Inverse Hölder Inequalities

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It is known that, for various classes of non-negative functions f, g, the Schwarz inequality

$$\left(\int_X fg\,d\mu\right)^2 \le \int_X f^2 d\mu \int_X g^2 d\mu$$

has an inverse of the form

(1.1) $$\int_X f^2 d\mu \int_X g^2 d\mu \le C_2^2 \left(\int_X fg\,d\mu\right)^2,$$

where C_2 is a positive constant which depends on the classes considered. For instance, if X is a finite interval, μ is the Lebesgue measure, and f,g are non-negative concave functions on X, it was shown by Blaschke and Pick [3] that $C_2 = 2$. If $(X, \mathcal{B}, \mu)$ is a positive measure space, and $f,g \in L^2(X, \mathcal{B}, \mu)$ and are such that

(1.2) $$0 < m_1 \le f \le M_1 < \infty, \quad 0 < m_2 \le g \le M_2 < \infty,$$

then

(1.3) $$C_2 = \frac{m_1 m_2 + M_1 M_2}{2\sqrt{m_1 m_2 M_1 M_2}}$$

[9,6,7].

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Similarly, the Hölder inequality

$$\int_X fg\,d\mu \le \left[\int_X f^p d\mu\right]^{\frac{1}{p}} \left[\int_X g^q d\mu\right]^{\frac{1}{q}}, \quad p^{-1} + q^{-1} = 1,$$

has an inverse of the form

(1.4) $$\left[\int_X f^p d\mu\right]^{\frac{1}{p}} \left[\int_X g^q d\mu\right]^{\frac{1}{q}} \le c_p \int_X fg\,d\mu ,$$

if the functions f and g are subjected to suitable restrictions. It was shown by Bellman [2] that the result of Blaschke and Pick generalizes to (1.4), with

(1.7) $$C_p = 6(p + 1)^{-\frac{1}{p}} (q + 1)^{-\frac{1}{q}} .$$

For functions satisfying the conditions (1.2), Diaz, Goldman and Metcalf [5] obtained an inequality equivalent to (1.4), with the value

(1.6) $$C_p = \frac{M_1^p M_2^q - m_1^p m_2^q}{[pm_2M_2(M_1M_2^{q-1}-m_1m_2^{q-1})]^{\frac{1}{p}}[qm_1M_1(M_2M_1^{p-1}-m_2m_1^{p-1})]^{\frac{1}{q}}}$$

for the constant. A closely related inequality had previously been obtained by Cargo and Shisha [4].

Diaz and Metcalf also showed that, for functions subject to (1.2), there exist inequalities of the type

(1.7) $$A\int_X f^p d\mu + B\int_X g^q d\mu \le C\int_X fg\,d\mu, \quad p^{-1} + q^{-1} = 1,$$

with suitable positive constants A,B,C. Since, by the geometric-arithmetic inequality,

$$(1.8)\quad (pA)^{\frac{1}{p}}(qB)^{\frac{1}{q}}\Big[\int_X f^p d\mu\Big]^{\frac{1}{p}}\Big[\int_X g^q d\mu\Big]^{\frac{1}{q}} \le A\int_X f^p d\mu + B\int_X g^q d\mu ,$$

(1.7) will be stronger than (1.4) in those cases in which

$$C = (pA)^{\frac{1}{p}}(qB)^{\frac{1}{q}}C_p .$$

In the present paper we shall examine the existence of inequalities of this "inverse" type from a more general point of view. Our basic result, to be proved in the following section, is that an "inverse" inequality which holds for functions $f \in H_1$, $g \in H_2$, where H_1 and H_2 are given sets of functions, must also hold for functions belonging to the convex hulls $C(H_1)$ and $C(H_2)$, respectively. This leads to two conclusions: (a) the sets H_1 and H_2 may be assumed to be convex in the first place; (b) to prove such an inequality for $f \in H_1$, $g \in H_2$, it is sufficient to establish its validity for subsets whose convex hulls coincide with H_1 and H_2, respectively. If these subsets consist of functions of simple character for which the integrals appearing in the inequalities can be easily computed, the determination of the exact constant in the inequality in question reduces to an elementary extremal problem.

In the subsequent sections we shall apply this procedure to obtain a number of "inverse" inequalities for various classes of functions.

2. We now state our basic result.

<u>Lemma 2.1</u>. <u>Let</u> (X, Σ, μ) <u>be a positive measure space, and let</u> $\phi_\nu(t)$ $(\nu = 1, \dots, n)$ <u>be non-negative continuous convex functions</u>

for $t > 0$. Let H_ν be a set of non-negative functions f_ν such that $\phi_\nu(f_\nu) \in L^1(X, \mathfrak{M}, \mu)$, and let $C(H_\nu)$ denote the convex hull of H_ν. If the inequality

$$(2.1) \qquad \sum_{\nu=1}^{n} A_\nu \int_X \phi_\nu(f_\nu)\, d\mu \le C \int_X \Big[\prod_{\nu=1}^{n} f_\nu\Big] d\mu$$

(A_ν, C positive constants) holds for all $f_\nu \in H_\nu$ ($\nu = 1, \ldots, n$), then it also holds for $f_\nu \in C(H_\nu)$.

This, of course, is of interest only if the integral on the right-hand side of (2.1) exists. This will therefore be assumed to be the case. We shall also assume that the constant C is the best of its kind, i.e., that there exist functions $f_1, \ldots, f_n$ such that (2.1) becomes false if C is replaced by a smaller number.

In the special case in which $\phi_\nu(t) = t^{p_\nu}$ ($p_\nu \ge 1$, $p_1^{-1} + \cdots + p_n^{-1} = 1$), the geometric-arithmetic inequality

$$(2.2) \qquad \prod_{\nu=1}^{n} x_\nu^{p_\nu^{-1}} \le \sum_{\nu=1}^{n} x_\nu p_\nu^{-1} \qquad (x_\nu > 0).$$

shows that

$$(2.3) \qquad \prod_{\nu=1}^{n} (p_\nu A_\nu)^{p_\nu^{-1}} \prod_{\nu=1}^{n} \Big[\int_X f_\nu^{p_\nu}\, d\mu\Big]^{p_\nu^{-1}} \le \sum_{\nu=1}^{n} A_\nu \int_X f_\nu^{p_\nu}\, d\mu \le C \int_X \Big[\prod_{\nu=1}^{n} f_\nu\Big] d\mu ,$$

i.e.

$$(2.4) \qquad \prod_{\nu=1}^{n} \Big[\int_X f_\nu^{p_\nu}\, d\mu\Big]^{p_\nu^{-1}} \le D \int_X \Big[\prod_{\nu=1}^{n} f_\nu\Big] d\mu ,$$

where D is a positive constant. The functions f_ν of Lemma 2.1 (with $\phi_\nu(t) = t^{p_\nu}$) are thus subject to an inverse Holder inequality. However, if (2.4) is obtained in this way, the question arises whether the constant D is the best possible of its kind. Evidently, this will be true only if the sign of equality holds in both inequalities (2.3) for a set of functions $f_1, \ldots, f_n$ such that $f_\nu \in H_\nu$. This difficulty is avoided if (2.4) is obtained by means of the following lemma.

Lemma 2.2 Let (X, Σ, μ) be a positive measure space, and let H_ν $(\nu = 1, \ldots, n)$ be a subset of $L^{p_\nu}(X, \Sigma, \mu)$, where $p_\nu \geq 1$, $p_1^{-1} + \cdots + p_n^{-1} = 1$. If the inequality (2.4) holds for functions $f_\nu \in H_\nu$, then it also holds for functions $f_\nu \in \mathrm{C} f H_\nu$.

We first prove Lemma 2.1. To simplify the writing, we assume $n = 2$; the extension of the argument to general n is obvious. To establish the result it is sufficient to show that

$$(2.5) \qquad A_1 \int_X \phi_1(F_1)\,d\mu + A_2 \int_X \phi_2(F_2)\,d\mu \leq C \int_X F_1 F_2\,d\mu$$

for all convex combinations

$$(2.6) \qquad \begin{aligned} F_1 &= \alpha_1 f_1^{(1)} + \cdots + \alpha_m f_1^{(m)} \quad (\alpha_k > 0,\ \alpha_1 + \cdots + \alpha_m = 1,\ f_1^{(k)} \in H_1) \\ F_2 &= \beta_1 f_2^{(1)} + \cdots + \beta_M f_2^{(r)} \quad (\beta_r > 0,\ \beta_1 + \cdots + \beta_M = 1,\ f_2^{(r)} \in H_2), \end{aligned}$$

provided (2.1) (with $n = 2$) holds for $f_1 = f_1^{(k)}$ and $f_2 = f_2^{(r)}$ $(k = 1, \ldots, m;\ r = 1, \ldots, M)$.

By (2.6) and Jensen's inequality, we have

$$\phi_1(F_1) \leq \sum_{k=1}^{m} \alpha_k \phi(f_1^{(k)}),$$

and a similar inequality for $\phi_2(F_2)$. Hence,

$$\phi \equiv A_1 \int_X \phi_1(F_1)\,d\mu + A_2 \int_X \phi_2(F_2)\,d\mu$$

$$\leq A_1\Big(\sum_{r=1}^{M} \beta_r\Big) \sum_{k=1}^{m} \alpha_k \int_X \phi_1(f_1^{(k)})\,d\mu + A_2\Big(\sum_{k=1}^{m}$$

$$= \sum_{k=1}^{m} \sum_{r=1}^{M} \alpha_k \beta_r \Big[A_1 \int_X \phi_1(f_1^{(k)})\,d\mu + A_2 \int_X \phi_2(f_2^{(r)})\,d\mu\Big].$$

Since (2.1) holds for $f_1 = f_1^{(k)}$ and $f_2 = f_2^{(r)}$, it follows that

$$\phi \leq C \sum_{k=1}^{m} \sum_{r=1}^{M} \alpha_k \beta_r \int_X f_1^{(k)} f_2^{(r)}\,d\mu .$$

Because of (2.6), this is equivalent to

$$\phi \leq C \int_X F_1 F_2\,d\mu .$$

This completes the proof of Lemma 2.1.

The proof of Lemma 2.2 is similar, except that Jensen's inequality has now to be replaced by Minkowski's inequality. Again, we simplify the writing by setting $n = 2$; the modifications required in the general case are evident. By (2.6) and Minkowski's inequality, we have

$$\Big[\int_X F_1^{p_1}\,d\mu\Big]^{p_1} \leq \sum_{k=1} \alpha_k \Big[\int_X (f_1^{(k)})^{p_1}\,d\mu\Big]^{p_1} ,$$

and thus

$$\Psi \equiv \left[\int_X F_1^{p_1} d\mu\right]^{\frac{1}{p_1}} \left[\int_X F_2^{p_2} d\mu\right]^{\frac{1}{p_2}}$$

$$\leq \left\{\sum_{k=1}^{m} \alpha_k \left[\int_X (f_1^{(k)})^{p_1} d\mu\right]^{\frac{1}{p_1}}\right\}\left\{\sum_{r=1}^{M} \beta_r \left[\int_X (f_2^{(r)})^{p_2} d\mu\right]^{\frac{1}{p_2}}\right\}$$

$$= \sum_{k=1}^{m} \sum_{r=1}^{*} \alpha_k \beta_r \left[\int_X (f_1^{(k)})^{p_1} d\mu\right]^{\frac{1}{p_1}} \left[\int_X (f_2^{(r)})^{p_2} d\mu\right]^{\frac{1}{p_2}} .$$

But $f_1^{(k)} \in H_1$, $f_2^{(r)} \in H_2$, and we have assumed that (2.4) holds for all $f_1 \in H_1$, $f_2 \in H_2$. Hence,

$$\Psi \leq \sum_{k=1}^{m} \sum_{r=1}^{M} \ldots$$

and Lemma 2.2 is established.

3. As a first application of Lemma 2,1 we consider the case in which $C(H_\nu)$ consists of the functions f_ν for which $\phi_\nu(f_\nu) \in L^1(X/X^K)$ and which are subject to the inequalities

(3.1) $$0 < m_\nu \leq f_\nu \leq M_\nu < \infty .$$

We shall also assume that the functions $\phi_\nu(t)$ are increasing for $t > 0$. The fact that $\phi_\nu(f_\nu)$ can be approximated in L^1 by step-functions is equivalent to saying that f_ν may be approximated by expressions of the form

(3.2) $$F_\nu = \sum_{k=1}^{m} \alpha_k^{(\nu)} \ldots^{(k)}, \quad \alpha^{(\nu)} \geq 0, \quad \alpha_1^{(\nu)} + \ldots + \alpha_m^{(\nu)} = 1,$$

where

(3.3) $f/^{*'} = m_y + (M_y - m_r)\chi(S_k)$,

S, € T", and χC(S_V) denotes the characteristic function of S_k. We may thus conclude from Lemma 2.1 that, in order to obtain an inequality of the type (2.1) for functions f_y satisfying (3.1), it is sufficient to do so for functions of the form (3.3).

With the help of this procedure, we shall prove the following result for the case n = 2. This restriction is, unfortunately, not merely a matter of convenience; for larger n, the difficulties encountered in determining the exact constants mount rapidly.

Theorem 3,1. Let (X, ^ A) be a finite positive measure space, and let <j^(t) (V= 1,2) denote a function which vanishes for t = 0 and is continuous, non-decreasing and convex for t >^ 0. If

(3.4) $0 < m_y \le f_v \le M, < \infty, = 1, 2,$

and (f)$_v$ (f^)eL^1 (X, X, A), then, for any two positive constants A-JJA^, we have the sharp inequality

(3.5) $A_x \int_X (^(f^dA. + A_2 \int_X <|)_2(f_2)d/t \le C \int_X f^d$,

where

(3.6) C = max[6_1^6_2, 6_3, 6_4]

and

(3.7) 6_1 = ^ f ^ A (m l) + A_2(|>$_2$(m_2)], 6_2 - M^ A ^ C M ^ + A_2(J)$_2$(M_2)],

(3.8) 6_3 = M^jtA^CM^ + A_2(f)$_2$(m_2)]$_3$ 5_4 = ^ [A ^ ^) + A ^ (M_2)]

In view of what was said before, the exact constant C in (3.5) can be characterized by

$$(3.9) \qquad C = \sup \frac{\lambda_1 \int_X \phi_1(f_1)\,d\mu + \lambda_2 \int_X \phi_2(f_2)\,d\mu}{\int_X f_1 f_2\,d\mu} ,$$

where

$$(3.10) \qquad f_1 = m_1 + (M_1 - m_1)\,\chi(S_1), \quad f_2 = m_2 + (M_2 - m_2)\,\chi(S_2)$$

and S_1, S_2 may be identified with any set in Σ.

Since $\mu(X)$ is finite, we may normalize μ by the condition $\mu(X) = 1$; evidently, this does not affect the value of the right-hand side of (3.9). If we set $\eta_k = \mu(S_k)$, $k = 1,2$, we then have $0 \leq \eta_k \leq 1$ and

$$(3.11) \qquad \int_X \phi_k(f_k)\,d\mu = \phi_k(m_k) + \eta_k[\phi_k(M_k) - \phi_k(m_k)], \quad k = 1,2.$$

Furthermore,

$$(3.12) \qquad \int_X f_1 f_2\,d\mu = m_1 m_2 + m_1(M_2 - m_2)\,\eta_2 + m_2(M_1 - m_1)\,\eta_1 + (M_1 - m_1)(M_2 - m_2)\,\mu(S_1 \cap S_2)$$

Since

$$\mu(S_1 \cap S_2) \geq \max[0, \mu(S_1) + \mu(S_2) - \mu(X)] = \max[0, \eta_1 + \eta_2 - 1],$$

It follows from (3.12) that

(3.13) $\min \frac{\ldots}{\ldots} = m_1 m_2 + m_1 (M_2 - m_2)\gamma_2 + m_2 (M_1 - m_1)\gamma_1 + \gamma (M_1 - m_1)(M_2 - m_2),$

where

(3.14) $$\gamma = \max[0, \gamma_1 + \gamma_2 - 1].$$

In view of (3.11) and (3.13), (3.9) is equivalent to

(3.15) $$C = \sup_{0 \le \gamma_1, \gamma_2 \le 1} \phi(\gamma_1, \gamma_2),$$

where

(3.16)
$$\phi(\gamma_1, \gamma_2) = \frac{A_1\{\phi_1(m_1) + \gamma_1[\phi_1(M_1) - \phi_1(m_1)]\} + A_2\{\phi_2(m_2) + \gamma_2[\phi_2(M_2) - \phi_2(m_2)]\}}{m_1 m_2 + m_1(M_2 - m_2)\gamma_2 + m_2(M_1 - m_1)\gamma_1 + \gamma(M_1 - m_1)(M_2 - m_2)}$$

and γ is defined by (3.14).

In the square $0 \le \gamma_1, \gamma_2 \le 1$ the function $\phi(\gamma_1, \gamma_2)$ is, in both γ_1 and γ_2 a rational function of order 1, except along the diagonal $\gamma_1 + \gamma_2 = 1$, where--because of the expression (3.14)--its partial derivatives are discontinuous. Accordingly, $\phi(\gamma_1, \gamma_2)$ can attain its maximum in the square only at one of the corners or along the diagonal $\gamma_1 + \gamma_2 = 1$. Since $\phi(\gamma_1, 1 - \gamma_1)$ is again a rational function of order 1 for $\gamma_1 \in [0,1]$, its maximum coincides with the larger of the two values $\delta_3 = \phi(1,0)$, $\delta_4 = \phi(0,1)$. The maximum of $\phi(\gamma_1, \gamma_2)$ in the square is therefore attained at one of the four corners. Since the values of $\phi(\gamma_1, \gamma_2)$ at these points are the numbers $\delta_1, \delta_2, \delta_3, \delta_4$ in (3.7) and (3.8), this completes the proof of Theorem 3.1.

4. As an example for the application of Theorem 3.1, we derive the inequality

$$m_2M_2[M_1M_2^{q-1} - m_1m_2^{q-1}]\int_X f_1^p\,d\mu + m_1M_1[M_2M_1^{p-1} - m_2m_1^{p-1}]\int_X f_2^q\,d\mu$$

(4.1)

$$\le [M_1^pM_2^q - m_1^pm_2^q]\int_X f_1f_2\,d\mu$$

($p^{-1} + q^{-1} = 1$) of Diaz, Goldman and Metcalf [5]. If we set $\phi_1(t) = t^p$, $\phi_2(t) = t^q$, $h_1 = m_2M_2[M_1M_2^{q-1} - m_1m_2^{q-1}]$, $h_2 = m_1M_1[M_2M_1^{p-1} - m_2m_1^{p-1}]$, a computation shows that the constants (3.8) take the values

$$\delta_3 = \delta_4 = M_1^pM_2^q - m_1^pm_2^q. \tag{4.2}$$

If δ_1 and δ_2 are the constants (3.7), it is found, after some simplification, that

$$\delta_1 - \delta_3 = M_1^pM_2^q[(\lambda_1^{-1} - 1)(\lambda_2^{-1} - 1) - \lambda_1^{-1}\lambda_2^{-1}(\lambda_1^{1-p} - 1)(\lambda_2^{1-q} - 1)] \tag{4.3}$$

and

$$\delta_2 - \delta_3 = M_1^pM_2^q[\lambda_1\lambda_2(1 - \lambda_1^{p-1})(1 - \lambda_2^{q-1}) - (1 - \lambda_1)(1 - \lambda_2)], \tag{4.4}$$

where $\lambda_1 = m_1/M_1$, $\lambda_2 = m_2/M_2$.

Since, for $r \ge 0$, $x \ge 1$, $x(x^r - 1) \ge r(x - 1)$, we have (because of $p \ge 1$, $q \ge 1$, $\lambda_1 \le 1$, $\lambda_2 \le 1$)

$$\lambda_1^{-1}\lambda_2^{-1}(\lambda_1^{1-p} - 1)(\lambda_2^{1-q} - 1) \ge (p - 1)(q - 1)(\lambda_1^{-1} - 1)(\lambda_2^{-1} - 1)$$

$$= (\lambda_1^{-1} - 1)(\lambda_2^{-1} - 1).$$

It thus follows from (4.3) that $5_{\pm} - 63 \leq 0$. Similarly, the inequality $x(l - x^r) \leq r(l - x)$ $(r \geq 0,\ 0 \leq x \leq 1)$ shows that

$$\lambda_1\lambda_2(1 - \lambda_1^{p-1})(1 - \lambda_2^{q-1}) \leq (p - 1)(q - 1)(1 - \lambda_1)(1 - \lambda_2) =$$

$$= (1 - \lambda_1)(1 - \lambda_2).$$

In view of (4.4), this implies $5_2 - §_3 \leq 0$. Accordingly, the constant C defined in (3.6) has, in our case, the value (4.2), and this establishes the inequality (4.1).

It is of interest to find the cases in which (4.1) becomes an equality. As shown above, $§_3$ = (j)(l,0) and 5_4 = <j)(0,l), where $\phi(\zeta_1, \zeta_2)$ is the function (3.16). Since §(y±> 1 ' f^ is a rational function of order 1 in ^ for ^,e[0,l], (j)(7\jl ~/^i) will reduce to a constant if (j)(0,l) = (|)(l,0), i.e., if 5_3 = 6^. In view of (4.2), we thus have §(^^, l - ? ±) =63 for all -^1.e[0,l]. The maximum $§_3$ of (j)(^1, ^2) is therefore attained at all points of the diagonal^? ^ + ^2 = 1^ an<^ i fc i s easily seen that this implies equality in (4.1) whenever f_1 and f_2 are of the form (3.10), and the measurable sets S_1', S_2 satisfy the conditions

(4.5) M(S₁) + μ(S₂) = ^L(X), utSj/l S₂) = 0.

Because of (1.8), inequality (4.1) implies an inverse Hölder inequality of the form (1.4), where C_p has the value (1.6). Equality in (1.4) is possible only if there is equality in both (1.4) and the geometric-arithmetic inequality used in (1.8).

An examination of these cases shows that there will be equality in (1.4) (with the value (1.6) for C_p) if f_1 and f_2 are of the form (3.10), where the sets S_1, S_2 satisfy (4.5) and the additional condition

$$\frac{\mu(S_1)}{\mu(S_2)} = \frac{\lambda_1}{\lambda_2}\left\{\frac{1 - \lambda_2\lambda_1^{p-1} - q^{-1}[1 - (\lambda_2\lambda_1^{p-1})^q]}{1 - \lambda_1\lambda_2^{q-1} - p^{-1}[1 - (\lambda_1\lambda_2^{q-1})^p]}\right\},$$

where $\lambda_1 = m_1/M_1$, $\lambda_2 = m_2/M_2$.

5. The inequalities (4.1) and (1.4) can be sharpened if, in addition to (3.4), the functions f_1, f_2 are subjected to certain other restrictions. As an example, we derive the following inverse Hölder inequality.

Theorem 5.1. Let $(X, \mathcal{S}, \mu)$ be a finite positive measure space, and let $f_1 \in L^p(X, \mathcal{S}, \mu)$, $f_2 \in L^q(X, \mathcal{S}, \mu)$, where $p, q > 0$, $p^{-1} + q^{-1} = 1$. If, in addition, f_1 and f_2 satisfy the conditions (3.4), and if the numbers ζ_1, ζ_2 $(0 \le \zeta_\nu \le 1)$ are defined by

(5.1) $$\int_X f_\nu \, d\mu = [m_\nu + \zeta_\nu(M_\nu - m_\nu)]\,\mu(X), \quad \nu = 1, 2,$$

then

(5.2) $$\left[\int_X f_1^p \, d\mu\right]^{\frac{1}{p}} \left[\int_X f_2^q \, d\mu\right]^{\frac{1}{q}} \le D \int_X f_1 f_2 \, d\mu$$

where

$$D = \frac{[m_1^p + (M_1^p - m_1^p)\gamma_1]^{\frac{1}{p}}\,[m_2^q + (M_2^q - m_2^q)\gamma_2]^{\frac{1}{q}}}{m_1 m_2 + m_1(M_2 - m_2)\gamma_2 + m_2(M_1 - m_1)\gamma_1 + \delta(M_1 - m_1)(M_2 - m_2)} \tag{5.3}$$

and

$$\delta = \max[0, \gamma_1 + \gamma_2 - 1]. \tag{5.4}$$

This result is a direct consequence of Lemma 2.2 and the fact [8] that the functions f_ν satisfying (3.4) and (5.1) (with a fixed γ_ν) form a convex set which is spanned by the functions

$$F_\nu = m_\nu + (M_\nu - m_\nu)\chi(S_\nu), \qquad \nu = 1,2 \tag{5.5}$$

where $S_\nu \in \Sigma$, and

$$\mu(S_\nu) = \gamma_\nu \mu(X), \qquad \nu = 1,2. \tag{5.6}$$

Accordingly, the constant p in (5.2) is given by

$$D = \sup \frac{[\int_X F_1^p d\mu]^{\frac{1}{p}}\,[\int_X F_2^q d\mu]^{\frac{1}{q}}}{\int_X F_1 F_2 d\mu}. \tag{5.7}$$

By (5.5) and (5.5),

$$\int_X F_1^p d\mu = [m_1^p + \gamma_1(M_1^p - m_1^p)]\mu(X),$$

$$\int_X F_2^q d\mu = [m_2^q + \gamma_2(M_2^q - m_2^q)]\mu(X).$$

The smallest possible value of the denominator in (5.7) was earlier found to be equal to (3.13). Inserting these values in (5.7), we obtain (5.3). In view of the derivation of (3.13), there will be equality in (5.2) in the following two cases:

(a) $\mu(S_1) + \mu(S_2) \le \mu(X)$, $\mu(S_1 \cap S_2) = 0$; (b) $\mu(S_1) + \mu(S_2) > \mu(X)$, $\mu(S_1 \cap S_2) = \mu(S_1) + \mu(S_2) - \mu(X)$ (i.e., S_1 and S_2 overlap as little as possible). This completes the proof of Theorem 5.1.

It is of interest ot observe the behavior of (5.3) as $m_1 \to 0$, $m_2 \to 0$. According to (5.4), the denominator of (5.3) will vanish if $\mu_1 + \mu_2 \le 1$, and this shows that there cannot be an inequality of the type of (5.2) in this case. If $\mu_1 + \mu_2 > 1$, we obtain the following result:

If $f_1 \in L^p(X, \mathfrak{S}, \mu)$, $f_2 \in L^q(X, \mathfrak{S}, \mu)$, $0 < f_i \le 1$ $(i = 1, 2)$, and

$$\int_X (f_1 + f_2 - 1)\, d\mu > 0,$$

then

$$\left(\int_X [illegible]\right) \left(\int_X [illegible]\right) \int (f_1 + [illegible])\, d\mu \le \int f_1 f_2\, d\mu .$$

6. In the present section we consider inverse inequalities for concave functions.

Theorem 6.1. Let $f_1, \ldots, f_n$ be continuous non-negative concave functions on a real interval I. If $p_\nu > 0$ $(\nu = 1, \ldots, n)$, $p_1^{-1} + \cdots + p_n^{-1} = 1$, then

$$\prod_{\nu=1}^{n} \left[\int_I f_\nu^{p_\nu}\, dx \right]^{p_\nu^{-1}} \le C_n \int_I \prod_{\nu=1}^{n} f_\nu\, dx \tag{6.1}$$

where

(6.2) $$C_n = \frac{(n+1)!}{([f]D^2 \prod_{\nu=1}^{n} (p_\nu + 1)^{\frac{1}{p_\nu}}}$$

There will be equality in (6.1) if $f_\nu = x$ for [$\frac{n}{2}$] of the subscripts ν, and $f_\nu = 1 - x$ in the other cases.

For $n = 2$, this reduces to the result of Bellman quoted in Section 1. It may also be noted that we derive our result without the assumption, made by Bellman, that the f_ν vanish at the end-points of I.

To simplify the writing, we shall assume that I is the interval [0,1]; evidently, this amounts only to a trivial normalization which does not affect (6.1). We shall obtain Theorem 6.1 as a corollary of the following stronger result.

Theorem 6.2. Let $f_1, \ldots, f_n$ be continuous non-negative concave functions in [0,1], and let

(6.3) $$\int_0^1 f_\nu dx = \frac{1}{2}, \qquad \nu = 1, \ldots, n.$$

If $p_\nu > 0$ and

(6.4) $$\sum_{\nu=1}^{n} \frac{1}{p_\nu} = 1,$$

then

(6.5) $$\sum_{\nu=1}^{n} \left(1 + \frac{1}{p_\nu}\right) \int_0^1 f_\nu^{p_\nu} dx \leq B_n \int_0^1 \left(\prod_{\nu=1}^{n} f_\nu\right) dx,$$

where

(6.6) $$B_n = \frac{(n+1)!}{([\frac{n}{2}]!)^2}$$

There will be equality in (6.5) if $f_\nu = x$ for $[\frac{n}{2}]$ of the subscripts ν, and $f_\nu = 1 - x$ in the other cases.

Inequality (6.1) is obtained from (6.5) by means of the geometric-arithmetic inequality (2.2), according to which

(6.7) $$\prod_{\nu=1}^{n}\left[(1+p_\nu)\int_0^1 f_\nu^{p_\nu}\,dx\right]^{1/n} \le \frac{1}{n}\sum_{\nu=1}^{n}(1+p_\nu)\int_0^1 f_\nu^{p_\nu}\,dx.$$

Fortunately, the functions f_ν for which (6.5) becomes an equality--x and $1 - x$--also give equality in (6.7); indeed, for both $f_\nu = x$ and $f_\nu = 1 - x$, we have

$$(1+p_\nu)\int_0^1 f_\nu^{p_\nu}\,dx = 1,$$

and the equality in (6.7) follows from (6.4). As a result, these functions also give equality in (6.1). We also note here that the normalization (6.3) has no effect on (6.1).

It is evidently sufficient to prove Theorem 6.2 in the case in which the curves $y = f_\nu(x)$ are concave polygonal lines. Such functions f_ν can be written in the form

(6.8) $$f_\nu(x) = \sum_{k=1}^{n} \alpha_k^{(\nu)} g(x, t_k^{(\nu)}), \quad \alpha_k^{(\nu)} > 0,$$

where the $t_k^{(\nu)}$ are numbers in [0,1] and $g(x,t)$ is defined by

$$g(x,t) = \frac{x}{t}\ (0 \le x \le t),\quad g(x,t) = \frac{1-x}{1-t}\ (t \le x \le 1),\quad t \in (0,1),$$

(6.9)

$$g(x,0) = 1 - x,\quad g(x,1) = x$$

Since, for all $t \in [0,1]$,

$$\int_0^1 g(x,t)\,dx = \frac{1}{2},$$

we have

$$\int_0^1 f_\nu(x)\,dx = \frac{1}{2} \sum_{k=1}^{n} \alpha_k^{(\nu)}.$$

The function f_ν will thus be normalized in accordance with (6.3) if

$$\sum_{k=1}^{n} \alpha_k^{(\nu)} = 1. \tag{6.10}$$

In view of (6.8), (6.10) and Lemma *2.1,* it is sufficient to prove (6.5) in the special case in which

$$f_\nu(x) = g(x,t_\nu),\quad t_\nu \in [0,1].$$

Since

$$\int_0^1 g^{p_\nu}(x,t)\,dx = \frac{1}{p_\nu + 1},$$

(6.4) shows that (6.5) reduces in this case to

$$\psi(t_1,\dots,t_n) \ge B_n^{-1}, \tag{6.11}$$

where

$$\psi(t_1,\dots,t_n) = \int_0^1 \Big[\prod_{\nu=1}^{n} g(x,t_\nu)\Big]dx \tag{6.12}$$

and B_n is the constant (6.6).

Since V ^ i * . . ., t_n) is continuous for t^cfl^O] (= l, . . . , n), it is sufficient to prove (6.11) under the assumption $0 < t_y < 1$. If te(O,l) and g(x,t) is defined by (6.9), then t(l - t)g(x,t) is the Green's function of the differential operator L y S y'' for the interval [0,1] and the boundary conditions y(0) = y(l) = 0, Hence,

$$\hat{\ }\frac{2}{dt} \ [\ f_{o}^{1} R(x)g(x,t)dx] = - \frac{\hat{\ }(\hat{\ }}{t(\quad t)}, \quad te(O,l),$$

for any function R(x) which is continuous in [0,1]. Applying this to (6.12), we find that

$$t_k(1 - V - \hat{\ }\frac{2}{dtk}2\ y\hat{\ }' - 'K) = - [\ i_{\circ}^{1} < \overline{i}_{V\hat{\ }k}\ g(x,t_\nu)]dx,$$

and this shows that]ν is a concave function of $t_{,\kappa}$ in (0,1). Hence, Ψ cannot have a local minimum for $t_{,\kappa}$e(0,l). Applying this argument, in turn, to all the variables $t_1, \ldots, t_n$, we arrive at the conclusion that the expression (6.12) can attain its minimum only if each of the variables t_ν is either 0 or 1. By (6.9), this corresponds either to g(x,t^) = x or g(x,ty) = 1 - x. In view of (6.11) and (6.12) we thus have

$$B_n^{-1} = \min \Psi(t_1, \ldots, t_n) = \min_{1 \le k \le n} \int_0^1 x^k(l - x)^{n \hat{\ } n} dx$$

$$= \min_{1 \le k \le n} \frac{kl\ (n-k)\ 1}{(n+1)!} - \frac{{}^{U}2^{Jm)}}{(n+1)!},$$

where the minimum is attained for $k = [\frac{n}{2}]$. Hence, the constant B_n has the value (6.6), and the proof of Theorem 6.2 (and its corollary, Theorem 6.1) is complete.

7. A function is said to be superharmonic in a region D of the Euclidean space E^m $(m \geq 1)$ if f has continuous first and second partial derivatives with respect to the coordinates and if $\Delta f \leq 0$, where Δ is the Laplace operator in E^m. For $m = 1$, the notions of concavity and superharmonicity coincide, and it is therefore natural to ask whether there exist results analogous to Theorem 6.1 for superharmonic functions in spaces of dimension $m \geq 2$. This question was considered by Bellman [2], p. 42], who showed that, for functions f_1 and f_2 of this type, there exists an inequality

$$(7.1) \qquad \left(\int_D f_1^p dV\right)^{p^{-1}} \left(\int_D f_2^q dV\right)^{q^{-1}} \leq C \int_D f_1 f_2 dV,$$

where $p^{-1} + q^{-1} = 1$, dV is the volume element, and C is given by

$$(7.2) \qquad C = \sup_{\zeta, \eta \in D} \frac{\left[\int_D g^p(z,\zeta)\, dV\right]^{p^{-1}} \left[\int_D g^q(z,\eta)\, dV\right]^{q^{-1}}}{\int_D g(z,\zeta) g(z,\eta)\, dV},$$

where $g(z,\zeta)$ is the harmonic Green's function of D with zero boundary values. The question whether or not the inequality (7.1) is meaningful thus depends on the finiteness of the right-hand side of (7.2). We shall show that, if $m = 2$ and D is a disk, C = co. Accordingly, no inequality of the type (7.1) can exist for superharmonic functions in a disk.

We take D to be the unit disk, and we use complex notation. We shall compute the right-hand side of (7.2) under the assumption that *? = 0 and p>_q; evidently, this is sufficient for our purpose. Since g(z,O) =—log|z|, the denominator of (7.2) has the form

(7.3) --A (5) •= - j^r iog|z}g(z,^)dV.
D

To evaluate this integral, we set

$$u(r) = \frac{1}{J}(1 - r^{9}) - \frac{r^2}{4}\log r, \quad r = |z|,$$

and we observe that $\wedge^2 u$ = —log r and u(l) = 0. Applying Green's formula, and noting that $S]_2$ g = 0 and g = u = 0 for r = 1, we obtain

$$A(\zeta) \quad \int_D (\nabla^2 u)g(z,t)dV = -\lim_{\epsilon \to 0}\Big[\int_{|z|=\epsilon} \Big(g\frac{\partial u}{\partial n} - u\frac{\partial q}{\partial n}\Big)ds + \int_{|z-\zeta|=\epsilon} \Big(g\frac{\partial u}{\partial n} - u\frac{\partial q}{\partial n}\Big)ds\Big]$$

$$= -2\pi u(\zeta) = \frac{\pi}{2}[(1 - |\zeta|^2) + |\zeta|^2 \log |\zeta|],$$

and thus

(7.4) $$A(\zeta) \le K_1(1 - |\zeta|^2),$$

where K_1 is a positive constant.

To compute the numerator of (7.2), we note that

$$(7.5)\quad B_p(\zeta) \equiv \int_D g^p(z,\zeta)\,dV_z = \int_D \left(\log\left|\frac{1-z\bar{\zeta}}{z-\zeta}\right|\right)^p dV_z,$$

or, with the substitution

$$t = \frac{\zeta - z}{1-\bar{\zeta}z},$$

$$B_p(\zeta) = (1-|\zeta|^2)^2 \int_D \left(\log\frac{1}{\rho}\right)^p \frac{dV_t}{|1-\bar{\zeta}t|^4}, \quad t = \rho e^{i\theta}.$$

Because of

$$2^{\pi}\left(\quad 7 \wedge 4 = 2. \sum_{V=0}^{\infty} (v+i)^2 isi^{2v}/^{v}, \right.$$
$$\mathfrak{3} \quad u\text{-}jtr$$

this yields

$$B_p(i) = 27r(l - |3|^2)^2 \sum_{V=0}^{\infty} (v+1)^2 |t|^{2V} \int_0^1 \left(\log\frac{1}{\rho}\right)^p \rho^{2\nu+1} d\rho$$

$$= 2^{-p}\pi\,\Gamma(p+1)(1-|*|^2)^2 \sum_{V=0}^{\infty} (\nu+1)^{1-p}|\zeta|^{2\nu}.$$

If $p = q = 2$, we thus have

$$B_2(\wedge) \geq K_2(l - |S|^2)\log\frac{1}{1-|\zeta|^2},$$

where K_2 is a positive constant. If $p > 2$, we use the fact

that

$$\sum_{V=0}^{\infty} \qquad - \ 1$$

and therefore

$$B_p(\zeta) \geq K_3(1 - |\zeta|^2)^{2},$$

where K_3 is another constant. In view of (7.2), (7.3), (7.4) and (7.5), we finally obtain

$$C \geq \frac{[B_p(\zeta)]^{\frac{1}{p}}\,[B_q(0)]^{\frac{1}{q}}}{A(\zeta)} \geq K_4(1 - |\zeta|^2)^{\frac{2}{p} - 1}$$

for $p > 2$, and

$$C \geq K_5\left[\log \frac{1}{1-|\zeta|^2}\right]^{\frac{1}{2}}$$

for $p = 2$. Hence, $C = \infty$ in all cases.

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