Representation of Additive
and Biadditive Functionals

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This paper is concerned with obtaining integral representations of a class of nonlinear functionals on function spaces of measurable functions. These functionals are known as additive functionals and their representation has been studied in recent papers of Martin and Mizel [1], Chacon and Friedman [2] and Friedman and Katz [3]. The class of additive functionals studied in this paper is the same as in [1] and has been found to be useful in the theory of fading memory in Continuum mechanics, (Coleman and Mizel [4]). Such functionals also occur in the functional analytic study of ordinary differential equations. These and other applications will be dealt with elsewhere

Apart from these applications the representation theorems obtained here are of intrinsic interest and provide generalizations of results established in Halmos [5] and Bartle and Joichi [7] concerning certain nonlinear operators on function spaces.

In this paper we propose to make a systematic study of the representation of additive functionals under varied continuity constraints. In addition since the applications to fading memory and nonlinear differential equations often require functionals of several variables, we define multiadditive functionals and study their representation. We mention in this connection that bilinear functionals over the Cartesian products of some important Banach spaces have been studied by Morse and Transue [9] and others.

After stating below the basic definitions and notations, we analyze

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in Sections 1 and 2 the representation of additive functionals over subspaces of the space of measurable functions on finite nonatomic and cx-finite nonatomic measure spaces. In Sections 3 and 4 these results are generalized to the case of multiadditive functionals.

Throughout the paper (T^I^/i) with or without a suffix i is a nonatomic measure space with E a cx-algebra of subsets of the set T and pi a non-zero measure. M denotes the vector space of real valued measurable functions on (T^S^/x) with the usual identification of two functions which are equal a.e,t R denotes the real line and **and** R^n the n-dimensional space.

Definition 1. ££ L IJE^ a subspace of M then a^ function F cm L is called an additive functional if (1) P(f + g) = F(f) + F(g) whenever f, g are two functions in L such that f(x)g(x) = 0 a.e.j and (2) F(f) = F(g) if. f,g are eauimeasurable functions, le.jLf for every Borel set B in R /i(f $^{-1}$(B)) = fi(g~ ^{1}(B)).

Functionals that satisfy condition (2) are called statistical in [1].

Definition 2. Let $(T_i, £_i, {}_3p_i)$ i = 1,2 be two measure spaces and let M. be the space of measurable functions on T. . If L. is a vector subspace of M_i for i = 1,2 then ci functional F on L^X L_2 is said to be biadditive provided F(*,,y) and Fix,*) are additive for every function yeL_o and XGL.. . More generally one defines n-additive functionals for n > 2.

Before proceeding to the representation theorem we restate for convenience of reference a useful theorem established in [1].

Theorem 1. Let (T.L,u) be a finite nonatomic measure space such that M(T) ^ 0. Let p be an additive functional on L (n) which is continuous in the sense that whenever $\{x_n\}_{n\geq 1}$ in $L_\infty(\mu)$ is a sequence converging boundedly a.e., to the function x in $L_\infty(\mu)$ then $F(x_n) \to F(x)$. Then there exists a unique continuous function $f:R\to R$ such that $f(0)=0$ and for all $x \in L_\infty(\mu)$

$$F(x) = \int_T f(x)\,d\mu.$$

1. Representation of Additive functionals in the Case of Finite Nonatomic Measure Space.

The theorems established in this section are similar to Theorem 1 except that different continuity conditions are imposed on the additive functional. We state first a proposition which is a generalization of the above theorem and states that the above theorem is true even if $x_n \to x$ boundedly in measure.

Proposition 1. Let (T,Σ,μ) be as in Theorem 1 and let F be an additive functional on $L_\infty(\mu)$. Then $F(x_n) \to F(x)$ whenever the sequence of functions x_n in $L_\infty(\mu)$ converges boundedly in measure to the function x if and only if F admits an integral representation of the form $F(x) = \int_T f(x)d\mu$ for some continuous function on $R \to R$ such that $f(o) = 0$. Such a representing function f is unique.

Proof: Since a.e. convergence on a finite measure space implies convergence in measure it follows from Theorem 1 that F admits a unique integral representation satisfying the conditions in Theorem 1. Conversely suppose f is a continuous function on $R\to R$ such that $f(o) = 0$. Let F be defined on $L_\infty(\mu)$ by the formula $F(x) = \int_T f(x)d\mu$.

Since the additivity of F is obvious it suffices to show that $x_n \to x$ boundedly in measure implies $F(x_n) \to F(x)$. Since f is continuous and x_n, x are totally measurable functions it follows by

Theorem II.2.12 in Dunford and Schwartz [6] that $f(x_n) \to f(x)$ boundedly in measure. Thus $f(x_n)$ and $f(x)$ are in $L_1(\mu)$ and $F(x_n) \to F(x)$.

Next we proceed to the case of not necessarily bounded a.e. convergence and convergence in measure.

Theorem 2. Suppose (T, Σ, μ) is as in Theorem 1 and let $F: M \to R$ be an additive functional on M. Then $F(x_n) \to F(x)$ whenever x_n is a sequence in M such that $x_n \to x$ in M a.e. if and only if there exists a continuous function f on $R \to R$ such that $f(0) = 0$ and range f is a bounded set in R, for which $F(x) = \int_T f(x)\,d\mu$ for all $x \in M$. Such a representation is unique.

Proof: Let f be a function having the properties stated in the theorem. Consider the functional $F: M \to R$ defined by $F(x) = \int_T f(x)\,d\mu$. Since range f is bounded and f is continuous the functional F is well defined and it is clear that F is additive. Now consider a sequence $\{x_n\}$ in M which converges to the measurable function x a.e. Since f is continuous $f(x_n) \to f(x)$ a.e. Since range f is bounded it follows by the theorem on dominated convergence that $F(x_n) \to F(x)$. Conversely suppose that F is an additive functional on M satisfying the given continuity condition. Then clearly $F_1 = F|L_\infty$ verifies the hypothesis in Theorem 1. Thus there exists a unique continuous function f on R into R such that $f(0) = 0$ and for all x in $L_\infty(\mu)$, $F(x) = \int_T f(x)\,d\mu$. Suppose that range f is unbounded. Then there exists a sequence of reals r_n such that $|r_n| \to \infty$ and $1 \le |f(r_n)| \to \infty$. Since the measure space is nonatomic there exists a decreasing sequence of measurable sets E_n such that $\mu(E_n) = \frac{\mu(T)}{|f(r_n)|}$. Let $x_n = r_n \chi_{E_n}$. Clearly $x_n \in M$ and $x_n \to 0$ a.e. However since $x_n \in L_\infty(\mu)$, $F(x_n) = \int_T f(x_n)\,d\mu = \pm \mu(T)$ is a contradiction. Thus range f is bounded.

Next let x be a nonnegative function in M. There exists a sequence s_n of simple functions such that $s_n \uparrow x$ a.e. Since f is continuous $f(s_n) \to f(x)$ boundedly a.e. Thus $f(x) \in L_1(\mu)$ and $\int_T f(x)d\mu = \lim \int_T f(s_n)d\mu = \lim F(s_n) = F(x)$. Thus the representation of F is valid.

The uniqueness of F is clear by applying Theorem 1 to the functional F_1.

As a corollary we obtain the following representation theorem for additive functionals on the topological vector space M with the topology of convergence in measure.

Corollary. F is an additive functional on M for which $F(x_n) \to F(x)$ whenever $x_n \to x$ in measure if and only if there exists a continuous function $f: R \to R$ with range f bounded and $f(0) = 0$ such that for all $x \in M$, $F(x) = \int_T f(x)d\mu$. Such a representation is unique.

The proof is similar to that of Proposition 1 and will thus be omitted.

Remark 1. Theorem 2 is valid even if the continuity condition is replaced by: $x_n \to x$ almost uniformly $\Rightarrow F(x_n) \to F(x)$. The existence of a representation for this case follows from the proof of Theorem 1 (see [1]). The necessity and sufficiency of boundedness of range f follow respectively from arguments in Theorem 2 and the observation that almost uniform convergence implies convergence in measure.

Next we turn our attention to the vector subspaces $L_p(\mu)$ $(1 \le p < \infty)$ of the vector space M. We equip the vector spaces $L_p(\mu)$ with the usual L_p-norm.

Theorem 3. If (T, Σ, μ) is a finite nonatomic measure space and if F is an additive functional on $L_p(\mu)$ then $F(x_n) \to F(x)$ whenever

$x_n \to x$ a.e. if and only if there exists a continuous function $f: R \to R$ such that (1) $f(0) = 0$, (2) range f is a bounded set in R and, (3) for every $x \in L_p(\mu)$, $F(x) = \int_T f(x)d\mu$. Such a representation of F is unique.

Proof: Let F be an additive functional on $L_p(\mu)$ satisfying the convergence condition in the theorem. Considering the additive functional $F_1 = F|L_\infty(\mu)$ and noting that $F_1(x_n) \to F_1(x)$ if a sequence $x_n \in L_\infty(\mu)$ converges boundedly a.e. to $x \in L_\infty(\mu)$ we see by Theorem 1 that there exists a unique continuous function $f: R \to R$ with $f(0) = 0$ and such that for all $x \in L_\infty(\mu)$, $F_1(x) = \int_T f(x)d\mu$. Now it is claimed that f must satisfy condition (2). Suppose that f does not satisfy condition (2). Then there exists a sequence $\{r_n\}$ in R such that $|r_n| \to \infty$ and $1 \le |f(r_n)| \uparrow \infty$. Let $\{B_n\}$ be a decreasing sequence of measurable sets in T such that $\mu(B_n) = \frac{1}{|f(r_n)|}\mu(T)$. Since $\mu(B_n) \to 0$ the sequence of functions $r_n \chi_{B_n} \to 0$ a.e. Clearly $r_n \chi_{B_n} \in L_p(\mu)$ and $F(r_n \chi_{B_n}) = \int_T f(r_n \chi_{B_n})d\mu = \pm \mu(T)$. Thus f has a bounded range.

Since the verification of the remaining assertion is routine the proof of the theorem is complete.

Corollary. The above theorem is true even if convergence a.e. is replaced by convergence in measure.

The proof of this is similar to that for the Corollary of Theorem 2.

Theorem 4. If F is an additive functional on $L_p(\mu)$ where μ is nonatomic and $0 < \mu(T) < \infty$ then F is continuous on the Banach space $L_p(\mu)$ if and only if there exists a continuous function $f: R \to R$ such that (1) $f(0) = 0$ (2) $\overline{\lim}_{|t| \to \infty} \frac{|f(t)|}{|t|^p} < \infty$ and (3) for $x \in L_p(\mu)$, $F(x) = \int_T f(x)d\mu$.

Proof: Let F be a continuous additive functional on the Banach space $L_p(\mu)$. Passing on to the restriction $F_1 = F|L_\infty(\mu)$ and appealing to Theorem 1 it is at once verified that there exists a unique continuous function $f: R \to R$ such that $f(0) = 0$ and for all $x \in L_\infty(\mu)$, $F(x) = F_1(x) = \int_T f(x)d\mu$. We claim that f satisfies condition (2). For if not there exists a sequence $\{r_n\}$ in R such that $1 \le |f(r_n)| = n|r_n|^p$. Let $\{E_n\}$ be a sequence of measurable sets in T such that $\mu(E_n) = \frac{1}{n|f(r_n)|}\mu(T)$. Since $\|r_n \chi_{E_n}\|_p^p \le \frac{\mu(T)}{n} \to 0$, $r_n \chi_{E_n} \to 0$ in L_p-norm. However since $F(r_n \chi_{E_n}) = \int_T f(r_n \chi_{E_n})d\mu = \mu(T) \neq 0$, $F(r_n \chi_{E_n}) \not\to 0$ contradicting the continuity of F. Hence f satisfies condition (2) of the theorem.

We proceed next to verify that the function f represents F as in (3) of the theorem. We note that if $x \in L_p(\mu)$ then $f(x) \in L_1(\mu)$. For condition (2) implies there exist constants c and k such that $|t| \ge c$ implies $|f(t)| \le k|t|^p$. Thus if $E_1 = \{t \mid |x(t)| < c\}$ and $E_2 = T \sim E_1$ then $|f(x)|$ is bounded on E_1 and $|f(x\chi_{E_2})| \le k|x\chi_{E_2}|^p$. Since $x \in L_p(\mu)$ implies $|x|^p \in L_1(\mu)$ it follows that $f(x) \in L_1(\mu)$.

Now let $x \in L_p(\mu)$. Since $L_\infty(\mu)$ is a dense subset of the Banach space $L_p(\mu)$ there exists a sequence $x_n \in L_\infty(\mu)$ such that $\|x_n - x\|_p \to 0$. As already observed in the first paragraph of the proof, $F(x_n) = \int_T f(x_n)d\mu$ and since F is continuous $F(x) = \lim F(x_n) = \lim \int_T f(x_n)d\mu$. Thus it is sufficient to show that $f(x_n) \to f(x)$ in the space $L_1(\mu)$. This will be accomplished by applying Vitali's convergence theorem for a statement of which we refer to Theorem 7.13, Bartle [8]. First note that according to that theorem the condition $\|x_n - x\|_p \to 0$ is equivalent on the finite measure space (T, Σ, μ) to the assertions (i) $x_n \to x$ in measure, (ii) for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if $E \in \Sigma$ and $\mu(E) < \delta(\epsilon)$, then $\int_E |x_n|^p d\mu < \epsilon$

for all $n \geq 1$.

Now by the continuity of f we deduce from (i) the assertion (i') $f(x_n) \to f(x)$ in measure. Moreover since f satisfies condition (2), there exist constants, c and K such that $|f(t)| \leq K|t|^p$ whenever $|t| > c$. Let $K_1 = \sup_{|t| \leq c} |f(t)|$. Then we deduce from (ii) the assertion (ii') for each $\epsilon > 0$ there is a $\delta'(\epsilon) > 0$ such that if $\mu(E) < \delta'(\epsilon)$ then $\int_E |f(x_n)| d\mu < \epsilon$ for all $n \geq 1$. This statement is clearly valid if we define $\delta'(\epsilon) = \min\{\delta(\epsilon/2K), \frac{\epsilon}{2K_1}\}$

Finally we note that by the case $p = 1$ of Vitali's theorem, (i') and (ii') suffice on the finite measure space (T, Σ, μ) to imply that $f(x_n) \to f(x)$ in $L_1(\mu)$. Thus we obtain

$$F(x) = \lim F(x_n) = \lim \int_T f(x_n) d\mu = \int_T f(x) d\mu,$$

which is the desired representation.

Conversely if f is a continuous function on $R \to R$ satisfying conditions (1) and (2) then the functional $F(x) = \int_T f(x) d\mu$ is well defined on $L_p(\mu)$ and is clearly additive. By arguing as in the preceding paragraph it is seen that F is continuous as well, and the proof is complete.

2. Representation of Additive Functionals in the σ-finite Nonatomic Case.

We proceed next to the case in which (T, Σ, μ) is a σ-finite nonatomic measure space with $\mu(T) = \infty$.

Remark 2. We note that there exist no nontrivial additive functionals on M. For if $X_A \in M$ is such that $0 < \mu(X_A) < \infty$ we can find a sequence $\{X_{A_i}\}_{i \geq 1}$ of pariwise disjoint measurable sets such that $\mu(X_{A_i}) = \mu(X_A)$. If F is any additive functional on M then because of condition (2) in Definition 1 for any constant K, $F(K X_{A_i})$ is independent of i. Thus if $F(K\chi_{v_i}) \neq 0$ then $(F(K\chi_{\bigcup_{i\geq 1} v_i})$ is infinite. Since F is real valued function this is a contradiction. Thus $F(S) = 0$ for simple functions S. With any reasonable continuity condition the last equation in turn implies $F(x) = 0$ for all measurable functions x. The same argument applies to functions $x \in L_\infty(\mu)$. For this reason we confine our attention to the case of additive functionals on $L_p(\mu)$, $1 \leq p < \infty$.

Theorem 5. Let F be an additive functional on $L_p(\mu)$. Then $F(x_n) \to F(x)$ whenever the sequence $\{x_n\}$ in $L_p(\mu)$ is such that $x_n \to x$ a.e. for some function $x \in L_p(\mu)$ if and only if there exists a continuous function $f: R \to R$ such that (1) $f(0) = 0$ and range f is a bounded set in R, (2) for some constant k and for all real numbers $|f(r)| \leq k|r|^p$, i.e. $\lim \frac{|f(r)|}{|r|^p} < \infty$ in view of condition (1), and (3) for all $x \in L_p(\mu)$, $F(x) = \int_T f(x)d\mu$. Such representation of the functional F is unique.

Proof: Let f be a real valued continuous function on R satisfying conditions (1) and (2) in the above statement. Since f satisfies condition (2) it follows by Theorem 1 in Halmos [5] that $x \in L_p(\mu)$ implies $f(x) \in L_1(\mu)$. Since f is continuous $f(x_n) \to f(x)$ a.e. whenever $x_n \to x$ a.e. Thus since range f is bounded it follows by Lebesgue's theorem on dominated convergence that $\int_T f(x_n) d\mu \to \int_T f(x) d\mu$ whenever $x_n, x \in L_p(\mu)$ and $x_n \to x$ a.e. Hence if F is the functional defined by $F(x) = \int_T f(x) d\mu$ then F has the convergence property in the theorem and further it is verified that F is additive on $L_p(\mu)$. Thus the proof of the if part is complete.

Conversely, suppose F is an additive functional on $L_p(\mu)$ satisfying the continuity condition in the theorem. If B is any measurable set in T with $0 < \mu(B) < \infty$ consider the space

$L_p(T,\Sigma,\mu_B)$ where μ_B is the contraction of μ to B. Let us define a functional F_B on $L_p(\mu_B)$ by setting $F_B(y) = F(y')$ where y' is the function in $L^p(\mu)$ such that $y'|B = y|B$ and $y'|T\sim B = 0$. It is easily seen that F_B is a well defined functional on $L_p(\mu_B)$, so that as a consequence of Theorem 3 the additive functional F_B admits a unique integral representation $F_B(y) = \int_B f(y)d\mu$ where f is a continuous function on $R \to R$ such that $f(0) = 0$ and f has bounded range. Further we note that the function f determined by F^B is independent of B, for if C is another set in Σ such that $0 < \mu(C) \le \mu(B)$ then by the nonatomicity of the measure space there exists a measurable set $B_1 \subset B$ such that $\mu(B_1) = \mu(C)$. Now since for any real number r, $r\chi_{B_1}$ and $r\chi_C$ are equimeasurable, $F(r\chi_{B_1}) = F(r\chi_C)$. Thus if f,g are the functions determined by F_B and F_C then $F(r\chi_{B_1}) = F(r\chi_C)$ i.e. $F_B(r\chi_{B_1}) = F_C(r\chi_C)$. Thus if f,g are functions representing F_B and F^C the preceding equation implies $f(r)\mu(B_1) = g(r)\mu(C)$. Since $\mu(B_1) = \mu(C)$ it follows that f = g.

With f chosen as above let us consider any simple function S. Denoting the support of S by N(S) it follows that $F(S) = F_{N(S)}(S) = \int f(S)d\mu$. Next we verify that f satisfies condition (2). For if not there exists a sequence of real numbers $r_n \to 0$ such that $|f(r_n)| \ge n|r_n|^p$. Let $\{B_n\}$ be a sequence of measurable sets such that $\mu(B_n) = \frac{1}{|f(r_n)|}$. Then the sequence of functions $\{r_n\chi_{B_n}\}$ are in $L_p(\mu)$ and $\|r_n\chi_{B_n}\|_{L_p} \to 0$. Thus $F(r_n\chi_{B_n}) \to 0$. However $F(r_n\chi_{B_n}) = \int_T f(r_n\chi_{B_n})d\mu = \pm 1$ which is a contradiction. Thus f satisfies the condition (2).

Now let us consider an arbitrary function $x \in L_p(\mu)$ and let x_P and x_N be its positive and negative parts. Since F is additive $F(x) = F(x_P) + F(x_N)$. Let $\{T_n\}$ be an increasing sequence of sets in Σ

such that $0 < \mu(T_n) < \infty$ and $T = \bigcup_{n=1}^{\infty} T_n$. Clearly $\{x_p \chi_{T_n}\}$ and $\{x_N \chi_{T_n}\}$ are sequences in $L_p(\mu)$ such that $x_p \chi_{T_n} \to x_p$ a.e. Hence $F(x_p) = \lim_{n\to\infty} F(x_p \chi_{T_n}) = \lim_{n\to\infty} \int_T f(x_p \chi_{T_n}) d\mu$. Since f satisfies condition (2), $f(x) \in L_1(\mu)$ if $x \in L_p(\mu)$. In particular $f(x_p) \in L_1(\mu)$ and since $|f(x_p \chi_{T_n})| \le |f(x_p)|$ and $f(x_p \chi_{T_n}) \to f(x_p)$ a.e. we have by the dominated convergence theorem $F(x_p) = \lim_{n\to\infty} F(x_p \chi_{T_n}) = \int_T f(x_p) d\mu$. A similar argument verifies the equation $F(x_N) = \int_T f(x_N) d\mu$. Thus $F(x) = \int_T f(x) d\mu$ for all $x \in L_p(\mu)$. The proof is complete.

We remark that Theorem 5 is valid even if the continuity condition is replaced by $F(x_n) \to F(x)$ whenever the sequence $\{x_n\}$ in $L_p(\mu)$ converges in measure to x in $L_p(\mu)$. The proof is very similar in details to that of the preceding theorem except that instead of appealing to Theorem 3 one appeals to the corollary following Theorem 3.

Theorem 6. Let F be an additive functional on $L_p(\mu)$ for some p, $1 \le p < \infty$. Then F is continuous on the Banach space $L_p(\mu)$ if and only if F admits the following integral representation. For all $x \in L_p(\mu)$, $F(x) = \int_T f(x) d\mu$ where f is a continuous function on $R \to R$ such that (1) $f(0) = 0$, (2) $|f(r)| \le k|r|^p$ for all real numbers r and for some constant k.

The proof is very similar in details to that of Theorem 5 except that instead of applying Theorem 3 we need to apply Theorem 4.

Next we proceed to the representation of multiadditive functionals. We confine our attention to the case of biadditive functionals since the passage to M-additive functionals for $M > 2$ is a straightforward generalization of the biadditive situation.

3. Representation of Biadditive Functionals in the Finite Nonatomic Case.

The measure spaces (T^ 2^, M_i) i = 1,2 in this section are finite nonatomic and 0 < $M_i(T_i)$ < oo . The product measure associated with these measure spaces is denoted by Mj, d$ M_2 • To facilitate the presentation we adhere to the following notation in the rest of the paper.

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Let <p be a function on R -> R. We define the following properties of (p.

(z) <p(a,O) = <p(O,b) = 0 for all real numbers a,b.

(BB) <p is bounded on bounded sets in R^2.

(BS) <p is bounded on finite strips Sc = {(a,b)j|a| < c},

Sd = { (a,b) | |b|-< d) ; sup|<p(a,b) | on these strips will be denoted by a(c) and ,8(d) respectively.

~~Remark~~ 3^. Suppose that <p is separately continuous, i.e. <p(a, •) and <p(*,b) are continuous for each a,beR. If for i = 1,2 f_1 are measurable functions on the measure spaces (T1, L.1,/i1) then φ(f_1, f_2) is measurable on the product of these measure spaces. For if E ^ L T consider the function *(pic, X^ , f₀)* = <p(0,f_o) + *((pic Y^ , f_n)* - <p(O,f_o)) = cp(0,f_o) + ^(c_n ^ , f j . It is easily verified that this function is measurable on the product space. Moreover using the fact that <p(O,f_2) =0 it is easily verified that *<p(f^, f_2)* is measurable whenever f_1 is a simple function. Finally since every measurable function on (T_1,£,/i_1) is the a.e. limit of a sequence of simple functions it follows from the continuity of <p in its first argument that the assertion holds for cp(f_1, f_2), In the proofs to follow we shall assume this fact without special mention.

Theorem Z' Let B_i> i = 1,2 ^£ the vector spaces £f essentially bounded measurable functions on the finite nonatomic measure spaces $(T_i, \mathcal{L}_i, \mu_i)$. Let N be a biadditive functional on $B_1 \times B_2$ which is continuous in the following sense. If $\{x_1^n\}$ is a sequence in B_1 such that $x_1^n \to x_1 \in B_1$ boundedly a.e. then $N(x_1^n, x_2) \to N(x_1, x_2)$ for each $x_2 \in B_2$. Likewise if $\{x_2^n\}$ is a sequence in B_2 such that $x_2^n \to x_2 \in B_2$ boundedly a.e., then $N(x_1, x_2^n) \to N(x_1, x_2)$ for each $x_1 \in B_1$.

For each such N there exists a unique function $\varphi: R^2 \to R$ which is separately continuous, satisfies (z) and (BB), and which for each $(x_1, x_2) \in B_1 \times B_2$ satisfies

$$(*) \quad N(x_1, x_2) = \int_{T_1 \times T_2} \varphi(x_1, x_2)\, d\mu_1 \otimes \mu_2 .$$

Conversely if φ is a separately continuous function satisfying conditions (z) and (BB) then the functional $N(x_1, x_2) = \int_{T_1 \times T_2} \varphi(x_1, x_2)\, d\mu_1 \otimes \mu_2$ is biadditive and continuous in the sense mentioned above.

Proof: Let $x_1 \in B_1$. Then by the biadditivity of N the functional $N(x_1, \cdot)$ is additive and satisfies the hypotheses of Theorem 1. Thus there exists a unique continuous function $f_{x_1}: R \to R$, $f_{x_1}(0) = 0$, such that

$$N(x_1, x_2) = \int_{T_2} f_{x_1}(x_2)\, d\mu_2 \qquad \text{for all } x_2 \in B_2 . \qquad (1)$$

Define $\bar{\varphi}: R^2 \to R$ by

$$\bar{\varphi}(c, d) = f_{c1}(d) \qquad \text{where } 1 = \chi_{T_1} . \qquad (2)$$

Now the biadditivity and the continuity property of N clearly imply that $\bar{\varphi}$ is separately continuous and has the property (z). We proceed next to establish that for every measurable set E_1 of T_1

$$f_{c\chi_{E_1}}(\cdot) = \frac{\mu_1(E_1)}{\mu_1(T_1)}\, \bar{\varphi}(c, \cdot) \qquad (3)$$

Observe that by the biadditivity of N we have for a fixed $x_2 \in B_2$ and for each simple function $x_1 = \sum_{i=1}^{k} c_i \chi_{E_i}$, the E_i denoting disjoint measurable sets in T_1:

$$N\left(\sum_{i=1}^{k} c_i \chi_{E_i}, x_2\right) = \sum_{i=1}^{k} N(c_i \chi_{E_i}, x_2) \tag{4}$$

In particular if $c_i = c_1$ and $\mu_1(E_i) = \mu_1(E_1)$ then we obtain that

$$N\left(c_1 \chi_{\bigcup_{i=1}^{k} E_i}, x_2\right) = kN(c_1 \chi_{E_1}, x_2) \tag{5}$$

Hence if $\{E_i\}_{i=1}^{k}$ is a partition of T_1

$$N(c_1 \chi_{E_1}, x_2) = \frac{1}{k} N(c_1 1_1, x_2) = \frac{\mu_1(E_1)}{\mu_1(T_1)} N(c_1 1_1, x_2). \tag{6}$$

In particular for x_1 chosen to be a constant function we deduce from (6) that (3) holds whenever $\mu_1(T_1)$ is an integral multiple of $\mu_1(E_1)$. By applying additivity once again we deduce from (6) that (3) is also verified whenever $\frac{\mu_1(E_1)}{\mu_1(T_1)}$ is a rational number. Finally, by using the continuity property of N it is verified that (3) holds in general. With (3) thus verified we can rewrite (4) as follows,

$$\begin{aligned} N\left(\sum_{i=1}^{k} c_i \chi_{E_i}, x_2\right) &= \sum_{i=1}^{k} \frac{\mu_1(E_i)}{\mu_1(T_1)} N(c_i 1_1, x_2) \\ &= \sum_{i=1}^{k} \frac{\mu_1(E_i)}{\mu_1(T_1)} \int_{T_2} \bar{\varphi}(c_i, x_2)\, d\mu_2 \\ &= \frac{1}{\mu_1(T_1)} \int_{T_1 \times T_2} \bar{\varphi}\left(\sum_{i=1}^{k} c_i \chi_{E_i}, x_2\right) d\mu_1 \otimes \mu_2 . \end{aligned} \tag{7}$$

If now the function φ is defined by setting

$$\varphi = \frac{1}{\mu_1(T_1)} \bar{\varphi}$$

we see that (*) is established in those cases in which $x_1 \in B_1$ is a simple function.

Assuming for a moment that φ has the property (BB) let us show that (*) holds in general. Let $x_1 \in B_1$. Thus there exists a sequence of simple functions $x_1^n \in B_1$ such that x_1 is the bounded a.e. limit of x_1^n. We have by the biadditivity and the results in the preceeding paragraph that

$$N(x_1,x_2) = \lim_{n\to\infty} N(x_1^n,x_2) = \lim_{n\to\infty} \int_{T_1\times T_2} \varphi(x_1^n,x_2)\,d\mu_1\,d\mu_2 \tag{9}$$

for every $x_2 \in B_2$. Let us consider the functions $h_n(s_1,s_2) = \varphi(x_1^n(s_1),x_2(s_2))$ $s_1\in T_1$, $s_2\in T_2$. It follows by the separate continuity of φ that the $h_n(s_1,s_2)$ converge pointwise to $h(s_1,s_2) = \varphi(x_1(s_1),x_2(s_2))$ outside a set of the form $(N_1\times T_2)\cup(T_1\times N_2)$ where the N_i are null sets in T_i. From the property (BB) of φ we conclude that $h_n(s_1,s_2) \to h(s_1,s_2)$ boundedly outside a set $(N_1\times T_2)\cup(T_1\times N_2)$. Thus by Lebesgue's dominated convergence theorem it follows from (9) that

$$N(x_1,x_2) = \int_{T_1\times T_2} \varphi(x_1,x_2)\,d\mu_1\,d\mu_2 \tag{10}$$

for all $x_1\in B_1$ and $x_2\in B_2$, establishing (*).

Next we proceed to show that φ has the property (BB). Assuming that this is false there exists a rectangle $Q = \{(c,d) \mid |c|\le K_1, |d|\le K_2\}$ such that $\varphi(c,d)$ is unbounded on Q. Since φ is separately continuous we note that for fixed c^*, d^*, $k_{c^*} = \max_{|d|\le K_2} |\varphi(c^*,d)|$ and $l_{d^*} = \max_{|c|\le K_1} |\varphi(c,d^*)|$ are well defined. However by assumption both $A_1 = \{k_c \mid |c| \le K_1\}$ and $A_2 = \{l_d \mid |d| \le K_2\}$ are unbounded.

Let $\{\theta_j\}$ be a sequence of positive numbers such that $\sum_{j\ge 1}\theta_j = 1$ and $\sum_{j\ge n+1}\theta_j \le \frac{1}{2}\theta_n$. We now choose inductively a sequence of points $\{(c_j,d_j)\}_{j\ge 1}$ in Q as follows. Start by selecting c_1 so that

$k_{c_1} > 4\theta_1^{-1}$, and then taking d_1 to be such that $|\varphi(c_1,d_1)| = k_{c_1}$.
In general having chosen (c_i, d_i) $1 \le i \le n-1$, choose c_n so that

$$k_{c_n} \ge 2\theta_n^{-1}\sum_{i=1}^{n-1} \ell_{d_i}\theta_i + 2^{n+1}\theta_n^{-1}\alpha_n$$

where

$$\alpha_n \ge 3/2 \sum_{i=1}^{n-1} k_{c_i} 2^{-i}\theta_i + \sum_{i=1}^{n-1} 2^{-i} \sum_{j=1}^{i-1} \ell_{d_j}\theta_j + n$$

and then take d_n to be such that $|\varphi(c_n, d_n)| = k_{c_n}$. Let $\{E_i\}_{i\ge 1}$ and $\{F_j\}_{j\ge 1}$ be sequences of disjoint measurable sets in T_1 and T_2 respectively such that $\mu_1(E_i) = 2^{-i}\mu_1(T_1)$ and $\mu_2(F_j) = \theta_j\mu_2(T_2)$. Define $x_2 = \sum d_j \chi_{F_j}$. Clearly $x_2 \in B_2$. Furthermore the sequence of functions $x_1^n = \sum_{i=1}^{n} c_i \chi_{E_i}$ and the function $x_1 = \sum_{i\ge 1} c_i \chi_{E_i}$ are in B_1 and $x_1^n \to x_1$ boundedly a.e. Thus $N(x_1^n, x_2) \to N(x_1, x_2)$ as $n \to \infty$. Consider the integral representation which we have established for $N(x_1, x_2)$ when either x_1 or x_2 is a simple function. This permits us to write

$$N(x_1^n, x_2) = \int_{T_1\times T_2} \varphi\left(\sum_{i=1}^{n} c_i \chi_{E_i}, \sum_{j\ge 1} d_j \chi_{F_j}\right) d\mu_1 \otimes \mu_2$$

$$= \sum_{i=1}^{n} \sum_{j\ge 1} \varphi(c_i, d_j)\mu_1(E_i)\mu_2(F_j). \qquad (11)$$

Furthermore we note that for each $1 \le i \le n$

$$\left|\sum_{j=1}^{i-1} \varphi(c_i,d_j)\mu_1(E_i)\mu_2(F_j)\right| \le \sum_{j=1}^{i-1} \ell_{d_j} 2^{-i}\theta_j \mu_1(T_1)\mu_2(T_2)$$

$$\le 2^{-i}(\theta_i/2\, k_{c_i} - 2^{i}\alpha_i)\mu_1(T_1)\mu_2(T_2) \qquad (12)$$

Also

$$\left|\sum_{j\ge i+1} \varphi(c_i,d_j)\mu_1(E_i)\mu_2(F_j)\right| \le \sum_{j\ge i+1} k_{c_i} 2^{-i}\theta_j \mu_1(T_1)\mu_2(T_2)$$

$$\le 2^{-i}\theta_i/2\, k_{c_i}\mu_1(T_1)\mu_2(T_2) \qquad (13)$$

From (11), (12) and (13) it is clear that

$$|N(x_1^n,x_2)| \ge \Big|\sum_{j\ge 1}\varphi(c_n,d_j)e_j\,\mu_1(E_n)\,\mu_2(T_2)\Big|$$

$$-\sum_{i=1}^{n-1}\sum_{j\ge 1}|\varphi(c_i,d_j)|\,\mu_1(E_i)\,\mu_2(T_2)$$

$$\ge [k_{c_n}\theta_n-\sum_{j=1}^{n-1}\ell_{d_j}\theta_j-k_{c_n}\theta_n/2]\,\mu_1(E_n)\,\mu_2(T_2)$$

$$-\sum_{i=1}^{n-1}2^{-i}[k_{c_i}\theta_i+\sum_{j=1}^{i-1}\ell_{d_j}\theta_j+k_{c_i}\theta_i/2]\,\mu_1(T_1)\,\mu_2(T_2)$$

$$\ge n$$

which contradicts the fact that $N(x_1^n,x_2)\to N(x_1,x_2)$. Thus φ has property (BB).

The proof of the converse is quite simple. It is enough to notice that the argument leading to (10) establishes that a finite valued N is actually defined by (*). The rest is a routine verification.

As a corollary of the preceding theorem we obtain the following representation when the functional N is required to satisfy a stronger continuity property.

Corollary. Suppose that N is a biadditive functional on $B_1 \times B_2$ with B_1 as in the preceding theorem. Suppose further that whenever the sequences $x_i^n \in B_i$, $i=1,2$ are such that $x_i^n \to x_i$ boundedly a.e. then $N(x_1^n,x_2^n)\to N(x_1,x_2)$. For each such N there exists a unique jointly continuous function $\varphi: R^2 \to R$ satisfying conditions (z) and (BB) which represents N in the sense of (*) in the theorem. The converse statement is also true.

Proof: Since N satisfies the hypothesis of Theorem 7 there exists a unique function $\varphi: R^2 \to R$ which is separately continuous, has the properties (z) and (BB) and represents the function B in the sense of (*).

Further by considering constant functions in B_1 and B_2 one easily sees that the continuity condition in the corollary implies the joint continuity of φ.

The proof of the converse is exactly similar to the proof of the converse part in Theorem 7.

For functionals N which are continuous relative to (unbounded) a.e. convergence we have the following representation theorem.

Theorem 8. Let B_i be as in Theorem 7. Let N be a biadditive functional on $B_1 \times B_2$ which is separately continuous with respect to a.e. convergence. Then there exists a separately continuous function $\varphi : R^2 \to R$ satisfying properties (z) and (BS) and such that for every pair $(x_1, x_2) \in B_1 \times B_2$ the representation

1 2 r n v m 1 2 1 2

$T_1 \times T_2$

is valid. Such a representation is unique.

Conversely if φ is a separately continuous function having properties (z) and (BS) then for all $(x_1, x_2) \in B_1 \times B_2$ the above integral is well defined and is indeed a biadditive functional on $B_1 \times B_2$ which is separately continuous with respect to a.e. convergence.

Proof: Let N be a biadditive functional on $B_1 \times B_2$ satisfying the given continuity condition. Clearly N also satisfies the corresponding continuity condition in Theorem 7. Thus there exists a unique separately continuous function $\varphi : R^2 \to R$ such that φ satisfies condition(z) and represents N, i.e.

$$N(x_1, x_2) = \int_{T_1 \times T_2} \varphi(x_1, x_2)\, d\mu_1 \otimes \mu_2$$

for all $(x_1, x_2) \in B_1 \times B_2$. We proceed to show that φ has the property (BS). Suppose that φ lacks the property (BS). Then there exists a strip $C_1 = \{(c,d) \mid a < d < b\}$ or a strip $C_2 = \{(c,d) \mid a < c < b\}$ such that <p is unbounded on C_1, or C_2. Let us assume for the sake of definiteness that <p is unbounded on C_1. Let $\{\theta_i\}_{i\geq 1}$ be a sequence of positive reals such that $\sum_{i\geq 1} \theta_i = 1$ and $\sum_{i\geq n+1} \theta_i \leq 1/2\, \theta_n$. We choose inductively a sequence of points $\{(c_i, d_i)\}_{i\geq 1}$ in C_1 as follows. Denote

$$k_c = \max_{a\leq d\leq b} |\varphi(c,d)|, \quad \ell_d = \sup_{-\infty<c<\infty} |\varphi(c,d)| .$$

That k_c is well-defined follows from the separate continuity of φ; that ℓ_d is finite follows from the continuity of N(»,dl~) (see Theorem 3). The assumption that <p is unbounded on C_1 implies that k_c is unbounded as a function of c and ℓ_d is unbounded as a function of d, a < d < b. Choose c_1 such that

$$k_{c_1} \geq 4\theta_1^{-1}$$

and then take $d_1 \in [a,b]$ such that $|\varphi(c_1, d_1)| = k_{c_1}$. In general having chosen $\{(c_i, d_i)\}$ $1 \leq i \leq n-1$ select c_n such that

$$k_{c_n} > 2\theta_n^{-1} \sum_{i=1}^{n-1} \theta_i \ell_{d_i} + n\theta_n^{-1}$$

and then choose d_n such that $|\varphi(c_n, d_n)| = k_{c_n}$. Let $x_2 \in B_2$ be defined as follows.

$$x_2 = \sum_{j\geq 1} \ell_{d_j}^{-1} \chi_{F_j} \quad \text{where } \{F_j\}_{j\geq 1} \text{ is a}$$

disjoint sequence of measurable sets such that $\mu_2(F_j) = \theta_j \mu_2(T_2)$.
nested

Let $\{E_i\}$ be a sequence of measurable sets in T_1 such that $\mu_1(E_i) = 2^{-i}\mu_1(T_1)$. Proceeding as in Theorem 7 it is verified by direct computation that if $x_1^n = c_n \chi_{E_n}$

$$|N(x_1^n, x_2)| \geq n\,\mu_1(T_1)\,\mu_2(T_2) .$$

However the continuity hypothesis on N implies $N(x_1^n, x_2) \to 0$ since

$x_1^n \to 0$ a.e. Hence we have a contradiction and the proof of property (BS) is complete.

The converse assertion is an easy application of the bounded convergence theorem and the proof is omitted.

Remark 4. It might be noted that the proof implies that the N defined in the theorem admits extensions to the spaces $B_1 \times M_2$ and $M_1 \times B_2$ retaining separate continuity with respect to a.e. convergence, on their respective domains.

Next we proceed to represent biadditive functionals on $L_p(\mu_1) \times L_p(\mu_2)$ for $1 \leq p < \infty$.

Theorem 9. A functional N defined on $L_p(\mu_1) \times L_p(\mu_2)$ is biadditive and separately continuous with respect to the norm topology if and only if there exists a separately continuous function $\varphi: R^2 \to R$ satisfying the growth condition

(G^p) $\qquad |\varphi(c,d)| \leq K(1+|c|)^p(1+|d|)^p$ for some constant K and all real numbers c,d,

such that

$$(*) \qquad N(x_1, x_2) = \int_{T_1 \times T_2} \varphi(x_1, x_2)\, d\mu_1 \otimes \mu_2$$

for all pairs $(x_1, x_2) \in L_p(\mu_1) \times L_p(\mu_2)$. Such a representation is unique.

Conversely if φ is a separately continuous function on $R^2 \to R$ having the property (G^p) then the preceding integral exists for all pairs $(x_1, x_2) \in L_p(\mu_1) \times L_p(\mu_2)$ and defines a separately continuous biadditive functional on $L_p(\mu_1) \times L_p(\mu_2)$.

Proof. Let N be such a biadditive separately continuous functional on $L_p(\mu_1) \times L_p(\mu_2)$. Let N_1 be the restriction of N to the subspace $B_1 \times B_2 \subset L_p(\mu_1) \times L_p(\mu_2)$ where B_i is the vector space of essentially bounded functions on (T_1, L_1, μ_1). By Lebesgue's bounded convergence theorem it is verified that N_1 satisfies the hypothesis of Theorem 7. Thus there exists a unique separately continuous function $\varphi: R^2 \to R$ with the properties (α) and (β) such that for $(x_1, x_2) \in B_1 \times B_2$

$$N(x_1,x_2) = \int_{T_1 \times T_2} \varphi(x_1,x_2)\, d\mu_1 \otimes \mu_2. \qquad (1)$$

Next we remark that for a fixed c (for a fixed d) there exists a constant K_c (L_d) such that $|\varphi(c,d)| \le K_c[1 + |d|]^p$ for all d ($|\varphi(c,d)| \le L_d[1 + |c|]^p$ for all c). For if one of the above inequalities is false, say if there does not exist a constant K_c, then there exists a sequence $\{d_n\}_{n \ge 1}$ such that $|d_n| \to \infty$ and $|\varphi(c,d_n)| > n|d_n|^p$. Hence by Theorem 4 the continuity of $N(c1_{T_1}, \cdot)$ is contradicted. Thus there do exist such constants K_c and L_d, which we will hereafter take to be $K_c = \sup_d \frac{|\varphi(c,d)|}{(1+|d|)^p}$, $L_d = \sup_c \frac{|\varphi(c,d)|}{(1+|c|)^p}$.

Using the above inequalities the representation (1) can be extended to the case of pairs (x_1, x_2), x_1 simple and x_2 in $L_p(\mu_2)$ or x_1 in $L_p(\mu_1)$ and x_2 simple as follows. Let x_1 be simple. Then since μ_2 is a finite measure, B_2 is a dense subset of $L_p(\mu_2)$ and there exists a sequence $\{x_2^n\} \subset B_2$ such that $x_2^n \to x_2$ in the L_p-norm. Further we can select x_2^n such that $|x_2^n| \le |x_2|$ and $x_2^n \to x_2$ pointwise a.e. Since $(x_1, x_2^n) \in B_1 \times B_2$ it follows from (1) that

$$N(x_1, x_2^n) = \int_{T_1 \times T_2} \varphi(x_1, x_2^n)\, d\mu_1 \otimes \mu_2$$

Now $|\varphi(x_1(\cdot), d)| \le K_{x_1}(1 + |d|)^p$ where $K_{x_1} = \sup\{K_c\}_{c \in F}$, F being the range of the simple function x_1. Since $\varphi(x_1, x_2^n) \to \varphi(x_1, x_2)$ a.e. $\mu_1 \otimes \mu_2$, it follows from Lebesgue's dominated convergence theorem that

$$N(x_1,x_2) = \lim_{n\to\infty} N(x_1^n,x_2^n) = \lim_{n\to\infty} \int_{T_1\times T_2} \varphi(x_1^n,x_2^n)\,d\mu_1\otimes\mu_2$$

$$= \int_{T_1\times T_2} \varphi(x_1,x_2)\,d\mu_1\otimes\mu_2 \qquad (2)$$

Using the above representation we prove that φ satisfies the growth condition (G_1). Let $\{\theta_j\}_{j\geq 1}$ be a sequence of positive real numbers such that $\sum \theta_j = 1$ and $\sum_{j\geq n+1} \theta_j < \frac{1}{2}\theta_n$. Suppose that φ does not satisfy (G_1). We shall choose a sequence $\{(c_i,d_i)\}_{i\geq 1}$ as follows. Select c_1 such that

$$K_{c_1} \geq 8\,\theta_1^{-1}(1+|c_1|)^p$$

and then take d_1 to be such that

$$|\varphi(c_1,d_1)| \geq \frac{3}{4} K_{c_1}(1+|d_1|)^p .$$

In general having chosen (c_i,d_i) $1 \leq i \leq n-1$ choose c_n such that

$$K_{c_n} \geq 4\,\theta_n^{-1} \sum_{j=1}^{n-1} K_{d_j}(1+|c_n|)^p \theta_j/(1+|d_j|)^p + 2^{n+1} a_n \qquad (3a)$$

where

$$a_n \geq \frac{3}{2}\sum_{i=1}^{n-1} K_{c_i}\,\theta_i 2^{-i}/(1+|c_i|)^p + \sum_{i=1}^{n-1} 2^{-i} \sum_{j=1}^{i-1} K_{d_j}\,\theta_j/(1+|d_j|)^p + n \qquad (3b)$$

and then take d_n to be such that

$$|\varphi(c_n,d_n)| \geq \frac{3}{4} K_{c_n}(1+|d_n|)^p \qquad (4)$$

Let $\{E_i\}_{i\geq 1}$ and $\{F_j\}_{j\geq 1}$ be sequences of disjoint measurable sets in T_1 and T_2 respectively such that

$$\mu_1(E_i) = 2^{-i}\mu_1(T_1)/(1+|c_i|)^p , \quad \mu_2(F_j) = \theta_j\mu_2(T_2)/(1+|d_j|)^p .$$

Define $x_2 = \sum_{j\geq 1} d_j \chi_{F_j}$. Clearly $x_2 \in L_p(\mu_2)$. Furthermore the sequence of simple functions $x_1^n = \sum_{i=1}^{n} c_i \chi_{E_i}$ and the function $x_1 = \sum_{i\geq 1} c_i \chi_{E_i}$ are in $L_p(\mu_1)$ and $x_1^n \to x_1$ in $L_p(\mu_1)$. Thus $N(x_1^n,x_2) \to N(x_1,x_2)$

as $n \to \infty$. Consider the integral representation which we have established for $N(x_1^n, x_2)$ when either x_1 or x_2 is a simple function. This permits us to write

$$N(x_1^n, x_2) = \int_{T_1 \times T_2} \varphi\Big(\sum_{i=1}^{n} c_i \chi_{E_i}, \sum_{j\ge 1} d_j \chi_{F_j}\Big)\, d\mu_1 \otimes \mu_2 = \sum_{i=1}^{n} \sum_{j\ge 1} \varphi(c_i, d_j)\mu_1(E_i)\mu_2(F_j) \tag{5}$$

Furthermore we note that for each $1 \le i \le n$

$$\Big|\sum_{j=1}^{i-1} \varphi(c_i, d_j)\mu_1(E_i)\mu_2(F_j)\Big| \le \sum_{j=1}^{i-1} L_{d_j}(1+|c_i|)^p \theta_j/(1+|d_j|)^p \mu_1(E_i)\mu_2(T_2) \tag{6}$$

Also

$$\Big|\sum_{j\ge i+1} \varphi(c_i, d_j)\mu_1(E_i)\mu_2(F_j)\Big| \le \sum_{j\ge i+1} K_{c_i}(1+|d_j|)^p \theta_j/(1+|d_j|)^p \mu_1(E_i)\mu_2(T_2)$$
$$\le K_{c_i}\theta_i/2\ \mu_1(E_i)\mu_2(T_2) \tag{7}$$

From (5), (6), and (7) it is clear that

$$|N(x_1^n, x_2)| \ge \Big|\sum_{j\ge 1}\varphi(c_n, d_j)\mu_1(E_n)\mu_2(F_j)\Big| - \sum_{i=1}^{n-1}\sum_{j\ge 1}|\varphi(c_i, d_j)|\mu_1(E_n)\mu_2(F_j)$$
$$\ge \Big[\varphi(c_n, d_n)\theta_n/(1+|d_n|)^p - \sum_{j=1}^{n-1} L_{d_j}(1+|c_n|)^p\theta_j/(1+|d_j|)^p$$
$$- \sum_{j\ge n+1} K_{c_n}\theta_j\Big]\mu_1(E_n)\mu_2(T_2)$$
$$- \sum_{i=1}^{n-1}\Big\{K_{c_i}\theta_i + \sum_{j=1}^{i-1} L_{d_j}(1+|c_i|)^p\theta_j/(1+|d_j|)^p$$
$$+ \sum_{j\ge i+1} K_{c_i}\theta_j\Big\} 2^{-i}/(1+|c_i|)^p \mu_1(T_1)\mu_2(T_2) \ge n$$

which contradicts the fact that $N(x_1^n, x_2) \to N(x_1, x_2)$. Thus φ satisfies (G_1^p).

Using this property of φ the domain of validity of representation (2) can be extended to all of $L_p(\mu_1) \times L_p(\mu_2)$. Since φ satisfies (G_1^p) it is clear that $(x_1, x_2) \in L_p(\mu_1) \times L_p(\mu_2)$ implies $\varphi(x_1,x_2) \in L_1(\mu_1 \times \mu_2)$. Let x_1^n be a sequence in B_1 such that $x_1^n \to x_1$ in $L_p(\mu_1)$. We can choose x_1^n such that $|x_1^n| \leq |x_1|$ and $x_1^n \to x_1$ a.e. Thus $\varphi(x_1^n, x_2) \to \varphi(x_1, x_2)$ a.e. and

$$|\varphi(x_1^n,x_2)| \leq K(1 + |x_1^n|)^p \, (1 + |x_2|)^p$$
$$\leq K(1 + |x_1|)^p \, (1 + |x_2|)^p .$$

Hence by applying the dominated convergence theorem we obtain

$$N(x_1,x_2) = \lim_{n\to\infty} N(x_1^n,x_2) = \int_{T_1 \times T_2} \varphi(x_1,x_2)\, d\mu_1 \otimes \mu_2 .$$

The essential part in the converse namely $\varphi(x_1, x_2) \in L_1(\mu_1 \times \mu_2)$ is once again an application of the bounded convergence theorem and similar in details to the argument in the preceeding paragraph. Hence the details are omitted.

Remark 5. It is easily verified that the theorem remains true if separate continuity is replaced throughout by joint continuity.

Before concluding this section we state the results when the biadditive functional on $L_p(\mu_1) \times L_p(\mu_2)$ is continuous with respect to a.e. convergence or convergence in measure. Since the proofs are essentially similar to these in preceding theorems we omit them.

Theorem 10. Suppose N is a biadditive functional on $L_p(\mu_1) \times L_p(\mu_2)$. Then N is separately continuous with respect to a.e. convergence if and only if there exists a separately continuous function $\varphi: R^2 \to R$ ~~satisfying the properties~~ (Z) ~~and~~ (BS) ~~and the growth~~ condition*

(G_1^p) $\quad \alpha(c) \leq K(1 + |c|)^p$ and $\beta(d) \leq K(1 + |d|)^p$ for some fixed K and, all real number c,d

* $\alpha(c)$ and $\beta(d)$ are defined in (BS).

such that

$$(*) \qquad N(x_1,x_2) = \int_{T_1\times T_2} \varphi(x_1,x_2)\, d\mu_1 \otimes \mu_2 .$$

Remark 6. The theorem remains true even if a.e. convergence is replaced by convergence in measure or separate continuity is replaced by joint continuity wherever it occurs.

4. Representation of Biadditive Functionals in the σ-finite Case.

In this section the representation theorems obtained in the preceding section are generalized to the σ-finite case. The passage from the finite to the σ-finite case for biadditive functionals turns out to be very similar to that carried out for additive functionals in Section 2. For reasons mentioned in Remark 2 we choose the domain of the biadditive functional to be $L_p(\mu_1)\times L_p(\mu_2)$ $(1 \le p < \infty)$. It is assumed in this section that the measure spaces (T_i,Σ_i,μ_i) $i = 1,2$ are σ-finite nonatomic and that $\mu_i(T_i) = \infty$.

Theorem 11. Suppose that N is a biadditive functional defined on $L_p(\mu_1)\times L_p(\mu_2)$. Then N is separately continuous with respect to the norm topology if and only if there exists a real valued continuous function φ on R^2 into R satisfying the condition (Z) and the growth condition (G^2) $|\varphi(c,d)| \le k|c|^p|d|^p$ for some constant k and all real numbers c,d, such that the representation

$$(*) \qquad N(x_1,x_2) = \int_{T_1\times T_2} \varphi(x_1,x_2)\, d\mu_1 \otimes \mu_2 \quad \text{is valid}.$$

The function φ is uniquely determined by the equation (*).

Proof: Let E_1, E_2 be any two measurable sets in T_1 and T_2 such that $0 < \mu_i(E_i) < \infty$. Consider the measure spaces $(T_i, \Sigma_i, \mu_{i,E_i})$ where

μ_{iE_i} is the contraction of μ_i to E_i. Consider the functional N_{E_1,E_2} canonically determined by N defined on $L_p(\mu_{1E_1}) \times L_p(\mu_{2E_2})$ by the formula $N_{E_1,E_2}(x_1,x_2) = N(x_1^*,x_2^*)$ where x_i^* are the functions in $L_p(\mu_i)$ $i = 1,2$ defined by $x_i^*|E_i = x_i$, $x_i^*|T_i - E_i = 0$. It follows from the properties of N that N_{E_1,E_2} is biadditive on $L_p(\mu_{1E_1}) \times L_p(\mu_{2E_2})$ and is separately continuous. Applying Theorem 9 we see that there exists a separately continuous function φ_{E_1,E_2} on $R^2 \to R$ having the properties (2) and (G_P) such that

$$N_{E_1,E_2}(x_1,x_2) = \int_{E_1 \times E_2} \varphi_{E_1,E_2}(x_1,x_2)\, d\mu_{1E_1} \otimes \mu_{2E_2} .$$

Now noting that $N(x_1,\cdot)$ and $N(\cdot,x_2)$ take the same values on equimeasurable functions (condition (2) in definition 1) and that the measure spaces are nonatomic we see by an argument similar to that in the second paragraph of Theorem 5 that the representing function φ_{E_1,E_2} is independent of the sets E_i. Let $\varphi_{E_1,E_2} = \varphi$ for all E_i such that $E_i \in S_i$, $0 < \mu_i(E_i) < \infty$. With φ chosen as above let us consider two simple functions x_i on (T_i, S_i, μ_i) $i = 1,2$. Since the supports of the x_i have finite measure it follows from the preceeding remarks that in this situation

$$N(x_1,x_2) = \int_{T_1 \times T_2} \varphi(x_1,x_2)\, d\mu_1 \otimes \mu_2 . \qquad \textbf{(1)}$$

Next we observe that for each pair of real numbers c^*, d^* the function φ satisfies

$$|\varphi(c^*,d)| \le K_{c^*}|d|^p \text{ for all } d, \quad |\varphi(c,d^*)| \le L_{d^*}|c|^p \text{ for all } c. \qquad (2)$$

For let c^* be given and choose $E_1 \in S_1$ such that $\mu_1(E_1) = 1$. Then $N(c^*\chi_{E_1},\cdot)$ is a continuous additive functional on $L_p(\mu_2)$. Hence from Theorem 6 it follows that there exists a continuous function $\psi_{c^*}: R \to R$ such that $|\psi_{c^*}(d)| \le K_{c^*} \cdot |d|^p$ for some constant K_{c^*} and all $d \in (-\infty, \infty)$ and such that

$$N(c^* \chi_{E_1}, x_2) = \int_{T_2} \psi_{c^*}(x_2)\, d\mu_2 \quad \text{for } x_2 \in L_p(\mu_2).$$

Further the representing function ψ_{c^*} is unique. In particular if $E_2 \in \Sigma_2$, $\mu_2(E_2) = 1$ then

$$\begin{aligned} N(c^* \chi_{E_1}, d\chi_{E_2}) &= \int_{T_2} \psi_{c^*}(d\chi_{E_2})\, d\mu_2 \\ &= \psi_{c^*}(d) \end{aligned} \tag{3}$$

From equations (1) and (3) it follows that

$$\begin{aligned} \psi_{c^*}(d) &= N(c^* \chi_{E_1}, d\chi_{E_2}) \\ &= \int_{T_1 \times T_2} \varphi(c^* \chi_{E_1}, d\chi_{E_2})\, d\mu_1 \otimes \mu_2 \\ &= \varphi(c^*, d)\, \mu_1(E_1)\, \mu_2(E_2). \end{aligned}$$

Thus $\varphi(c^*, d) = \psi_{c^*}(d)$. Thus the first equation in (2) is a consequence of the properties of ψ_{c^*}. An entirely similar argument yields the second inequality in (2). Hereafter we shall take K_{c^*} and L_{d^*} to be defined as follows

$$K_{c^*} = \sup_{d \ne 0} |\varphi(c^*, d)|/|d|^p, \quad L_{d^*} = \sup_{c \ne 0} |\varphi(c, d^*)|/|c|^p.$$

Notice that since φ satisfies the growth conditions in (2), it follows by Theorem 6 that for each set $E \in \Sigma_1$, $0 < \mu_1(E) < \infty$, and any $x_2 \in L_p(\mu_2)$

$$N(c\chi_E, x_2) = \int_{T_1 \times T_2} \varphi(c\chi_E, x_2)\, d\mu_1 \otimes \mu_2,$$

with a similar formula applying for $N(x_1, d\chi_F)$. By the additivity of

N this leads to the representation

$$N(x_1,x_2) = \int_{T_1\times T_2} \varphi(x_1,x_2)\,d\mu_1\otimes\mu_2 \qquad x_1\in L_p(M_1),\ x_2\in L_p(M_2) \tag{4}$$

whenever x_1 or x_2 is a simple function.

Now we proceed to show that φ satisfies the growth condition (G_3). The proof of this assertion is quite similar to an argument occurring in the proof of Theorem 9, If (G_3) is invalid then $\{K_c/|c|^p \mid c\neq 0\}$ is unbounded. Let $(\theta_j)_{j\ge 1}$ be a sequence of positive real numbers such that $\sum_{j\ge 1}\theta_j = 1$, $\sum_{j\ge n+1}\theta_j \le 2^{-n}\theta_n$. We choose inductively a sequence $\{(c_i,d_i)\}_{i\ge 1}$ as follows. Select $c_1\neq 0$ such that

$$K_{c_1} \ge 8\,\theta_1^{-1}|c_1|^p$$

and then take $d_1 \neq 0$ to be such that

$$|\varphi(c_1,d_1)| \ge 3/4\, K_{c_1}|d_1|^p$$

In general having chosen (c_i,d_i) $1\le i\le n-1$ choose $c_n\neq 0$ such that

$$K_{c_n} \ge 4\,\theta_n^{-1} \sum_{j=1}^{n-1} L_{d_j}|c_j|^p\,\theta_j/|d_j|^p + \text{[illegible]} \tag{5a}$$

where the quantity a_n is required to satisfy

$$a_n \ge 3/2 \sum_{i=1}^{n-1} K_{c_i}\theta_i 2^{-i}/|c_i|^p + \sum_{i=1}^{n-1} 2^{-i}\Big(\sum_{j=1}^{i-1} L_{d_j}\theta_j/|d_j|^p\Big) + n, \tag{5b}$$

and then take $d_n \neq 0$ to be such that

$$|\varphi(c_n,d_n)| \ge 3/4\, K_{c_n}|d_n|^p \tag{6}$$

Let $(E_i)_{i\ge 1}$ and $\{F_j\}_{j\ge 1}$ be sequences of disjoint measurable sets in T_1 and T_2 respectively such that

$$\mu_1(E_i) = 2^{-i}/|c_i|^p \qquad \text{and} \qquad \mu_2(F_j) = \theta_j/|d_j|^p.$$

Define $x_2 = \sum_{j\ge1} d_j \chi_{F_j}$. Clearly $x_2 \in L_p(\mu_2)$. Furthermore the sequence of simple functions

$x_1^n = \sum_{i=1}^{n} c_i \chi_{E_i}$ and the function $x_1 = \sum_{i\ge1} c_i \chi_{E_i}$

are in $L_p(\mu_1)$, and $x_1^n \to x_1$ in $L_p(\mu_1)$. Thus $N(x_1^n, x_2) \to N(x_1, x_2)$ as $n \to \infty$. Consider the integral representation (6) which we have established for N when either of its arguments is a simple function. This permits us to write

$$N(x_1^n, x_2) = \int_{T_1 \times T_2} \varphi\Big(\sum_{i=1}^{n} c_i \chi_{E_i}, \sum_{j\ge1} d_j \chi_{F_j}\Big)\, d\mu_1 \otimes \mu_2 = \sum_{i=1}^{n} \sum_{j\ge1} \varphi(c_i, d_j)\, \mu_1(E_i)\, \mu_2(F_j) \tag{7}$$

Furthermore we note that for each $1 \le i \le n$

$$\Big|\sum_{j=1}^{i-1} \varphi(c_i, d_j)\, \mu_1(E_i)\, \mu_2(F_j)\Big| \le \sum_{j=1}^{i-1} L_{d_j} |c_i|^p \theta_j / |d_j|^p\, \mu_1(E_i) \tag{8}$$

Also

$$\Big|\sum_{j\ge i+1} \varphi(c_i, d_j)\, \mu_1(E_i)\, \mu_2(F_j)\Big| \le \sum_{j\ge i+1} K_{c_i} |d_j|^p \theta_j / |d_j|^p\, \mu_1(E_i) \le K_{c_i} \theta_i / 2\, \mu_1(E_i) \tag{9}$$

From equations (5)-(9) it is clear that

$$|N(x_1, x_2)| \ge \Big|\sum_j \varphi(c_n, d_j)\, \mu_1(E_n)\, \mu_2(F_j)\Big| - \sum_{i=1}^{n-1} \Big|\sum_{j\ge1} \varphi(c_i, d_j)\, \mu_1(E_i)\, \mu_2(F_j)\Big|$$

$$\ge \Big[|\varphi(c_n, d_n)|\, \theta_n / |d_n|^p - \sum_{j=1}^{n-1} L_{d_j} |c_n|^p \theta_j / |d_j|^p - K_{c_n} \theta_n / 2\Big] \mu_1(E_n)$$

$$- \sum_{i=1}^{n-1} \Big[|\varphi(c_i, d_i)|\, \theta_i / |d_i|^p + \sum_{j=1}^{i-1} L_{d_j} |c_i|^p \theta_j / |d_j|^p + K_{c_i} \theta_i / 2\Big] \mu_1(E_i)$$

$$\ge n$$

which contradicts the fact that $N(x_1^n, x_2) \to N(x_1, x_2)$.

Conversely suppose that φ is a separately continuous function with the properties (z) and (G_p). Then it is clear that for all pairs $(x_1, x_2) \in L_p(\mu_1) \times L_p(\mu_2)$, $\varphi(x_1, x_2) \in L_1(\mu_1 \otimes \mu_2)$ and that the functional N defined by

$$N(x_1, x_2) = \int_{T_1 \times T_2} \varphi(x_1, x_2)\, d\mu_1 \otimes \mu_2 \quad \text{is biadditive.}$$

Now suppose $x_1^n \to x_1$ in $L_p(\mu_1)$ and $x_2 \in L_p(\mu_2)$. Since $x_1^n \to x_1$ in $L_p(\mu_1)$ and $\varphi(\cdot, d)$ verifies conditions (1) and (2) of Theorem 6, it follows that if ψ_n, ψ are the functions on T_2 defined a.e. by

$$\psi_n(t_2) = \int_{T_1} \varphi(x_1^n, x_2(t_2))\, d\mu_1$$

and

$$\psi(t_2) = \int_{T_1} \varphi(x_1, x_2(t_2))\, d\mu_1$$

then $\psi_n \to \psi$ pointwise on T_2. Since $\varphi(x_1^n, x_2)$ and $\varphi(x_1, x_2)$ are integrable with respect to $\mu_1 \otimes \mu_2$, the functions ψ_n and ψ are in $L_1(\mu_2)$ and recalling that φ satisfies (G_p) it follows that $|\psi_n(y)| \leq K|x_2(y)|^p \|x_1^n\|^p$ and $|\psi(y)| \leq K|x_2(y)|^p \|x_1\|^p$. Since the set $\{\|x_1^n\|^p_{n \geq 1}, \|x_1\|\}$ is a bounded set and since $x_2 \in L_p(\mu_2)$, we may apply the dominated convergence theorem to deduce

$$\int_{T_2} \psi_n\, d\mu_2 \to \int_{T_2} \psi\, d\mu_2, \text{ i.e } \quad N(x_1^n, x_2) \to N(x_1, x_2).$$

The proof of the theorem is complete.

Similarly one can establish that the analogues in the σ-finite case of Theorem 10 and Remark 5 continue to be true provided that condition (G_p) in Theorem 10 is replaced by

(G_4^p) $a(c) \leq K|c|^p$, $j8(d) \leq K|d|^p$ for some K and all real c,c

where a(c) and]8(d) are as defined in (BS) .

In conclusion we wish to point out that although our arguments have been given for real valued functionals they apply equally well to complex valued functionals. Next we mention a few problems not considered in this paper$_o$ We have only partial results concerning integral representations of additive functionals when the measure space contains atoms. These results are similar to Theorem 1.8 in [1]• In several theorems in this paper we considered the domain to be an L_p space. Some of these admit straightforward generalizations to the case when L_p is replaced by an Orlicz function space, Luxemburg [10] or Zanen [11]^but our investigation of this matter is not complete. The additive functionals considered here admit a natural generalization to vector valued additive functionals.Further the case in which the measure is a vector valued measure also arises in a natural way.

We hope to discuss these and other related problems elsewhere.

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