The Schwarzian Derivative and Disconjugancy of n-th Order Linear Differential Equations

by

Meira Lavie

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1. Introduction

The paper deals with the number of zeros of a solution of the n-th order linear differential equation

$$(1.1) \quad y^{(n)}(z) + p_{n-2}(z)y^{(n-2)}(z) + \dots + p_0(z)y(z) = 0 \qquad n=2,3,\dots,$$

where the functions $p_j(z)$ $(j=0,1,\dots,n-2)$ are assumed to be regular in a given domain D of the complex plane. The differential equation (1.1) is called disconjugate in D, if no (nontrivial) solution of (1.1) has more than (n-1) zeros in D. (The zeros are counted by their multiplicity.)

The ideas of this paper are related to some papers by Nehari [5], [7], in which second order differential equations were considered. In [5], Nehari pointed out the following basic relationship. The function

$$(1.2) \qquad f(z) = \frac{y_1(z)}{y_2(z)},$$

where $y_1(z)$ and $y_2(z)$ are two linearly independent solutions of

$$(1.3) \qquad y''(z) + p(z)y(z) = 0,$$

is univalent in D, if and only if no solution of equation (1.3) has more than one zero in D, i.e., if and only if (1.3) is disconjugate in D.

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The coefficient $p(z)$ of (1.3) is expressed in terms of the function (1.2) by the identity

$$2p(z) = \{f,z\} \tag{1.4}$$

where $\{f,z\}$ denotes the Schwarzian derivative of $f(z)$ with respect to z, namely

$$\{f,z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 . \tag{1.5}$$

It is well known that the Schwarzian derivative (1.5) is invariant under a linear transformation

$$Tf = \frac{Af+B}{Cf+D}, \quad AD-BC \neq 0. \tag{1.6}$$

Thus, (1.4) is independent of our choice of the solutions $y_1(z)$ and $y_2(z)$ in (1.2).

By making use of the duality relationship between disconjugancy of (1.3) and univalence of (1.2) (for the necessary condition), and of an integral inequality (for the sufficient condition), Nehari proved the following theorem ([5], Theorem 1), which we state here as a disconjugancy criteria. <u>In order that equation (1.3) be disconjugate in</u> $|z|< 1$, <u>it is necessary that</u>

$$|p(z)| \leq \frac{3}{(1-|z|^2)^2}, \quad |z|< 1, \tag{1.7}$$

<u>and sufficient that</u>

$$|p(z)| \leq \frac{1}{(1-|z|^2)^2}, \quad |z|< 1. \tag{1.8}$$

Both conditions are sharp as shown by the Koebe function and by an example due to E. Hille [4].

2. (n-2) parameter family of univalent functions.

Our study of equation (1.1) starts with a problem suggested to us by Z. Nehari.* In view of [5], [7], what are, if any, the function-theoretic aspects of disconjugancy of n-th order linear differential equations. In the following, we shall prove that a disconjugate equation (1.1) is related to an (n-2) parameter family of univalent functions.

In analogy with (1.2), we consider the function

$$f(z,a_1,a_2,\dots,a_{n-2}) = \frac{y_1(z)}{y_2(z)} \tag{2.1}$$

where $y_1(z)$ and $y_2(z)$ are two linearly independent solutions of (1.1), which vanish on a given set S of $(n-2)$ points $a_1,a_2,\dots,a_{n-2}$ of D. (Some of these zeros may coincide, giving rise to zeros of higher order). The existence of at least two such linear independent solutions is an immediate consequence of the existence of a foundamental set of n linearly independent solutions $\eta_1(z),\eta_2(z),\dots,\eta_n(z)$ of equation (1.1). Indeed, setting now, $y(z) = \sum_{k=1}^{n} \alpha_k \eta_k(z)$, and writing

$$y(a_j) = 0\,, \qquad j = 1,2,\dots,(n-2), \tag{2.2}$$

one obtains a system of (n-2) homogeneous equations for the n unknown constants α_k, and there always exist at least two linearly independent solutions of (2.2). In case of a zero of higher order, e.g. $a_1 = a_2 = \dots = a_m$, $1 < m \leq n-2$, (2.2) is replaced by

$$y(a_1) = 0,\ y'(a_1) = 0,\ \dots\ y^{m-1}(a_1) = 0,\ y(a_{m+1}) = 0,\dots,y(a_{n-2}) = 0$$

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* The author wishes to express her thanks to Professor Z. Nehari for his valuable advice offered during many discussions.

and the same conclusion follows. Moreover, if $m = n-2$ there exist exactly two linearly independent solutions which vanish $(n-2)$ times at the point $a_1 \epsilon D$, but for $1 \leq m < n-2$ it does not follow from the general existence theorem that any three solutions of (1.1) which vanish on a set of $(n-2)$ points are linearly dependent. In the following lemma we give two sufficient conditions which guarantee such a situation.

Lemma 1.

If there exist more than two linearly independent solutions of equation (1.1), which vanish on the set S of $(n-2)$ points of D, then equation (1.1) is conjugate in D and at least one of the functions of the type (2.1) is nonunivalent in D.

Proof.

Assume there exist three linearly independent solutions $y_1(z)$, $y_2(z)$ and $y_3(z)$, which vanish on S. Let $b \epsilon D$, such that $y_2(b) \neq 0$, and set

$$y^*(z) = \alpha_1 y_1(z) + \alpha_2 y_2(z) + \alpha_3 y_3(z) \qquad y^*(b) = 0, \; y^{*\prime}(b) = 0.$$

It follows that there always exists a nontrivial solution $y^*(z)$ which vanishes at least n times in D. Moreover, $y^*(z)/y_2(z)$, which is a function of the type (2.1) is nonunivalent in D.

We are now ready to formulate the connection between the function (2.1) and the equation (1.1).

Theorem 1.

Equation (1.1) is disconjugate in D if and only if the function (2.1) is univalent in D for any choice of two linearly independent solutions $y_1(z)$ and $y_2(z)$ which vanish on any given set $(a_1, a_2, \ldots, a_{n-2})$ of $(n-2)$ points of D.

Proof.

i) Disconjugancy implies univalence. If $f(b_1) = f(b_2) = -\beta\alpha^{-1}$ it follows from (2.1) that the solution $\alpha y_1(z) + \beta y_2(z)$ has n zeros in D at the points $a_1, a_2, \ldots, a_{n-2}, b_1, b_2$, and (1.1) is conjugate in D.

ii) Univalence implies disconjugancy. Suppose there exist a solution $y_1(z)$ which vanishes at $a_1, a_2, \ldots, a_n$. There always exists a solution $y_2(z)$, which vanishes at $a_1, a_2, \ldots, a_{n-2}$ and is linearly independent on $y_1(z)$. Now if

$$(2.3) \qquad y_2(a_{n-1}) \neq 0 \quad y_2(a_n) \neq 0$$

then the function (2.1) is nonunivalent in D. So suppose (2.3) is false and denote by Σ the set of common zeros of $y_1(z)$ and $y_2(z)$. We may assume without loss of generality that $a_{n-1} \in \Sigma$. Let now $b \in D$, such that $b \notin \Sigma$. There exists a solution $y_3(z) = \alpha_1 y_1(z) + \alpha_2 y_2(z)$, which vanishes at b and at all the points of Σ. Moreover, there exists another solution $y_4(z)$ which vanishes at b and at $a_1, \ldots, a_{n-3}$ and is linearly independent on $y_3(z)$. Now $y_4(a_{n-2}) \neq 0$, $y_4(a_{n-1}) \neq 0$. Because suppose $y_4(a_{n-2}) = 0$, then by our lemma $y_4(z)$ is a linear combination of $y_1(z)$ and $y_2(z)$ i.e. $y_4(z) = \beta_1 y_1(z) + \beta_2 y_2(z)$, but being independent on $y_3(z)$ it follows from $y_3(b) = 0, y_4(b) = 0$ that $y_1(b) = y_2(b) = 0$, which contradicts our assumption that $b \notin \Sigma$. So $y_4(z)$ does not vanish at a_{n-2} nor at a_{n-1}. Considering now the function

$$(2.4) \qquad f(z, a_1, \ldots a_{n-3}, b) = \frac{y_3(z)}{y_4(z)},$$

it follows that (2.4) is nonunivalent in D.

3. Quantities invariant under linear transformations

Our next goal is to express the coefficients of (1.1) in terms of the function (2.1). In case of a disconjugate equation (1.1), a different choice of the two solutions in (2.1) would have resulted in a function Tf, where T is a linear transformation of the type (1.6). Hence, any identity connecting the coefficients of the disconjugate equation (1.1) with the function (2.1) should be expressed by quantities invariant under the transformation (1.6). The simplest quantity of this type is the Schwarzian derivative

$$s(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{3.1}$$

Other invariant quantities may be obtained by differentiating (3.1) and by producing various combinations of $s(z)$ and its derivatives. But, basicly, all these invariant quantities are derived from $s(z)$. A different way of obtaining invariant quantities is by expanding the function

$$g(z) = \frac{f'(\zeta)}{f(z+\zeta)-f(\zeta)}$$

into a power series of the form

$$g(z) = \frac{1}{z} + \sum_{j=0}^{\infty} I_j[f(\zeta)]z^j . \tag{3.2}$$

It is easily checked that the coefficients $I_j[f(\zeta)]$ [$j=1,2,\ldots,$) are invariant under a linear transformation (1.6). Examination of the first coefficients shows that

$$I_1[f(\zeta)] = -\frac{1}{3!}\left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right] = -\frac{s(\zeta)}{6}, \tag{3.3}$$

$$(3.4)\qquad I_2[f(\zeta)] = -\frac{1}{4!}\left[\frac{f^{(4)}}{f'} - \frac{4f'''f''}{(f')^2} + 3\left(\frac{f''}{f'}\right)^3\right] = -\frac{s'(\zeta)}{24} .$$

While for $I_j[f(\zeta)]\,(j=2,3,\ldots)$ we have, by a recent result due to D. Ahronov [1] the recurssion formula

$$(3.5)\qquad (j+3)I_{j+1}[f(\zeta)] = I_j'[f(\zeta)] - \sum_{k=1}^{j-1} I_k[f(\zeta)]I_{j-k}[f(\zeta)] \quad j = 1,2,\ldots,$$

In view of (3.3), it now follows from (3.5) that all the invariants $I_j[f]$ $(j=1,2,\ldots,)$ are also derived from (3.1). This information arises the question whether there exists at all an invariant quantity which is not derived from the Schwarzian derivative $s(z)$. In the following theorem we answer this question in the negative, provided the function $f(z)$ is meromorphic in D, with simple poles at most, and such that $f'(z) \neq 0$. We shall say that such functions belong to the <u>restricted</u> <u>class</u> in D (see [8]), and denote the class by $RC(D)$. Evidently, for $f(z)\in RC(D)$, (3.1) is a regular function.

<u>Theorem 2</u>.

<u>Let</u> $f(z)\in RC(D)$, <u>and let</u>

$$(3.6)\qquad E[f(z)] = E[f(z), f'(z), \ldots, f^{(n)}(z)] = I(z)$$

<u>be a differential operator of order</u> n, <u>operating on</u> $f(z)$. <u>If</u> (3.6) <u>is invariant when</u> f <u>is subject to a linear transformation</u> (1.6), <u>namely if</u>

$$(3.7)\qquad E[Tf(z)] = E[f(z)] = I(z), \qquad z\in D,$$

<u>then</u> $I(z)$ <u>is derived from</u> $s(z)$, <u>and</u> $E[f(z)]$ <u>is identical with</u>

a differential operator of order $(n-3)$, operating on $s(z)$, i.e.

$$(3.8)\qquad I(z) = E[f(z)] = E^*[s(z)] = E^*[s(z), s'(z), \ldots, s^{(n-3)}(z)].$$

Proof.

Let $z_0 \in D$. We may assume without loss of generality that

$$(3.9)\qquad f(z_0) = 0, \quad f'(z_0) = 1, \quad f''(z_0) = 0.$$

Because if (3.9) is not true and $f(z_0) = \alpha$, $f'(z_0) = \beta \neq 0$ and $f''(z_0) = 2\gamma$, then the function

$$F(z) = \frac{\beta[f(z) - \alpha]}{\beta^2 + \gamma[f(z) - \alpha]} \qquad \alpha \neq \infty$$

satisfies (3.9), and by (3.7) we have $E[F(z)] = E[f(z)]$. If $\alpha = \infty$, then apply first a transformation $f \rightarrow f^{-1}$ and then proceed as before. Setting now $z = z_0$ in (3.1) and (3.4), it follows by (3.9) that

$$(3.10)\qquad s(z_0) = f'''(z_0),$$

and

$$(3.11)\qquad s'(z_0) = f^{(4)}(z_0).$$

By differentiation of (3.4) and by induction we obtain

$$(3.12)\qquad s^{(m)}(z) = \frac{f^{(m+3)}(z)}{f'(z)} + \frac{P_{m+2}[f^{(m+2)}(z), \ldots, f'(z)]}{[f'(z)]^{m+2}}, \quad m=0,1,2,\ldots,$$

where P_{m+2} is a polynomial of order $(m+2)$, in which the highest degree of $f'(z)$ is m. Using (3.9), it follows now from (3.10), (3.11), and (3.12) that

$$(3.13)\qquad s^{(m)}(z_0) = f^{(m+3)}(z_0) + P_{m+2}[f^{(m+2)}(z_0), \ldots, f^{(5)}(z_0), s'(z_0), s(z_0), 0, 1], \quad m=2,3,\ldots,$$

By elimination and induction, (3.13) implies

$$(3.14) \qquad f^{(k)}(z_o) = s^{(k-3)}(z_o) + Q_{k-1}[s^{(k-4)}(z_o),\dots,s(z_o)], \quad k=3,4,\dots,$$

where Q_{k-1} $(k = 5,6,\dots)$ is a polynomial of order $(k-1)$ free of terms of order 0 and 1, and $Q_2 = Q_3 = 0$. Insertion of (3.9) and (3.14) in (3.6) yields now.

$$(3.15) \qquad I(z_o) = E[0,1,0,s(z_o),s'(z_o),f^{(5)}(z_o),\dots,f^{(n)}(z_o)] =$$
$$= E^*[s(z_o),s'(z_o),\dots,s^{(n-3)}(z_o)].$$

As (3.15) holds for every $z_o \in D$, it implies the identity (3.8).

4. Relations between the coefficients of (1.1) and the Schwarzian derivative

If equation (1.1) is disconjugate in D, then by Theorem 1 and 2 any connection between the coefficients of (1.1) and the function (2.1) has to be expressed in terms of the Schwarzian derivative of (2.1). But, in case equation (1.1) is conjugate in D, (2.1) may or may not belong to $RC(D)$ and Theorem 2 may not be applied. To take care of this problem, we replace the $(n-2)$ parameter family of functions (2.1) by an one-parameter sub-family of functions $f(z,a)$ defined by

$$(4.1) \qquad f(z,a) = \frac{y_1(z)}{y_2(z)}$$

where $y_1(z)$ and $y_2(z)$ are linearly independent solutions of (1.1), which vanish $(n-2)$ times at the point $a \in D$. Now, even if (1.1) is conjugate in D $\frac{df(z,a)}{dz}\Big|_{z=a} = f'(a,a) \neq 0$, and $f(z,a) \in RC(N(a))$ where $N(a)$ is some neighborhood in D of the point a. Denote by

(4.2) $$s(z,a) = \{f(z,a), z\},$$

and let

$$s^{(r)}(z,a) = \frac{d^r s(z,a)}{dz^r} \qquad r = 1,2,\ldots,$$

it follows that $s(z,a)$ and $s^{(r)}(z,a)$ are regular functions in $N(a)$. We are ready now to establish a relation between some of the coefficients of (1.1) and the derivatives of $s(z,a)$.

Theorem 3. *Assume*

(4.3) $$p_{n-2}(z) \equiv 0,\ p_{n-3}(z) \equiv 0, \ldots, p_{n-k+1}(z) \equiv 0,\ p_{n-k}(z) \not\equiv 0,\ 2 \le k \le n$$

where $p_j(z)\ (j=0,1,\ldots,n-2)$ *are the coefficients of equation* (1.1).

Then

(4.4) $$s(a,a) = 0,\ s'(a,a) = 0, \ldots, s^{(k-3)}(a,a) = 0 \qquad 3 \le k \le n.$$

and

(4.5) $$p_{n-k}(a) = \frac{(n+k-1)!}{k(k+1)!(n-2)!} s^{(k-2)}(a,a), \qquad 2 \le k \le n.$$

Proof.

Let $y_1(z)$ and $y_2(z)$ be two solutions of (1.1) which satisfy the following initial conditions.

(4.6) $$y_1(a) = 0,\ y_1'(a) = 0,\ \ldots,\ y_1^{(n-2)}(a) = 0,\ y_1^{(n-1)}(a) = (n-1)!$$

(4.7) $$y_2(a) = 0,\ y_2'(a) = 0,\ \ldots,\ y_2^{(n-3)}(a) = 0,\ y_2^{(n-2)}(a) = (n-2)!,\ y_2^{(n-1)}(a) = 0.$$

By (1.1), (4.3) and (4.6) it follows now that

(4.8) $$y_1(z) = (z-a)^{n-1}[1 + \alpha(z-a)^k + \ldots], \qquad 2 \le k \le n$$

with

$$(4.9)\qquad \alpha = \frac{y_1^{(n+k-1)}(a)}{(n+k-1)!} = -\frac{p_{n-k}(a)\,y_1^{(n-1)}(a)}{(n+k-1)!} = -\frac{p_{n-k}(a)\,(n-1)!}{(n+k-1)!},$$

and in a similar way,

$$(4.10)\quad y_2(z) = (z-a)^{n-2}[1 + \beta(z-a)^k + \ldots], \qquad 2 \le k \le n$$

with

$$(4.11)\qquad \beta = -\frac{p_{n-k}(a)\,(n-2)!}{(n+k-2)!}.$$

By inserting (4.8) and (4.10) in (4.1) we obtain

$$(4.12)\qquad f(z,a) = (z-a)[1 + (\alpha-\beta)(z-a)^k + \ldots], \quad 2 \le k \le n .$$

Hence,

$$(4.13)\qquad \begin{aligned} &f(a,a) = 0,\ f'(a,a) = 1,\ f''(a,a) = 0,\ \ldots f^{(k)}(a,a) = 0,\\ &f^{(k+1)}(a,a) = (k+1)!(\alpha-\beta), \quad 2 \le k \le n . \end{aligned}$$

By (4.13) it follows from (3.12) that

$$s^{(m)}(a,a) = f^{(m+3)}(a,a) \qquad m=0,1,2,\ldots(k-2), \quad 2 \le k \le n,$$

which implies (4.4) and (4.5).

Since any solution of (1.1), which has a zero of order (n-2) at the point a, is a linear combination of the two particular solutions (4.8) and (4.10), a different choice of the two solutions in (4.1) would replace f by Tf, where T is of the form (1.6). But $s(z,a)$ and $s^{(r)}(z,a)$ are invariant under the transformation (1.6), hence (4.4) and (4.5) hold for any choice of the solutions $y_1(z)$ and $y_2(z)$ regardless of the initial conditions (4.6) and (4.7).

Remark: If for $n = 2$, (4.1) is interpreted as (1.2), then (4.5) implies the known relation (1.4).

5. Linear transformations of equation (1.1)

We shall use now (4.4) and (4.5) in order to study the result of a linear transformation

$$(5.1) \qquad z = \frac{A\zeta + B}{C\zeta + D} \qquad AD - BC \neq 0$$

upon the differential equation (1.1). We start by considering the effect of (5.1) on $s(z,a)$.

Suppose $z = z(\zeta)$ is an one-to-one analytic transformation which maps the domain Δ onto D and $\alpha \in \Delta$ to $a \in D$, then

$$(5.2) \qquad f(z,a) = f[z(\zeta),a] = \varphi(\zeta,\alpha).$$

The Schwarzian derivative when subject to a transformation $z = z(\zeta)$ obeys the following rule.

$$(5.3) \qquad \sigma(\zeta,\alpha) = s(z,a)\left(\frac{dz}{d\zeta}\right)^2 + \{z(\zeta),\zeta\}$$

where

$$\sigma(\zeta,\alpha) = \{\varphi(\zeta,\alpha),\zeta\}, \quad s(z,a) = \{f(z,a),z\}.$$

If $z(\zeta)$ is of the form (5.1) then $\{z(\zeta),\zeta\} \equiv 0$, and

$$(5.3') \qquad \sigma(\zeta,\alpha) = s(z,a)\left(\frac{dz}{d\zeta}\right)^2 .$$

By (5.3'), we have now, $s(a,a) = 0$ if and only if $\sigma(\alpha,\alpha) = 0$. Differentiation of (5.3') with respect to ζ yields

$$(5.4) \qquad \sigma'(\zeta,\alpha) = s'(z,a)\left(\frac{dz}{d\zeta}\right)^3 + 2s(z,a)\frac{dz}{d\zeta}\frac{d^2z}{d\zeta^2} .$$

Suppose $s(a,a) = 0$, then $\sigma'(\alpha,\alpha) = s'(a,a)\left(\frac{dz}{d\zeta}\right)^3\Big|_{\zeta=\alpha}$ and $s'(a,a) = 0$ if and only if $\sigma'(\alpha,\alpha) = 0$. By successive differentiation of (5.3') and by assuming (4.4) we obtain

$$(5.5)\qquad \sigma(\alpha,\alpha) = 0,\ \sigma'(\alpha,\alpha) = 0,\ \ldots\ \sigma^{(k-3)}(\alpha,\alpha) = 0, \qquad 3 \le k \le n$$

$$\sigma^{(k-2)}(\alpha,\alpha) = s^{(k-2)}(a,a)\left(\frac{dz}{d\zeta}\right)^k\Big|_{\zeta=\alpha}, \quad 2 \le k \le n$$

which can be rewritten as

$$(5.6)\qquad \sigma^{(r)}(\alpha,\alpha) = s^{(r)}(a,a)\left(\frac{dz}{d\zeta}\right)^{r+2}\Big|_{\zeta=\alpha} \qquad r = 0,1,2,\ldots,k-2.$$

Formula (5.6) provides us now with a deeper understanding of the mechanism which determines the form into which equation (1.1) is transformed when subject to a linear transformation (5.1). We give here a new proof to two theorems stated by R. Haddas for the case $k = n$, ([3] Theorems 1 and 2).

Theorem 4.

Equation (1.1) with the additional assumption (4.3) is transformed by an one-to-one transformation

$$(5.7)\qquad z = z(\zeta),\quad w(\zeta) = y[z(\zeta)]\tau(\zeta), \qquad \tau(\zeta) \neq 0$$

into an equation of the same form, namely

$$(5.8)\qquad w^{(n)}(\zeta) + q_{n-2}(\zeta)w^{(n-2)}(\zeta) + \ldots + q_o(\zeta)w(\zeta) = 0$$

with

$$(5.9)\qquad q_{n-2}(\zeta)\equiv 0,\ \ldots, q_{n-k+1}(\zeta)\equiv 0,\ q_{n-k}(\zeta)=p_{n-k}[z(\zeta)]\left(\frac{dz}{d\zeta}\right)^k, 2 \le k \le n,$$

if and only if $z(\zeta)$ *is of the form* (5.1).

Proof.

Substitution of $z = z(\zeta)$, $\omega(\zeta) = y[z(\zeta)]$ in (1.1) leads us to

$$\omega^{(n)}(\zeta) + R_{n-1}(\zeta)\omega^{(n-1)}(\zeta) + \ldots + R_o(\zeta)\omega(\zeta) = 0 .$$

It is well known that the coefficient of $\omega^{(n-1)}(\zeta)$ can be removed by a suitable choice of the function $\tau(\zeta)$ in (5.7). Indeed, by setting $\tau(\zeta) = \exp[\int \frac{R_{n-1}(\zeta)}{n} d\zeta]$, one obtains equation (5.8). So what we really have to prove is that (4.3) implies (5.9), if and only if $z(\zeta)$ is linear. Let $y_1(z)$ and $y_2(z)$ be linearly independent solutions of (1.1) possessing zeros of order (n-2) at $a\epsilon D$, then $w_1(\zeta) = y_1[z(\zeta)]\tau(\zeta)$ and $w_2(\zeta) = y_2[z(\zeta(]\tau(\zeta)$ are independent solutions of (5.8) with zeros of order (n-2) at the point α $(z(\alpha) = a)$, and

$$\text{(5.10)} \qquad \frac{w_1(\zeta)}{w_2(\zeta)} = \frac{y_1[z(\zeta)]}{y_2[z(\zeta)]} = f[z(\zeta),a] = \varphi(\zeta,\alpha)$$

holds.

Suppose now $z(\zeta)$ is of the type (5.1). By Theorem 3, (4.3) implies (4.4) and (4.5) which imply (5.6). In view of (5.10), we may apply (4.5) to the coefficients $q_{n-2}(\zeta),\ldots,q_{n-k}(\zeta)$ of (5.8), and by (5.6), we obtain (5.9). Conversely, assume $z(\zeta)$ is not linear, then $\{z(\zeta),\zeta\} \not\equiv 0$, i.e., there exists a point $\alpha\epsilon\Delta$, such that $\{z(\zeta),\zeta\}|_{\zeta=\alpha} \neq 0$. For $3 \leq k \leq n$, it now follows from (5.3) and (4.4) that $\sigma(\alpha,\alpha) \neq 0$ which by (4.5) implies that $q_{n-2}(\alpha) \neq 0$. For $k = 2$, it is trivial that (5.3) implies (5.9) if and only if $z(\zeta)$ is a linear transformation.

Remark.

As noted by R. Haddas the necessary condition goes back to a theorem by Wilczynski [9].

6. Necessary condition for disconjugancy in the unit disk.

We shall use now the results of Theorems 1, 3 and 4 to obtain a necessary condition for disconjugancy of equation (1.1) in the unit disk.

Theorem 5.

Let the coefficients of (1.1) be regular for $|z| < 1$ and satisfy (4.3) there, and let equation (1.1) be disconjugate in the unit disk, then

$$|p_{n-k}(z)| \leq \frac{2(n+k-1)!}{k^2(n-2)!(1-|z|^2)^k}, \qquad 2 \leq k \leq n. \tag{6.1}$$

Proof.

By Theorem 1, disconjugancy of (1.1) implies univalence of the function (2.1). In particular, the function (4.1) is univalent in the unit disk for any $|a| < 1$. Setting $a = 0$ in (4.1) and choosing $y_1(z)$ and $y_2(z)$ as in (4.6) and (4.7), we obtain by (4.12).

$$f(z,0) = z + (\alpha-\beta)z^{k+1} + \dots, \qquad 2 \leq k \leq n \tag{4.12'}$$

where α and β are given by (4.9) and (4.11). But, for the univalent function (4.12')

$$|\alpha-\beta| \leq \frac{2}{k}, \tag{6.2}$$

and equality holds in (6.2) if and only if

$$f(z,o) = \frac{z}{(1-e^{i\theta}z^k)^{2/k}} \qquad 0 \leq \theta < 2\pi. \tag{6.3}$$

By (4.9) and (4.11) it follows now from (6.2) that

$$|p_{n-k}(o)| \leq \frac{2(n+k-1)!}{k^2(n-2)!}, \qquad 2 \leq k \leq n \tag{6.4}$$

which establishes (6.1) for $z = 0$. In order to prove (6.1) for any $|z| < 1$, we apply a linear transformation

$$(6.5) \qquad z = \frac{\zeta+a}{1+\bar{a}\zeta}, \quad |a| < 1$$

which maps $|\zeta| < 1$ onto $|z| < 1$. By Theorem 4, equation (1.1) is transformed into equation (5.8), and by (5.9)

$$(6.6) \qquad q_{n-k}(o) = p_{n-k}(a)\left(\frac{dz}{d\zeta}\right)^k\Big|_{\zeta=0} .$$

Since disconjugancy is preserved by the transformation (5.7), which is our case means that (1.1) is disconjugate in the unit circle if and only if (5.8) is, we may apply (6.4) to $q_{n-k}(o)$. Using the fact that for transformations of the unit circle on itself

$$(6.7) \qquad \left|\frac{dz}{d\zeta}\right| = \frac{1-|z|^2}{1-|\zeta|^2},$$

holds, we obtain (6.1).

In view of (3.14) and (5.6), it is possible to state Theorem 5 also as a necessary condition for univalence of $f(z)$ in $|z| < 1$.

<u>Theorem 5'.</u>

<u>Assume</u> $f(z)$ <u>is univalent for</u> $|z| < 1$ <u>and let</u> $s(z) = \{f(z), z\}$. <u>Suppose</u>

$$s(a) = s'(a) = \dots = s^{(m-1)}(a) = 0, \quad |a| < 1$$

<u>then</u>

$$|s^{(m)}(a)| \leq \frac{2(m+3)!}{(m+2)(1-|a|^2)^{m+2}}, \quad |a| < 1, \quad m = 0,1,2,\dots$$

<u>7. The equation $y^{(2m)} + py = 0$.</u>

For $k = n = 2$ (6.1) reduces to (1.7) which is the necessary condition given by Nehari for the disconjugancy of equation (1.3). The natural question to be asked next is whether it is possible to establish a sufficient condition for disconjugancy, which will generalize the sufficient condition (1.8). (Sufficient conditions of different type were given by Nehari in [6].) It is obvious that the easiest case to handle is that of equation

$$(7.1) \qquad y^{(n)}(z) + p(z)y(z) = 0$$

where we have only one coeficient. For (7.1) we have the following conjecture.

Conjecture.

Assume $p(z)$ *is regular in* $|z| < 1$. *In order that* (7.1) *be disconjugate in* $|z| < 1$, *it is sufficient that*

$$(7.2) \qquad |p(z)| \leq \frac{A(n)}{(1-|z|^2)^n}, \qquad |z| < 1$$

with a suitable constant $0 < A(n) \leq \frac{2(2n-1)!}{n^2(n-2)!}$. Unfortunately, we have not succeeded in proving this conjecture nor in disproving it. Yet, weaker results backing (7.2) a bit, were obtained for equations of even order

$$(7.3) \qquad y^{(2m)}(z) + p(z)y(z) = 0. \qquad m=1,2,\ldots$$

In the following theorem we prove that a condition of the type (7.2) quarantees the non-existence of a solution of equation (7.3) possessing two zeros of order m.

Theorem 6.

Assume $p(z)$ *is regular in* $|z| < 1$ *and satisfies*

$$(7.4) \qquad |p(z)| \leq \frac{B(2m)}{(1-|z|^2)^{2m}}, \qquad |z| < 1$$

where

(7.5) $$B(2) = 1,\ B(4) = 9,\ B(2m) = 9 \prod_{k=3}^{m} (4k-3) \qquad m = 3,4,\ldots,$$

then no solution of (7.3) has two zeros, of order m in $|z| < 1$.

To prove Theorem 6 we need an integral inequality, which will be established in the following lemma.

Lemma 2.

Let $U(x)$ be a real function with m continuous derivatives in the interval $[-1,1]$, possessing zeros of order m at the points $x = \pm 1$, then

(7.6) $$\int_{-1}^{+1} [U^{(m)}(x)]^2 dx > B(2m) \int_{-1}^{+1} \frac{[U(x)]^2 dx}{(1-x^2)^{2m}} \qquad m = 1,2,\ldots$$

where $B(2m)$ are constants defined in (7.5).

Proof.

For $m = 1$ (7.6) was proved by Nehari [5]. By a slight change in Nehari's proof we first establish the following inequality

(7.7) $$\int_{-1}^{+1} \frac{[V'(x)]^2}{(1-x^2)^{2k-2}} \geq [4k-3] \int_{-1}^{+1} \frac{[V(x)]^2}{(1-x^2)^{2k}}, \qquad k = 1,2,\ldots,$$

for real continuous function $V(x)$ with zeros of order k at ± 1. Expansion and integration by parts of the trivial inequality

$$\int_{-1}^{+1} \left[\frac{V'(x)}{(1-x^2)^{k-1}} + \frac{\gamma x V(x)}{(1-x^2)^k}\right]^2 dx \geq 0$$

leads us to

$$\int_{-1}^{+1} \frac{[V'(x)]^2 dx}{(1-x^2)^{2k-2}} \geq \gamma \int_{-1}^{+1} \frac{1+(4k-3-\gamma)x^2}{(1-x^2)^{2k}} [V(x)]^2 dx \;, \qquad k = 1,2,\ldots$$

Setting now $\gamma = 4k-3$, (7.7) follows. Equality may hold in (7.7) if and only if

$$(7.8) \qquad V(x) = C(1-x^2)^{2k-\frac{3}{2}} \qquad k = 1,2,\ldots$$

For $k = 1$, (7.8) does not satisfy our hypotheses, so equality in (7.7) is excluded, but for $k=2,3,\ldots$ equality may hold in (7.7). Applying now (7.7) successively to the functions $V(x) = U^{(m-1)}(x)$ $(k=1)$, $V(x) = U^{(m-2)}(x)$ $(k=2),\ldots$, $V(x) = U'(x)$ $(k=m-1)$, we obtain

$$\int_{-1}^{+1} [U^{(m)}(x)]^2 dx > \int_{-1}^{+1} \frac{[U^{(m-1)}(x)]^2}{(1-x^2)^2} dx \geq 1.5 \int_{-1}^{+1} \frac{[U^{(m-2)}(x)]^2}{(1-x^2)^4} dx \geq \ldots$$

$$\geq 1.5.9\ldots(4m-3) \int_{-1}^{+1} \frac{[U(x))]^2}{(1-x^2)^{2m}} dx \;.$$

Hence

$$(7.9) \qquad \int_{-1}^{+1} [U^{(m)}(x)]^2 dx > \prod_{k=1}^{m} (4k-3) \int_{-1}^{+1} \frac{[U(x)]^2}{(1-x^2)^{2m}} dx \;.$$

Now (7.9) differs from (7.6) only by a constant. To prove (7.6) one has to use Beesack's inequality, ([2] p. 494)

$$(7.10) \qquad \int_{-1}^{+1} [V''(x)]^2 dx > 9 \int_{-1}^{+1} \frac{[V(x)]^2 dx}{(1-x^2)^4}$$

which holds for real function $v(x)$ with two continuous derivatives in the interval $[-1,1]$, possessing zeros of second order at ± 1. Beesack mentions that for $v(x) = C(1-x^2)^{\frac{3}{2}}$ both sides of (7.10) go to ∞. But since $(1-x^2)^{\frac{3}{2}}$ does not satisfy our hypotheses, (7.10) holds always for the class of function defined above. Applying now (7.10) to $V(x) = U^{(m-2)}(x)$ we obtain

$$(7.10') \qquad \int_{-1}^{+1} [U^{(m)}(x)]^2 dx > 9 \int_{-1}^{+1} \frac{[U^{(m-2)}(x)]^2}{(1-x^2)^4} dx .$$

Proceeding now as before by applying (7.7) successively, (7.6) follows.

Remark.

By substituting ρx for x in (7.6) we obtain a modified form of inequality (7.6),

$$(7.6') \qquad \int_{-\rho}^{\rho} [U^{(m)}(x)]^2 dx > B(2m)\rho^{2m} \int_{-\rho}^{\rho} \frac{[U(x)]^2 dx}{(\rho^2-x^2)^{2m}}, \qquad m = 1,2,\ldots,$$

which holds for real function $U(x)$ with m continuous derivatives in the interval $[-\rho,\rho]$, possessing zeros of order m at $\pm\rho$.

Proof of Therem 6.

Suppose the theorem is false and there exists a solution $y(z)$ with two zeros z_1 and z_2 ($|z_1|,|z_2| < 1$) each of multiplicity m. By a suitable choice of the parameters α and θ in

$$(7.11) \qquad \zeta(z) = e^{i\theta}\frac{(z-\alpha)}{1-\bar{\alpha}z}, \quad |\alpha| < 1, \quad 0 \leq \theta < 2\pi$$

it is possible to map $|z|< 1$ onto $|\zeta|< 1$ and z_1 and z_2 on two symetric points of the real axes $\pm\rho$. By Theorem 4, the differential equation (7.3) is transformed into

$$(7.12) \qquad w^{(2m)}(\zeta) + q(\zeta)w(\zeta) = 0$$

with

$$q(\zeta) = p(z)\left(\frac{dz}{d\zeta}\right)^{2m} .$$

By (6.7) and (7.4) it follows that

$$(7.4') \qquad |q(\zeta)| = |p(z)| \cdot \left(\frac{1-|z|^2}{1-|\zeta|^2}\right)^{2m} \leq \frac{B(2m)}{(1-|\zeta|^2)^{2m}}, \quad |\zeta| < 1.$$

Thus, our assumption that (7.3) has a solution with two zeros of order m implies that (7.12) has a solution $w_1(\zeta)$ possessing two zeros of order m at $\pm\rho$, while (7.4') holds. We write now (7.12) for $w_1(\zeta)$, multiply by $\overline{w_1(\zeta)}$ and integrate along the real axes. This leads us to

$$\int_{-\rho}^{\rho} w_1^{(2m)}(x)\overline{w}_1(x)\,dx + \int_{-\rho}^{\rho} q(x)\,|w_1(x)|^2 dx = 0.$$

By integrating by parts m times, and by noting that all the integrated parts vanish, we obtain

$$(-1)^m \int_{-\rho}^{\rho} |w_1^{(m)}(x)|^2 dx = -\int_{-\rho}^{\rho} q(x)\,|w_1(x)|^2 dx.$$

Hence,

$$(7.13) \qquad \int_{-\rho}^{\rho} |w_1^{(m)}(x)|^2 dx = \left|\int_{-\rho}^{\rho} q(x)\,|w_1(x)|^2 dx\right| \leq \int_{-\rho}^{\rho} |q(x)|\,|w_1(x)|^2 dx .$$

Writing now $w_1(x) = u(x) + iv(x)$, we have $|w_1|^2 = u^2+v^2$, $|w_1^{(m)}|^2 = [u^{(m)}]^2 + [v^{(m)}]^2$, and (7.13) takes the form

$$(7.13') \qquad \int_{-\rho}^{\rho}\left([u^{(m)}(x)]^2 + [v^{(m)}(x)]^2\right)dx \leq \int_{-\rho}^{\rho} |q(x)|\,[u^2(x)+v^2(x)]\,dx$$

which by (7.4') implies

$$\int_{-\rho}^{\rho}\left([u^{(m)}(x)]^2 + [v^{(m)}(x)]^2\right)dx \leq B(2m)\int_{-\rho}^{\rho}\frac{u^2(x)+v^2(x)}{(1-x^2)^{2m}}dx <$$

(7.14)

$$< B(2m)\rho^{2m}\int_{-\rho}^{\rho}\frac{u^2(x)+v^2(x)}{(\rho^2-x^2)^{2m}}dx.$$

Since $w_1(x) = u(x) + iv(x)$ is supposed to have zeros of order m at $x=\pm\rho$, the same is true for $u(x)$ and $v(x)$ separately. By the remark following Lemma 2 we obtain, therefore

$$(7.15) \qquad \int_{-\rho}^{\rho}\left([u^{(m)}(x)]^2 + [v^{(m)}(x)]^2\right)dx > B(2m)\rho^{2m}\int_{-\rho}^{\rho}\frac{u^2(x)+v^2(x)}{(\rho^2-x^2)^{2m}}dx$$

which contradicts (7.14). Thus, we have proved that no solution of (7.3) can have two zeros of order m in the unit circle if $p(z)$ satisfies (7.4).

Remark 1.

For fourth order equation ($m=2$) Theorem 6 is included in Theorem 6 of [3], while for $m\geq 3$ our Theorem 6 may serve as a complimentary theorem to Theorem 6 of [3].

Remark 2.

As regards the sharpness of Theorem 6, the question is still open. It seems that for $m=2$ $B(4) = 9$ is the best constant, while for $m \geq 3$, $B(2m)$ are not the best.

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