ON CATEGORIES IN GENERAL TOPOLOGY

by

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results can be extended to more general fibred categories, but the special nature of our fibres makes our theory much simpler and, we hope, more lucid.

The projection functors of our paper are essentially equivalent to the pull-back stripping functors of [6], and the main results of [6] remain valid in our theory. We also generalize and complement results of fio].

The organization of the paper is as follows. Section 2 deals with preliminaries. In section 3, we define our categories and obtain some basic properties, and section 4 provides examples. Sections 5 - 7 deal with coadjoint[3] functors, couniversal maps, and coreflective full subcategories, and section 8 brings point separation axioms into our theory.[4]

We use-the terminology of f8J for categories, with some modifications (see section 2). The n^{th} item (theorem, lemma, etc.) of section m will be referred to as m.n. The Halmos symbol | signals the end, or the abscence, of a proof.

2. Preliminaries

We shall use parentheses only when necessary. Thus we usually write fx for f(x) . Morphisms of a category £ will also be called <u>maps</u> of £ . For objects A and B of C , we denote by C,(A,B) the set of all maps of £ from A to B . For a class IC of objects of £ , we denote by CJK] the full sub-

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3. The prefix co- is due to the general principles followed in fs] (see the preface, p. vi). Many authors do not use co- where f8J does, and vice versa.

4* Only parts of chapters I, II, V of f8] are used.

category of £ with K as its class of objects.

For every diagram scheme I , and each object A of a category £ , there is a constant diagram A^I : I —$> £ , with vertices A and arrows 1_A . A map f : A —> B of C induces a map f^{I_*} : A^I —> B^I of diagrams in an obvious way. With this notation, a limit of a diagram *A* : I —> £ consists of an object A of £ and[5] a map A : A^I —> Z\ of diagrams such that, for every map O> : C^I —£> /S of diagrams, there is a unique map f : C —^ A of £ for which & = A f^I .

A couniversal[3] morphism for a functor T : A. —^ IJ and an object B of J3 consists of an object S_{tt} of A and[5] a map y6L : B ~» T S_n of B such that, for every map v : B *—^ T A of B , there is exactly one map u : S —^ A of A for which v - (T u) P . If A is a subcategory of B and T the embedding functor, then a couniversal morphism for T and an object B of B is called a coreflection[3] for B in A. .

A functor T : il *—> J3 has a coadjoint[3] if and only if every object of J3 has a couniversal morphism for T • In particular, a subcategory A of a category jB is coreflective[3] if and only if its embedding functor has a coadjoint.

We call a monomorphism m of a category £ extremal if a factorization m « m' e , with m' monomorphic and e epimorphic, is only possible if e is isomorphic. Every coretraction (f8J, 1,4) is an extremal monomorphism.

The category of complete ordered sets will be denoted by £ • Objects of £ are ordered sets[2] in which every family[6] \^x i/ᵢ ^ᵢ ∊ I °^ elements has an infimum

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5« By abuse of language, one often "forgets" either the map or the object.

6. This will almost always include the empty family.

$\bigwedge x_i$ and a supremum $\bigvee x_i$. Maps of £ are order preserving mappings. A complete ordered set has a greatest and a least element, the infimum and the supremum of the empty family of elements. The dual ordered set X* of a complete ordered set is complete. We say that a map $f : X \to Y$ of 0 preserves infima if $f(\bigwedge x_i) = \bigwedge f x_i$ for every family (x_i) of elements of X .

Lemma 2.1. A map $f : X \to Y$ of £ preserves infima if and only if there is a map $g : Y \to X$ such that y <£ f x <£=£> gy <x , for all x £ X , y£Y f

3. Definition and properties

We assume that a category £ and a contravariant functor $T : £ \to £$ are given, and that all maps T f of JD , for maps f of C , preserve infima. As a rule, we shall just write f* for T f . We shall study a new category $£^T$ constructed from these data and called the T-fibred category over £ .

Objects of $£^T$ are all pairs (A,x) such that A is an object of £ and X € T A . Morphisms of $£^T$ are all triples (f,x,y) such that f is a map of £ and x ^ f * y with x £ I A and y £ T B if f £jC(A-B) . The composition of two morphisms of $£^T$ is defined by putting

$$(g,y,z)(f,x,y) = (g f,x,z)$$

if g f is defined in £ . If g f is not defined in £ , or if y' ≠ y , then (g,y',z)(f,x,y) is not defined in $£^T$.

It is easily seen that $£^T$ is a category, with $1_{(A,x)} = (1_A,x,x)$ for every object (A,x) . The domain of a morphism (f,x,y) of $£^T$, with f £ £(A,B) , is (A,x) and the codomain (B,y) • We usually just write f : (A,x) —^ (B,y) if we want to state that (f,x,y) is a morphism of $£^T$ with f££(A,B) .

Monomorphisms and epirnorphisms of $£^T$ are the morphisms (f,x,y) with f monomorphic and epimorphic respectively in £ . Extremal raonomorphisms of $£^T$ are the morphisms of the form (m,m* y,y) , m an extremal monomorphism of £ . Isomorphisms of $£^T$ are the morphisms (u,u* y,y) , u isomorphic in £ .

Putting P(A,x) = A and P(f,x,y) = f , for every object (A,x) and morphism (f,x,y) of $£^T$, clearly defines a functor P : $£^T$ —> £ . We call P the projection functor of $£^T$. We note that P is a faithful functor.

If the objects and maps of j) are considered as categories and functors, then the "functor P : $£^T$ — > £ becomes a fibration, in the terminology of [4] and f5], and the functors f* define a split cleavage of P • Since x ^y in T A , for an object A of £ , if and only if 1_A : (A,x) -—> (A,y) in $£^T$, the fibre P^{-1}(A) of A is isomorphic to T A , considered as a category.

In the terminology of f6], P is a pullback stripping functor. Conversely, a pullback stripping functor H : A~~~^IL ^s o** ^e form P Cj>, for a projection functor P : £ •—>£ and an equivalence of categories CP : A.—>£ • In this situation, it is easily seen that A has products, as required in f6J, if and only if £ has products and all maps f* , for maps f of £ , preserve infima.

Examples will be given in the next section.

For f in £(A,B) , we define f^ : T A —->T B by putting f^ x^ y «£=£• x^ f* y , for all x £ T A , y £ T B . By 2.1, the maps $f_\#$ are well defined.

Lemma 3*1« The maps f^ define a covariant functor from C to 0 .

Proof. For f – 1_A , we have f x,< y ^^=> x ^ f * y = y , for all x, y in T A , and thus $f_\#$ « 1^{TA} • If f £ £(A_fB) and g£C(B,c) , then

(g f)* 3C<z «=4> x< (g f)* z = f* g* z
<==> f^ x ^ g* Z <^=|> g^ $f_\#$ X < 2 ,

for all x£TA , z £ T C , and (g f)$_\#$ = g$_\#$ f$_\#$ follows |

We put T* A a (T A)* ', the dual ordered set, and T* f = $f_\#$: T* A –> T* B $_f$ for an object A and a map f : A -–> B of £ ', and we call T* : £* – > £ the dual functor of T J £ -–>JQ . By 2.1$_f$ the maps T* f preserve infima.

Theorem 3.2. (£*)^{m*} is isomorphic to the dual category ' •(£^{m})* .of JC^m .

Proof, y < f$_\#$x in T*B, for f in£(A_fB) and x 6T A , y ^ T B , if and only if x ^ f * y in T A • Thus f : (B,y) –> (A,x) in (£*)^{T*} if and only if f : (A,x) –^(B,y) in £^{T} • One sees easily thajb this establishes an isomorphism between (£*)^{T*} and (C^T)* [

This shows that our theory is completely self-dual, as long as £ is not specialized; We usually do not state the duals of our definitions and results.

Por an object A of £ , we denote by oi_h the least element, and by UJ$_A$ the greatest element, of T A $_f$ and we put <x A = (A, C*$_A$) and OJA = (A, OJ$_A$) , Por a map f in £(A,B) , we put <x f = (f,o($_A$,o($_B$) and CJf = (f,6Jf$_{Af}$WL$_B$) • Since (X$_k$ ^f*<X^ and OJ. « f*CcL , ocf and 6Jf are morphisms of C^T . This obviously defines functors <X and CO from £ to £^{T} •

Theorem 3»3* With the notations .just defined, the functor o< is coad.joint, and the functor U) .adjoint, to the projection functor P •

Proof. For objects A of £ and (B_f y) of £^⊥ , "ηf=(f,c<_A y) defines a bisection rrj :_jC(A, P(B,y)) —> £ {& A, (B_t y)) _f natural in A and in (B_f y) . This proves one half of 3*3; the other half is proved dually ∎

~~Theorem 3«4«~~ ~~A diagram~~ ,A : I —^ £^T ~~has a limit in~~ £^T ~~if and only if the~~ diagram P«A : I *-> £ has a limit in £ .

Proof. By 3.3 and fs], II.12.1, PA has a limit in £ if A has a limit in £^T • Conversely, let A i = (A_i, x_i) for each vertex i of I , and let A : A^I —> I>/\ be a limit of P Z\ in £ , with maps A_i •• A —> A_i . The morphisms (A_i ,x, x_i) of £^T , with z = ⋂ j W_i * x_i) in T A , clearly define a map Ã : (A_f x)^I ——) Z\» We want to show that this is the desired limit of Zi .

If D̃ : (C, z)^I —5 / _9 with morphisms (φ_i, z_9 x_i) , then the maps <p_i : C —> A_i define a map cp ~ Vq̃> : C^I —^ PZl , and thus C5 = A f^I for a unique map f : C —> A of £ • Now

$$z \leq \bigcap (f^* \lambda_i^* x_i) = f^*(\bigcap(\lambda_i^* x_i)) = f^* x ,$$

so that ^> = A (f,z,x)^ # As this equation implies a> = A f^-_f- it determines f , and hence (f,z,x) _f uniquely ∎

4. Examples

We simply state the ingredients for each example and name the result, leaving to the reader the easy verificationsthat in each case the given ingredients define a fibred category £^T .

Example 4*1* Let £ be any category. For an object A of JC , let A^o be a singleton, and for f €.C(A,B) , let f^o : B^o —^ A^o be the unique mapping. For the fibred category $£^o$ thus defined, the projection functor P : $J2^o$--^ JC is an isomorphism of categories.

Example 4.2« Let £ = S^ , the category of sets. For filters JE and $3?^1$ on a set A , we write $£^f$ ^ F if F^1 is finer than F , i.e. JP C £* • Then every non-empty family $(F_i)_{i \in I}$ of filters on <A has a supremum (^JF$_i$, consisting of all set unions ^Jx$_i$ with X$_i$ £• F$_i$ for all i £ I . For a mapping f : A-—>B and a filter £ on A , we denote by $f_\#$ F the filter on B generated by the sets f(x) , X £ F . This preserves suprema.

A convergence structure q on a set A is a relation q from the set F A of filters on A to the set A_f subject to the two Frechet axioms.

L1. If P q x and F' ^ P , then P^1 q x ,

L2. If x £A and if £ consists of all subsets X of A with x £ X , then P q x . We denote this filter by x •

We write q * ^ q , for convergence structures q and q^1 on a set A , if q^f is finer than q , i.e. if always F q^1 x —£ * P q x , With this definition, convergence structures on A form a complete ordered set Q A • For a mapping f : A —> B and q 6 Q B , we denote by f* q the convergence structure on A defined by F (q* f) x 4r=^> ($f_\#$ f) q (fx) , for all filters T on A and all xfcA . The fibred category i> thus defined is the category of convergence spaces, with continuous mappings as morphisms.

Example 4«5« Let again £ = S . For a set A , let Tp A be the set of

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all topologies of A , ordered by putting $\tau' \leq \tau$ if τ is finer than τ', i.e. if all τ'-open sets are also τ-open. For a mapping $f : A \to B$ and a topology σ of B, let $f^*\sigma$ be the topology with the sets $f^{-1}(V)$, V open for σ, as open sets. The fibred category S^{Tp} thus defined is the category of topological spaces, with continuous mappings as morphisms.

There are several examples similar to 4.2 and 4.3 in general topology.

Example 4.4. Let $\mathcal{E} = G$, the category of groups. For a group A , let Tpg A be the set of all topologies of A compatible with the group structure. The order relation of Tpg A , and the maps $f^* : \text{Tpg } B \to \text{Tpg } A$ for group homomorphisms $f : A \to B$, are defined as in 4.3. The resulting fibred category G^{Tpg} is the category of topological groups, with continuous, but not necessarily closed, group homomorphisms as morphisms.

This is a theme with many variations.

Example 4.5. Let $\mathcal{E} = S$, the category of sets. For a set A , let I_A be the diagonal of $A \times A$, and let R A be the set of all subsets U of $A \times A$ which contain I_A, ordered by set inclusion. For $f : A \to B$ in S and $V \in R B$, let $f^* V = (f \times f)^{-1}(V)$. An object $\rho = (A,U)$ of the resulting fibred category S^R may be considered as a reflexive relation $\rho : A \to A$, with graph U . We call S^R the category of reflexive relations.

If we let E A be the set of all graphs of equivalence relations on A and define $f^* V$ as before, we obtain a fibred category S^E, the category of equivalence relations. S^E clearly is a subcategory of S^R.

5. Fibred functors and coadjoints

We consider in this section two fibred categories $\underline{A}^T$ and $\underline{B}^S$. We use the notations of section 3 for $\underline{A}^T$ and for $\underline{B}^S$ since the context always will show e.g. which functor P is meant, or whether f^* is Tf or Sf.

We say that a functor $\Phi : \underline{A}^T \to \underline{B}^S$ is fibred over a functor $F : \underline{A} \to \underline{B}$ if $P\Phi = FP$. If this is the case, then $F = P\Phi L_0$, so that Φ determines F. Putting $\Phi(A,x) = (FA, \varphi_A x)$ for an object (A,x) of $\underline{A}^T$ defines maps $\varphi_A : TA \to SFA$, one for each object A of $\underline{A}$. If $f : (A,x) \to (B,y)$ in $\underline{A}^T$, it follows that $\Phi(f,x,y) = (Ff, \varphi_A x, \varphi_B y)$. Thus

$$(5.1) \qquad x \le f^* y \Longrightarrow \varphi_A x \le (Ff)^* \varphi_B y ,$$

for $f : A \to B$ in $\underline{A}$ and all $x \in TA$, $y \in TB$. For $f = 1_A$, this implies that φ_A preserves order. For $x = f^* y$, (5.1) becomes

$$(5.2) \qquad \varphi_A f^* y \le (Ff)^* \varphi_B y ,$$

for $f : A \to B$ in $\underline{A}$, and $y \in TB$.

Conversely, let a functor $F : \underline{A} \to \underline{B}$ and order preserving maps $\varphi_A : TA \to SFA$, one for each object A of $\underline{A}$, be given. If (5.2) is always satisfied, then (5.1) is always valid, and $\Phi(f,x,y) = (Ff, \varphi_A x, \varphi_B y)$, for $f : (A,x) \to (B,y)$ in $\underline{A}^T$, defines a fibred functor Φ over F.

We call a functor $\Psi : \underline{B}^S \to \underline{A}^T$ a <u>fibred coadjoint</u> of a functor $\Phi : \underline{A}^T \to \underline{B}^S$ if Φ and Ψ are fibred functors, and Ψ is coadjoint to Φ.

<u>Theorem 5.3.</u> <u>For a fibred functor</u> $\Phi : \underline{A}^T \to \underline{B}^S$ <u>over a functor</u> $F : \underline{A} \to \underline{B}$,

with maps ^ : T A —> S F A , the following statements are logically equivalent.

(i) F has a coadjoint, all maps *(jb* preserve infima. and <p_K f* = (F f)*#L
/A — - j £ / 25

for every map f : A —;> B gX A .

(ii) For every object B of B , a couniveraal morphism /3_ : B —> F G_
for F and a map *if* : *S* B -~>T G_{fi} jof θ, exist such that A : (B,y) —^
ϕ(G_B *(β* y) is a cpuniyeraal morphism for ^D_t for all y £ S B ,

(iii) *(t)* has a coad.joint, and *(f)LO~* — ωF .

(*v) *(T)* has a fibred coadjoint.

Proof. We prove (iv) ==£> (iii) =^> (i) ==^> (ii) =^> (iv), and we use the symbol X —| Y to denote that X is coadjoint to Y •

If M^ — \ *(p* and PS"'= G P , for G s JB —> A , then G « P *^CK*——{
V'(Pu)= F • Thus P ^—*&U1,* and G P — I CJF , so that *0u)* and *to* F
are naturally equivalent. If u : (F A_f *G>IU>M)* — ^ (^ ^>^*VA)* is an equivalence,
then u is isomorphic in B , and, *&.U* ~ ^U A * ^W P A ^s ^ F A * Tllus ^iv^ ==:^> ^iii).

If *Cp* has a coadjoint *y^$* then P^^oc — / F *f* , as above, *and Cp* preserver
limits. For an object A of A and a family $(x_i)_{i \in I}$ of elements of T A ,
the object (A,f)x_i) of A^ is a limit of a diagram with one vertex (A, *CO*) ,
and with arrows 1• : (A_f x.) -—^(A_f cU') . If *0* satisfies (iii), then *0* pre-
serves this situation, and thus *(p* (()x.) s [jfo) x.) • Similarly, a diagram

(A,x)————>(A,6J.)
1_A ⇓ A
(B.y)———1_B———>(B, ω_B)

in j1 , with f in ^(A^) , is a pullback if and only if x = f• y . This is easily verified. If ÇD satisfies (iii), then 0 preserves this pullback situation, and <p f* y ~ (P f) * <p y follows. Thus (iii) =~^ (i).

If (i) is valid, let β_B : B —> F G_B be a couniversal morphism for F •
By 2.1, y/fl y^y¹ <=$> y<(VTc y^f • for a11 y £-^{s B} and y^f£ T G B ,
defines a map Ç£L t S B —> T G_B of 0 . This map \L≥ satisfies

$$\psi_B\, y \leqslant f^* x \iff y \leqslant \beta_B^* \varphi_{G_B} f^* x = \beta_B^* (F f)^* \varphi_A x ,$$

for f : G_B '—;> A in ^f y £ S B , x £ T A »' Thus a bijective correspondence

$$\eta : \underline{A}^T((G_B, \psi_B\, y), (A, x)) \to \underline{B}^U((B,y),(F\ A,\ \wedge\ x))$$

is defined by /ft(t9 Us y, x) « ((F f)/5 t y, ^ x) . It follows immediately that i§ : (B_t y) —^^M^'y^-R^{y^} ^*s a couniversal morphism for (p • Thus (i) s=^> (ii).

Finally, if (ii) is valid, and if we use the conuniversal morphisms f£B and fy^fy> 9*r ^R^{y^} to construct coadjoints G of F and ^ of (t), in the usual way, 'then one obtains ^(g, y_f z) •« (G g, ^ y,]^, z) for any morphism g : (B,y) —>(C, z) of 1^S , so that S^ i^s fibred over G , and (ii) «^ (iv) |

Examples# The functor which assigns to every topological group the **underlying** topological space is fibred, over the forgetful functor F s £ •—> JJ • This functor satisfies (i) of 5.3 and thus has a fibred coadjoint, over the functor G which assigns to every set A the free group G A with A as set of generators*

Fibred functors Ǝu x A. —> A. over the identity functor, with P<f>= P , are of special importance. In this case, we may put β^ » 1.. in 5«3, (ii)_f and then Uy is determined by C/> y ^ y^f <£=> y ^ 01 y¹ • The functor which assigns to every topological space the underlying convergence space is one of many examples.

Theorem 5*3, (iii) raises a question. Is it possible that a fibred functor has a coadjoint, but not a fibred coadjoint? The author has not been able either to prove that this is impossible or to find an example.

6. Isocoreflective subcategories

We consider again a fibred category $\underline{C}^T$. We call a full subcategory $\underline{C}^T[\underline{K}]$ of $\underline{C}^T$ isocoreflective if every object (A,x) of $\underline{C}^T$ has a coreflection $\gamma_{(A,x)}$ in $\underline{C}^T[\underline{K}]$ with $P\gamma_{(A,x)} = 1_A$.

Theorem 6.1. A full subcategory $\underline{C}^T[\underline{K}]$ of $\underline{C}^T$ is isocoreflective if and only if K satisfies the following two conditions.

(i) If all objects (A,x_i), $i \in I$, are in K, for an object A of $\underline{C}$ and a family $(x_i)_{i \in I}$ of elements of TA, then $(A, \bigwedge x_i)$ always is in K.

(ii) For f in $\underline{C}(A,B)$ and (B,y) in K, $(A,f^* y)$ always is in K.

Proof. If K satisfies (i) and (ii), let $\gamma_{(A,x)} = (1_A, x, \chi_A x)$, where $\chi_A x$ is the infimum of all $x' \in TA$ with $x \leq x'$ and $(A,x') \in K$. $(A, \chi_A x)$ is in K by (i). If $f : (A,x) \to (B,y)$ with (B,y) in K, then $(A,f^* y)$ is in K by (ii), and thus $\chi_A x \leq f^* y$. But then

$$(f,x,y) = (f, \chi_A x, y)\gamma_{(A,x)}$$

in $\underline{C}^T$, and $\gamma_{(A,x)}$ is indeed a coreflection in $\underline{C}^T[\underline{K}]$.

Conversely, if $\underline{C}^T[\underline{K}]$ is isocoreflective, let $((A,x_i))_{i \in I}$ be a family of objects in $\underline{K}$. If $(1_A, \bigwedge x_i, x')$ is a coreflection for $(A, \bigwedge x_i)$ in $\underline{C}^T[\underline{K}]$, then $(l_f x^f{}_f x_i)$ is a morphism of $£^1$, and thus $[\,)x_i \wedge x^f \wedge x_i$, for all $i \in I$. But then x^f « $|\,\widehat{j}x_i$, and $(A, j\widehat{)}x_i)$ is in jC • This proves (i)$_f$ and (ii) is easily verified by the same method.J

If K satisfies (i) and (ii), then the $x \in TA$ with (A,x) tfrJC form a complete ordered set $T^f A$ for every object A of £ • If $T^1 f$ is the restriction of f^* to $T^1 B$ and $T^f A$ / for $f \in £(A,B)$, then $T^1 f$ preserves infima,

T^1 T^1 _ Tr i

Thus a fibred category £ is defined, and one sees easily that £ ~ £ |jCJ •
The fibred functor J s £ •—? £ defined by the inclusion maps j : $T^f A \to TA$ is, of course, the inclusion functor, J has a fibred coadjoint by 5.3, (i)«

V/e call a family K of objects of $£^T$ productive if every family of objects in K has a product in jC • If £ has products aid £ ^{T}jc] is coreflective, then K is productive by *3A* and f8J, V.5.1* If $f \in £(A,B)$ and $y \in TB$, then we call the object $(A_f\ f^*\ y)$ of $£^T$ the inverse intake of $(B_f y)$ by f.

Proposition 6«2» If K is productive, and if L is the class of inverse images of objects in JC by maps of £$_f$ then £ ^{T}jj is isocoreflective.

Proof, ^ satisfies condition (ii) of 6,1 trivially; we must verify (i). Let $((A, f_i{}^* y_i))_{i \in I}$ be a family of objects of L, with $f_i \in £(A,B_i)$ and $(B_i, y_i) \in K$ for $i \in I$ • Let $(B_t y)$ in K be a product of the objects $(B_i, y_i)_t$ with projections (p_i, y, y_i) • By 3.4, the maps $p_i : B \to B_i$ are the projections of a product, and $y = \bigcap (p_i{}^* y_i)$. If $f_i = p_i f$ for all $i \in I$, then

$$\bigcap (f_i{}^* y_i) = \bigcap (f^* p_i{}^* y_i) = f^*(\bigcap (p_i{}^* y_i)) = f^* y ,$$

and thus $(A_f \bigcap (f_i{}^* y_i)) = (A, f^* y)$ is in L

Examples. A convergence space (see 4.2) (A,q) is called a limit space [7], [3] if F q x and G q x , for x ∈ A and filters F and G on A , always implies (F ∩ G) q x . We call (A,q) a neighborhood space if (∩ P_i) q x , for x ∈ A and a non-empty family $(P_i)_{i \in I}$ of filters on A , whenever P_i q x for all i ∈ I . If K is the class of all limit spaces or of all neighborhood spaces, then $\mathcal{C}^Q(K)$ is an isocoreflective subcategory of $\mathcal{C}^Q$. Several similar examples, in $\mathcal{C}^Q$ and in other categories, are discussed in [9].

In the category $\mathcal{T}$ of topological spaces, T_i-spaces (i = 0, 1, 2) form productive classes. If this is K in 6.2, then L is the class of R_{i-1}-spaces studied in [2], for i = 1, 2. For i = 0, L consists of all topological spaces.

7. Hereditary and epicoreflective subcategories

We denote by M^e or $M^e(\mathcal{C})$ the class of all extremal monomorphisms of $\mathcal{C}$. We say that $\mathcal{C}$ is M^e-factored if every map f of $\mathcal{C}$ has a factorization f = m e with e epimorphic and m ∈ M^e . We call a class K of objects of $\mathcal{C}^T$ hereditary if, for a morphism m : (A,m* y) → (B,y) in $M^e(\mathcal{C}^T)$, with (B,y) ∈ K (and m ∈ $M^e(\mathcal{C})$) , there always is in K an object (A',x') isomorphic to (A,m* y) . We say that $\mathcal{C}^T[K]$ is epicoreflective if every object (A,x) of $\mathcal{C}^T$ has a coreflection $\gamma_{(A,x)}$ in $\mathcal{C}^T[K]$ with P $\gamma_{(A,x)}$ epimorphic in $\mathcal{C}$.

Proposition 7.1. If $\mathcal{C}$ is M^e-factored and $\mathcal{C}^T[K]$ epicoreflective, then K is hereditary.

Proof. If m : A → B is in M^e and (B,y) in K , let (e,m* y,x') be a coreflection for (A,m* y) in $\underline{C}^T[\underline{C}]$, with e : A —r>A' epimorphic in £ . Then m = f e for a map f of $\underline{C}$, and (f,x',y) is a morphism of £ • If £ is $\underline{\underline{M}}$ -factored, it follows that e is isomorphic in £ and f in M , Now x' = f* y , and e* f* y - m* y ^ e* x' ^ e* f*.y • Since e* is isomorphic, x' » f* y and m* y ~ e* x' follow. Thus e : (A,m* y) —> (A',x') is an isomorphism of $£^T$, with (A',x')£K j

Theorem 7»2« If £ is K-factored, colocally small, and has products, then $£^T[\underline{C}]$ is epicoreflective if and only if K is hereditary and productive*

Proof> 7.1 and the remark preceding 6*2 take care of the "only if" part.

For the "if" part, let (A,x) be an object of $£^T$. Let us call f t (A,x) —>(B,y) a K-quotient if f is epimorphic in £ and (B,y)£$\underline{C}$* If £ is colocally small, then there is a family $(f_i : (A,x)—^(B_i,y_i))_{i \in I}$ of K-quotients such that every K-quotient is of the form $(u^{-1}f_i, x, u^* y_i)$, for some i∈I and an isomorphism u of £ « If £ has products and is K -factored, let $p_i : (B,y)—^(B_i,y_i)$ be the projections of a product in $£^T$, and let f_i » p_i m e for all i £ I , with m£M and e : B —^>B_o epimorphic in $\underline{C}$ • If y_o = m* y_1 , then e : (B,x)—> (B_o,y_o) in $\underline{C}^T$.

If K is productive and hereditary, then we can carry out this construction so that (B_1,y_1) and (B_o,y_o) are in K. If g : (A,x) —^>(C,Z) in $£^T$, with (C,z) in K , then we factor g s $m_1 e_1$ in £ , with m_1 in ^ and e_1 : A — ^ C_1 epimorphic in £ , so that (C_1,z_1)£K for z_1= m_1* Z . Then $(e_1, x, £_1)$ is a K-quotient, and thus $e_1 = u^{-1} f_i$, z_1 » u* y_i , for some i£I and an isomorphism u : C_1 •—> B_i of C • Now g_1 : (B_o,y_o) —$^(C,z) is a morphism of $\mathcal{L}^T$, and $g = g_1 e$ in $\mathcal{L}$, for $g_1 = m^{-1} u p_1 m$. Since e is epimorphic, $g = g_1 e$ determines g_1, and (e,x,y_0) is a coreflection for (A,x) in $\mathcal{L}^T[\underline{K}]$, with $p(e,x,y_0) = e$ epimorphic in $\mathcal{L}$ |

We call a class $\underline{K}$ of objects of $\mathcal{L}^T$ ~~replete~~ if every object of $\mathcal{L}^T$ which is isomorphic to an object in $\underline{K}$ is itself in $\underline{K}$.

<u>Theorem 7.3.</u> <u>If</u> $\mathcal{L}$ <u>has products, then a full subcategory</u> $\mathcal{L}^T[\underline{K}]$ <u>of</u> $\mathcal{L}^T$ <u>is isocoreflective if and only if</u> $\underline{K}$ <u>is productive, hereditary and replete, and all objects</u> $(A, C\emptyset_k)$ <u>of</u> $\underline{C}^T$ <u>are in</u> $\underline{K}$.

<u>Proof</u>. The "only if" part follows directly from 6.1 and the remark preceding 6.2, using condition (i) for the empty family, and (ii) only for $f \in \mathcal{L}^e$.

By 6.2, we must only verify condition (ii) of 6.1 for the "if" part. Thus let $f : A \to B$ in $\mathcal{L}$ and $(B,y) \in \underline{K}$. Let (C,z) be a product of (A, tQ) and (B,y) in $\mathcal{L}^T$, with projections p_1 and p_2. Then $(C,z) \in \underline{K}$. By 5.4, C is a product of A and B in $\mathcal{L}$, and $z = p_2^* y$. Now let $p_1 m = 1$, $p_2 m = f$ in $\mathcal{L}$. Then $m : A \to C$ is in $\mathfrak{A}$, and $f^* y = m^* p_2 y = m^* z$. Since $\underline{K}$ is hereditary and replete, $(A,f^* y) = (A,m^* z)$ is in $\underline{K}$ |

<u>Examples and Remarks</u>. For $\mathcal{L} = \underline{S}$, the category of sets, the hypothesis of 7.2 is satisfied. Regular and completely regular spaces (assumed to be T_1) define epicoreflective subcategories of $\underline{S}^{Tb}$, the category of topological spaces. Other examples are given in the next section, and in [9]. The class of normal spaces is neither hereditary nor productive, and thus does not qualify.

Theorem 7.3 is an improved version of Theorem 2.8 (and Theorem B) of [6], and

7.2 generalizes Theorem C of J6J. The given proof of Theorem A of f6] is valid, with minor adjustments, for any fibred category 23^T over the category of sets*

Many interesting coreflections in general topology are not epicoreflections, and thus not yet covered by our theory* Among these are the various completions and compactifications with couniversal[3] mapping properties.

8« Point separators

We assume in this section that £ is concrete, i.e. equipped with a faithful functor P J £ —> j3, where 23 is the category of sets. If £ = £> f then P will be the identity functor. We call a class K of objects of $£^T$ infective if, for a morphism m : (A,x) •—> (B,y) of $£^T$ with (B,y)£LIC and P m injective, (A,x) always is in K .

We call a fibred functor $Ф$) i $£^T$ —>$£≥^R$ (see 4.5) over F a point separator on $£^T$. If LУ is a point separator, let (J9(A,x) = (P A, $<p_A$ x) for objects (A,x) of £ . We say that (A,x) is (^-separated if $<p'_A$ x = 3_{FA} , and we denote by K(Ф)) the class of all (fi -separated objects of $£^T$ • We call a class K of objects of $£^T$ point-separated if K ss K((D) for some point separator (j)o

Proposition 8.1* If (b is a point separator on £ , then %(0) is injective. If £ has products and P preserves products, then][(©) is productive.

Proof. If m : (A,x) —^ (B,y) in $£^T$, with (B,y)£K(0) and P m injective, then (p x < (p m) * <L» y « (P m) * I « 1 , and (A,x)g K((f)) .

If $((A_i,x_i))_{i \in I}$ is a family of objects in jC(0) and (A,x) their product, with projections p_i { A —> A_i and x = /\(p_i* x_i) , then

?K^X ≤ n%V^x_i) -< ⋂((F p_i)*φ_{A_i} x_i) = ⋂((F p_i)* I_{FA_i}) .

If P preserves products, then the mappings F p_i are the projections of a product, and P|((F p_i) * I_{FA_i}) = Ip_A follows. Thus (A,x)£ K(C))) I

Distinct point separators can have the same class IC(n)) • We compare point separators on £^T by saying that Cp is finer than (f), or Cp ^L 0_f if Cf>*_A x d<p_k* for all objects (A,x) of 'j?.. Clearly K((|)) C. Lift) if C/)^1 f£ 0 • We call a point separator (p coarse if, conversely, jt(0) C! ^(d)^f) always implies Cp ^ (p. A coarse point separator (7) is uniquely determined by jC(<}6) •

Theorem 8.2» jjf K is infective and ^ f^] coreflective, then jC is points separated, and K = JC(<φ>) for a unique coarse point separator (p.

Proof. If r_A : (A,x) —> (c,z) is a coreflection for (A,x) in £^T[K] , we put Cf_A x = (F r_A)• I_FC . Since r_A is determined by (A,x) up to an isomorphic factor, this determines φ_A x uniquely.

If f : (A,x) —>(B,y) in £^T , and if r_A and r_B : (B,y)—^>(C,z^f) are coreflections in £^T[K] » then r_B f = g r_A for a morphism g : (C,z) —$>{o\z^%) . But then (P r_A) * 1^ 4 (P r_A) * (P g) * I_p(,, - (P f) * (P r_fi) 1^, . Thus the maps | _A : I A - ^ R F A satisfy (5.l), and define a fibred functor Φ.

For r_A : (A,x) —^-(C.Z) as above, ^ x = (P r_A) * ^ = ^ only if F; r_A is injective. But then (A,x) ^ K since IC is injective. Conversely, if (A,X)(^K , then r_A is isomorphic, and (P r_A)* I_FC = I_FA . Thus K(0) = K .

The point separator φ just constructed clearly is coarse, and hence uniquely determined by K(0) = K |

One interesting aspect of point separators is their connection with cofibrations. We consider this only for £ = S , the category of sets, and a fibred category j5 # We call a morphism (f,x,y) of S a quotient morphism if f is surjective and $y = f_{\#} x$ (see 3.1)« We say that a morphism f : (A,x) —> (B,y) of S: is K-cofibred if, for every morphism g : (A,x) —> (C,z) of S. with $(C_f z)$(^*jC , there is a unique morphism h t (B,y) —$> (C,z) such that g = hf •

Proposition 8.3» !£ (b is a point separator on $£^T$ and K = jc(^)) , then a quotient morphism f : (A,x) —^ (B,y) jof £ which satisfies (p x s f*^> y is jC-jcofibred. Conversely, jX ^ is coarse, then every K-cofibred morphism f : (A,x) —^>$(B_f y)$ satisfies $\varphi_A x = f^* \varphi_B y$ •

Proof. If $<p_k x = f^*\wedge_{fi} y$, and if g j (A,x) —> (c,z) with $(C_f z)$ <£K_f then $f^* I_B$ ^ f * f_B y — $<f_k$ x ^ S* f> z = g* I_Q in R A • If $(f_f x,y)$ is a quotient morphism, it follows that g = h f for a unique mapping h g and that $h_{\#} y = h_{\#} f_{\#} x$ B $g_{\#}$ x< z in T C , so that h : (B,y) —> $(c_f z)$ in S^T, ,

For the second part, let h : (B,y)—^> (C,z) be the coreflection for $(B_t y)$ in S^TfK] . This exists by 8.1 and 7.2. If f : (A,X) —>(B,y) is K-cofibred, then h f : (A,x)—>(c,z) clearly is a coreflection in S^TTK] . But then $<p_A$ x * (h f)* I_C = f* h* I_C = f* G_B y by the construction of (f> in the Voof of 8.2 |

Examples and remarks. We may call a point separator (T) strict if all values $(fXk_f x)$ are equivalence relations. For every point separator LP , there is a finest strict point separator $(p_1$ coarser than (f> , and $JC(0_1)$ =jf(0) • Every coarse point separator is strict. Strict point separators for topological spaces have been studied in fioj. 8*1 and the first part of 8.3 generalize results of flOJ,

We define tv/o point separators on $\underline{S}$ (4»3), by stating when (x,y)(£ÇP$_A$ XT for points x, y of a topological space (A, T) •

T_o • (x,y)<£i $_A$ XT if x and y have the same T-neighborhoods.

T_1 • (x,y) £r<?∆~ if x is in the -closure of £y} .

The first example defines a coarse point separator. C_p-separated, spaces are T_o-spaces and T_1-spaces respectively. Other examples may be found in flOJ.

For a fibred category $\underline{S}^T$ over the category of sets, a class $\underline{K}$ of objects is point-separated if and only if $\underline{K}$ is injective and productive, by 7.2, 8.1 and 8.2, and then & fjKJ is epicoreflective. One example for $\underline{j}S^P$ (for which we do not have a convenient point separator) is the class of all. topological spaces in which every compact set is closed. Many other examples, and references to yet more point separation axioms, may be found in fl].

The classes of regular spaces and of completely regular spaces are not point-separated, since these classes are not injective (see fioj).

Point separators, and the resulting epicoreflective subcategories, are of interest not only for topological spaces, but also e.g. for limit spaces, uniform limit spaces, and Cauchy spaces. See fs>]#

9. A correction

We append to the present report a correction to [11]. We shall use the notations of [11] and of the present paper.

The uniform limit structures on a set E form a complete ordered set U E, with $\mathcal{J}_1 \leq \mathcal{J}_2$ if $\mathcal{J}_1$ is finer than $\mathcal{J}_2$, i.e. $\mathcal{J}_1 \subset \mathcal{J}_2$. For a mapping f : E → F and a uniform limit structure $\mathcal{J}$ on F, let $f^{-1}\mathcal{J}$ be the uniform limit structure on E consisting of all filters ϕ on E × E such that $(f \times f)_* \phi \in \mathcal{J}$. This results in a fibred category $\underline{S}^U$, the category of uniform limit spaces, with uniformly continuous mappings as morphisms.

For a uniform limit space (E, $\mathcal{J}$), let $\Delta_{\mathcal{J}}$ be the union of all filters $\dot{x} \times \dot{y}$ in $\mathcal{J}$, for points x, y of E. This is a principal filter on E × E, and (E, $\mathcal{J}$) is separated if and only if $\Delta_{\mathcal{J}} = \Delta$. We call (E, $\mathcal{J}$) saturated if $\dot{\Delta}_{\mathcal{J}}$ is in $\mathcal{J}$. Using 6.1, one verifies easily that $\underline{S}^U[\underline{K}]$ is isocoreflective if $\underline{K}$ is the class of all saturated uniform limit spaces.

Now comes the error in [11]. In order to construct the completion of (E, $\mathcal{J}$), the construction sketched in [11] should be applied, not to (E, $\mathcal{J}$), but to the saturated space (E, $\sigma\mathcal{J}$), where (1_E: (E, $\sigma\mathcal{J}$) → (E, $\mathcal{J}$)) is the coreflection for (E, $\mathcal{J}$) in the full subcategory $\underline{S}^U[\underline{K}]$ of saturated uniform limit spaces.

This correction does not affect the proof of Theorem 2 of [11] in any way, since Theorem 2 of [11] is concerned with a uniform space, or principal uniform limit space, (E, $\mathcal{J}$), and every principal uniform limit space is saturated.

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