Representations of Certain Isotropic Tensor Functions

by

Walter Noll

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1. Results. Let $\mathcal{V}$ be an inner product space of dimension n over the field $\mathcal{R}$ of real numbers. The inner product of u and v is written $u \cdot v$. We denote the $\frac{n(n+1)}{2}$-dimensional space of symmetric tensors (i.e. linear transformations) over $\mathcal{V}$ by $\mathcal{S}$ and the orthogonal group of $\mathcal{V}$ by $\mathcal{O}$. A function whose domain is a Cartesian product made up from $\mathcal{R}$, $\mathcal{V}$, and $\mathcal{S}$ and whose values are in $\mathcal{R}$, $\mathcal{V}$, or $\mathcal{S}$ is called isotropic if it is covariant under the action of the orthogonal group $\mathcal{O}$. For example, $g: \mathcal{S} \to \mathcal{R}$ is isotropic if

$$g(QAQ^T) = g(A) \tag{1}$$

holds for all $A \in \mathcal{S}$ and all $Q \in \mathcal{O}$. (Q^T denotes the transpose of Q.) The function $f: \mathcal{S} \to \mathcal{S}$ is isotropic if

$$f(QAQ^T) = Qf(A)Q^T \tag{2}$$

for all $A \in \mathcal{S}$ and all $Q \in \mathcal{O}$.

It is well known (cf. [1], p. 28 and p. 32) that functions of the two types just described are isotropic if and only if they have certain representations in terms of real valued functions of several real variables. Such representations have important applications in Continuum Mechanics.

It is the purpose of this note to prove the following representation theorems for isotropic functions of the type $\varphi: \mathcal{S} \times \mathcal{V} \to \mathcal{R}$ and $\psi: \mathcal{S} \times \mathcal{V} \to \mathcal{V}$:

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Theorem I: The function $\varphi : \mathcal{S} \times \mathcal{V} \rightarrow \mathcal{R}$ is isotropic, i.e., satisfies

$$\varphi(QAQ^T, Qu) = \varphi(A,u) \tag{3}$$

for all $A \in \mathcal{S}$, all $u \in \mathcal{V}$ and all $Q \in \mathcal{O}$ if and only if there is a function $\Phi : \mathcal{R}^{2n} \rightarrow \mathcal{R}$ such that for all $A \in \mathcal{S}$ and all $u \in \mathcal{V}$

$$\varphi(A,u) = \Phi(I_1(A), \ldots, I_n(A), u \cdot u, u \cdot Au, \ldots, u \cdot A^{n-1}u), \tag{4}$$

where $I_j(A)$ is the jth principal invariant of A.

Theorem II: The function $\psi : \mathcal{S} \times \mathcal{V} \rightarrow \mathcal{V}$ is isotropic, i.e., satisfies

$$\psi(QAQ^T, Qu) = Q\psi(A,u) \tag{5}$$

for all $A \in \mathcal{S}$, all $u \in \mathcal{V}$, and all $Q \in \mathcal{O}$, if and only if there are n isotropic functions $\varphi_k : \mathcal{S} \times \mathcal{V} \rightarrow \mathcal{R}$, $k = 0, 1, \ldots, n-1$, such that for all $A \in \mathcal{S}$ and all $u \in \mathcal{V}$

$$\psi(A,u) = \Big(\sum_{k=0}^{n-1} \varphi_k(A,u) A^k\Big) u. \tag{6}$$

2. Proofs. It is a matter of trivial verification to show that the functions φ or ψ satisfy (3) or (5) if they have representations (4) or (6), respectively. The existence of such representations for given isotropic functions remains to be shown.

To prove Theorem I it is sufficient to show that if $A, B \in \mathcal{S}$ and $u, v \in \mathcal{V}$ satisfy

$$I_j(A) = I_j(B), \quad j = 1, \ldots, n \tag{7}$$

and

$$u \cdot A^k u = v \cdot B^k v, \quad k = 0, \ldots, n-1, \tag{8}$$

then there exists a $Q \in \mathcal{O}$ such that

$$B = QAQ^T, \quad v = Qu. \tag{9}$$

It is well known (cf. [1], p. 28) that (7) implies the existence of a $Q_1 \in \mathcal{O}$ such that

$$B = Q_1 A Q_1^T \tag{10}$$

It is also well known (cf. [2], p. 156, Theorem 2), that the orthogonal projections E_i of the spectral resolution

$$A = \sum_{i=1}^{r} a_i E_i \quad (r \leq n) \tag{11}$$

can be expressed as polynomials of degree $\leq r$ in A:

$$E_i = p_i(A) \quad i = 1, \ldots, r. \tag{12}$$

Therefore, since $E_i^2 = E_i$ and $E_i \in \mathcal{S}$, we can infer from (8) and (10) that

$$\begin{aligned} |E_i u|^2 &= E_i u \cdot E_i u = u \cdot E_i^2 u = u \cdot p_i(A) u \\ &= v \cdot p_i(B) v = Q_1^T v \cdot p_i(A) Q_1^T v \\ &= Q_1^T v \cdot E_i(Q_1^T v) = |E_i Q_1^T v|^2 \end{aligned} \tag{13}$$

Let $\mathcal{U}_i$ = range E_i. Since $E_i u$ and $E_i Q_1^T v$ both belong to $\mathcal{U}_i$ and have, by (13), the same magnitude, there exist orthogonal transformations of the $\mathcal{U}_i$ which map $E_i u$ into $E_i Q_1^T v$. The direct sum $Q_2 \in \mathcal{O}$ of these transformations leaves every $\mathcal{U}_i$ invariant and hence satisfies

T (14)

It is clear from (14) that $E_i Q_2 u = U^\wedge Q_i^T v$ and hence, after summing over i, that

$$Q_2 u = Q?v. \tag{15}$$

Moreover, (11) and (14)? imply

$$A = Q_2 A Q^\wedge \tag{16}$$

It follows from (10), (16) and (15) that (9) holds with the choice Q ≐ Q-xQ^e CTy which completes the proof of Theorem I.

To prove Theorem II assume that Ae J8 with spectral resulution (11) and UG 19 are given. Consider, for some j, 1 ≤ j ≤ r, the vector $E_j u \in \mathcal{U}_j$ = range E_j. The only vectors in $\mathcal{U}_j$ that are left invariant by all those orthogonal transformations of $\mathcal{U}_j$ that leave E.u invariant are just the scalar multiples of E.u. Hence, if v.eU- and if

$$Q v_j = v_j \tag{17}$$

holds for all Qe O^ that satisfy

$$QE_j u = E_j u \quad \text{and} \quad Qw = w \quad \text{if} \quad w \in \mathcal{U}_i. \tag{18}$$

we can conclude there is a number β_j such that

$$v_j = \beta_j E_j u. \tag{19}$$

Now, if (18) holds, it is easily seen that $QAQ^T = A$ and $Qu = u$. For such a choice of Qe cr (5) states that

$$Qv = v, \text{ where } v = A\& (A,u). \tag{20}$$

Hence, if we put $v_j = E_j v$ and observe $E_j Q = Q E_j$, we see that (17) holds. We can conclude that (19) must be valid for $v_j = E_j v$. Summing over j we get

$$v = \sum_{j=1}^{r} \varphi_j E_j u . \qquad (21)$$

Substituting the polynomial representations (12) into (21) we obtain

$$v = \varphi(A,u) = \left(\sum_{k=0}^{n-1} c_k A^k \right) u , \qquad (22)$$

where the c_k are the coefficients of the polynomial $\sum_{j=1}^{r} \varphi_j p_j(x)$. These coefficients depend, of course, on the original choice of A and u.

We can define an equivalence relation on the set $\mathcal{A} \times \mathcal{V}$ by putting $(A,u) \sim (A_0,u_0)$ if there exist a $Q \in \mathcal{O}$ such that

$$A = Q A_0 Q^T , \quad u = Q u_0 .$$

Selecting a particular member (A_0,u_0) from each of the resulting equivalence classes, we can construct the representation (22) for the choice $\hat{A} = A^0$, $\hat{u} = u^0$. For any pair (A,u) equivalent to (A^0,u^0) we set $\varphi_k(A,u)$ equal to the coefficient c_k constructed with (A^0,u^0). It is clear that the $\varphi_k : \mathcal{A} \times \mathcal{V} \to \mathbb{R}$ thus obtained are isotropic. Moreover, we have

$$\varphi(A,u) = \varphi(Q A_0 Q^T, Q u_0) = Q \varphi(A_0,u_0) = Q \sum_{k=0}^{n-1} \varphi_k(A,u) A_0^k u_0 =$$

$$= \sum \varphi_k(A,u) (Q A_0 Q^T)^k Q u_0 ,$$

i.e., the desired representation (6). Q.E.D.

References

[1] Truesdell, C. and W. Noll, 'The Non-linear Field Theories of Mechanics,' Encyclopedia of Physics, vol. III/3. Springer-Verlag 1965.

[2] Halmos, P. R., 'Finite-Dimensional Vector Spaces,' Van' Nostrand 1958.