)(\sum a_i \otimes x_i).$$

So $\iota(b)E = E\varphi(b)$ for all b in N. To show now that ι is u.w. continuous. Let

$$\zeta_k = \sum_{i=1}^{n_k} a_i^{(k)} \otimes x_i^{(k)} \quad , \quad \eta_h = \sum_{j=1}^{n_h} b_j^{(h)} \otimes y_j^{(h)}$$

with $\sum \|\zeta_k\|^2 < \infty$ and $\sum \|\eta_h\|^2 < \infty$. Let a_α be a net converging u.w. to a in M. Then it is sufficient to show that A tends to zero where

$$A = \sum_{k,h} < \iota(a - a_\alpha)\zeta_k \, , \, \eta_h > .$$

We have

$$A = \sum_{k,h} \sum_{i,j} \langle \varphi(b_j^{(h)*}(a - a_\alpha)a_i^{(k)}) x_i^{(k)} , y_j^{(h)} \rangle$$

Now $b_j^{(h)*}(a - a_\alpha)a_i^{(k)}$ tends to zero u.w. As φ is normal, A tends to zero. Let $N \subset M$ be two von Neumann algebras acting on h. Let φ be a faithful, normal expectation of M on N.

<u>Proposition</u> 6.

<u>There exists a Hilbert space</u> k <u>such that</u>:

(1) h <u>can be imbedded in</u> k

(2) <u>There exists an u.w. continuous representation</u> ι <u>of</u> M <u>in</u> $L(k)$ such that $\varphi(A) = p_1\iota(A)p_1$ where p_1 is the projection <u>of</u> k <u>on</u> h.

(3) ι <u>is a</u>* <u>isomorphism</u>.

(4) p <u>commutes with all</u> $\iota(b)$ <u>with</u> b <u>in</u> N.

Proof: Let $k = M \otimes h$, if $\iota(a) = 0$ then $\iota(a^*a) = 0$ so $\varphi(a^*a) = 0$.

By faithfulness of φ this implies $a = 0$. Hence I is a * isomorphism of M in $L(k)$. The rest of proposition 6 is a restatement of lemma 5.

Theorem 7.

There exists an ampliation of M in $h \otimes k$ such that if N is any von Neumann subalgebra of M which is the range of a faithful normal expectation φ then there exists an isometry V in $(N \otimes I_k)'$ such that $\varphi \otimes I_k(\tilde{A}) = V\tilde{A}V^*$, $VV^* = I$, on putting $V^*V = P$ then P is in $(N \otimes I_k)'$, $\varphi \otimes I_k(\tilde{A}) P = P\tilde{A}P$. For all $\tilde{A}$ positive, $\tilde{A}P = 0$ implies $\tilde{A} = 0$.

Proof: Let s be a Hilbert space with cardinality greater or equal to the maximum of $\aleph_1$ and cardinality of a Hammel basis of $M \otimes h$. Define $T(\tilde{A}) = i(A) \otimes I_s$, $\tilde{\varphi} = \varphi \otimes I_s$. Then:
$\tilde{\varphi}(\tilde{A}) = (P_h \otimes I_s)T(\tilde{A})(P_h \otimes I_s)$. By proposition 1, T is spatial, There exists an isometry U of $h \otimes s$ onto $k \otimes s$ such that $\tilde{\varphi}(\tilde{A}) = U(\tilde{A})U^*$. Hence

$$\tilde{\varphi}(\tilde{A}) = P_{h\otimes s}\, U(A \otimes I_s)U^* P_{h\otimes s}$$

where $P_{h\otimes s}$ denotes the projection of $k \otimes s$ on $h \otimes s$. Moreover $P_{h\otimes s}$ commutes with all $U\tilde{B}U^*$ as B ranges over N (proposition 6). So $U^*P_{h\otimes s}U$ commutes with all $\tilde{B}$ for B in N.

Let $V = P_{h\otimes s}U$ then $VV^* = P_{h\otimes s}$ $(= I_{h\otimes s})$. Define $V^*V = P = U^*P_{h\otimes s}U$. Then P is in $(N \otimes I_s)'$. So $\tilde{\varphi}(\tilde{A}) = V\tilde{A}V^*$ for all A in M.; Claim: V is in $(N \otimes I_s)'$. Let B be in N, $\tilde{B} = \tilde{\varphi}(\tilde{B}) = V\tilde{B}V^*$ so $V^*\tilde{B} = P\tilde{B}V^* = \tilde{B}PV^* = (\tilde{B})V^*$ so V is in N. Now

$$P\tilde{A}P = V^*V\tilde{A}V^*V$$

$$= V^*\widetilde{<p}(\tilde{A})V$$

$$= V^*V\widetilde{<p}(\tilde{A}) = P\widetilde{<p}(\tilde{A})$$ Also,,

$$P\tilde{A}P = V^*\widetilde{(p}(\tilde{A})V = \widetilde{<p}(\tilde{A})V^*V$$

$$= \tilde{\varphi}(\tilde{A})\,P.$$

Let $\bar{P}$ be now the central carrier of P, I - F = ^(I-*$\bar{P}$) = (I-$\bar{P}$)P_{hQs} = O. So $\bar{P}$* = I. Hence if A $(\tilde{B})$ = $P\tilde{B}P$^ then 7\ is an isomorphism. If $\tilde{A}P = 0$ then $P\widetilde{<p}(\tilde{A})P = 0$ so $\widetilde{<p}(\tilde{A}) = 0$. By faithfulness if A is positive^ A = 0.

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