A Note on Cauchy»s Theorem

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Laws of balance in continuum physics are often expressed in equations of the form

$$\int_{\partial D} f(x, \mathbf{n}_D(x))\, dA_x = \int_D b(x)\, dV_x \ , \tag{1}$$

where D is an arbitrary regular region in some open region R of space and $\mathbf{n}_D$ is the unit outward normal vector to the surface S of D. We here assume that the functions f and b are scalar-valued with $f(x,\mathbf{n})$ defined for all x in R and all unit vectors $\mathbf{n}$ and $b(x)$ defined for all x in R.

A fundamental consequence of (1) was established by Cauchy [1823, 1827]:

Theorem: Suppose that $f(\cdot,\mathbf{n})$ (for each $\mathbf{n}$) and $b(\cdot)$ are contin-*uous on* R, *and let* (1) *hold for every tetrahedron* D *in* R. *Then there exists a continuous vector field* $\mathbf{f}$ *on* R *such that*

$$f(x,\mathbf{n}) = \mathbf{f}(x) \cdot \mathbf{n}$$

for all unit vectors $\mathbf{n}$ *and every* x *in* R.

We remark that the corresponding theorem for the case in which f and b are vector-valued (which is actually the case considered by Cauchy) follows as a corollary of the above theorem.

Within the context of mechanics Cauchy's theorem implies the existence of a stress tensor. Unfortunately, in this instance the requirement that $f(\cdot,\mathbf{n})$ be continuous is far too stringent for applications (e.g. to problems involving shock waves). In this note we show that the theorem remains true under much weaker hypoth-eses, hypotheses that may be expected to be valid in most applications.

Theorem: Suppose that f(«,n) (for each n) and b(«) are integrable over R, and let (1) hold for every tetrahedron D in R for which the left-hand integral exists. Then there exists c* vector field $\tilde{f}$ on R such that for each $\tilde{n}$

$$f(x,n) = f(x) \cdot n$$

for almost every x irv R.

Pro&f: We shall construct continuous functions satisfying Cauchy»s hypotheses by use of the Friedrichs-Sobolev averaging operator. For each 6 > 0 it is not difficult to exhibit a class C°° function p^{δ} such that $p^{\delta} \geq 0$, $p^{\delta} = 0$ outside the closed ball of radius 6, and

$$\int_E p^6 dV = 1 ,$$

where E denotes the entire space. Our assumptions suffice to quarantee that

$$f^6(x,\tilde{n}) = \int_E P^6(x - Y) f(y,\tilde{n}) dV_y$$

and

$$b^6(x) = \int_E p^{\delta}(x - y) b(y) dV_y$$

exist for all x in R. (Here we have extended $f(\#_3\tilde{n})$ and b(0 to all of E by requiring that they vanish outside of R.) The functions $f^{\delta}(«,\tilde{n})$ and $b^{\delta}(\bullet)$ are of class C°° and approach $f(*,\tilde{n})$ and b(0 in $L^1(R)$ as 6-*0.

Choose e > 0 and let R_ϵ denote the set of all y such that R contains the closed ball of radius *e* at y. (We require that e be such that R_ϵ is non-empty.) We now show that f^δ and b^6 obey (1) in R_ϵ whenever $6 < G$. Let D be any tetrahedron in R_ϵ and choose 6 with $0 < 6 < e$. If S is the boundary of D let

$$I = \int_S f^6(x, n_S(x))\,dA_x. \tag{2}$$

Then, using Fubini's Theorem,

$$I = \int_S \int_E f(y, n_S(x))\,p^6(x - y)\,dV_y\,dA_x$$

$$= \int_S \int_E f(x - z, n_S(x))\,p^6(z)\,dV_z\,dA_x$$

$$= \int_E p^6(z) \int_S f(x - z, n_S(x))\,dA_x\,dV_z$$

since $p^\delta(z)f(x - z, n)$ (for each fixed n) is integrable in (x,z). If we write D-z for the translate of the set D by amount z, then it is clear that D-z has boundary S-z and $n_S(x) = n_{S-z}(x-z)$. Thus

$$I = \int_E \rho^\delta(z) \int_S f(x-z, n_S(x-z))\,dA_x\,dV_z$$

$$= \int_E p^\delta(z) \int_{S-z} f(x, n_{S-z}(x))\,dA_x\,dV_z.$$

We may conclude from Fubini's Theorem that the inner integral exists for almost every z in R; hence (1) implies

$$I = \int_E P^6(z) \int_{D-z} b(x)\,dV_x\,dV_y$$

since $D-z$ is contained in R whenever $p^{\delta}(z) \neq 0$. Thus, as above,

$$I = \int_E p^{\delta}(z) \int_D b(x-z)\, dV_x\, dV_z$$

$$= \int_D b^{\delta}(x)\, dV_x . \qquad (3)$$

By (2) and (3) f^{δ} and b^{δ} satisfy (1) for every tetrahedron D in R_{ϵ}; hence Cauchy's Theorem implies the existence of a vector field $\underset{\sim}{f}^{\delta}$ on R_{ϵ} such that

$$f^{\delta}(x,\underset{\sim}{n}) = \underset{\sim}{f}^{\delta}(x) \cdot \underset{\sim}{n} \qquad (4)$$

for all x in R_{ϵ} and all unit vectors $\underset{\sim}{n}$. In fact, if $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_p$ is an orthonormal basis for E, then

$$\underset{\sim}{f}^{\delta}(x) = \sum_{i=1}^{p} f^{\delta}(x,\underset{\sim}{e}_i)\, \underset{\sim}{e}_i . \qquad (5)$$

Since $f^{\delta}(\cdot,\underset{\sim}{n})$ tends to $f(\cdot,\underset{\sim}{n})$ in $L^1(R)$ and hence in $L^1(R_{\epsilon})$ we conclude from (5) that $\underset{\sim}{f}^{\delta}$ tends to the vector field

$$\underset{\sim}{f} = \sum_{i=1}^{p} f(\cdot,\underset{\sim}{e}_i)\, \underset{\sim}{e}_i$$

in $L^1(R_{\epsilon})$. The equality (4) then implies that for any $\underset{\sim}{n}$

$$f(x,\underset{\sim}{n}) = \underset{\sim}{f}(x) \cdot \underset{\sim}{n}$$

for almost every x in R_{ϵ}. Fix $\underset{\sim}{n}$ and let N_{ϵ} denote the set of all x in R_{ϵ} such that

$$f(x,\underset{\sim}{\eta}) \neq f(x) \cdot \underset{\sim}{\eta} \quad . \tag{6}$$

Then (6) holds for a point x in the region R only if x is in

$$N = \bigcup_{n=1}^{\infty} N_{1/n} \quad .$$

But each $N_{1/n}$ is of zero volume and $\{N_{1/n}\}$ is an increasing sequence; hence N is of zero volume and the proof is complete.

References

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