MATERIALLY UNIFORM SIMPLE BODIES
WITH INHOMOGENEITIES

by

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Contents

*This paper supersedes an unpublished preliminary study written by the author in 1963. Section 34 of reference [2] is a summary of this study.

1. Introduction

The basic concepts of the theory of simple materials have been introduced in reference [1], (see also the exposition in [2], Chapter C III). Here I present a detailed study of the structure of bodies that consist of a uniform simple material yet are not necessarily homogeneous.

After assembling the necessary mathematical tools in Sects. 2-4, the concept of a simple body is introduced in Sect. 5. This concept is more inclusive than the one described in [1] because it can be appropriate not only to mechanical material properties, but also to thermal, optical, electrical, magnetic, or any other type of material properties. A body may be simple with respect to any particular or combination of particular such material properties. The physical theory relevant to these properties need not be made explicit.

In Sect. 6 a precise definition of a materially uniform simple body is given. The nature of the coherence of a uniform body with respect to the local material properties under consideration can be described mathematically in terms of what I call a material uniformity or in terms of what I call a uniform reference. In general, neither of these is uniquely determined by the simple body structure, but the degree of non-uniqueness can be delimited precisely. There may or may not exist uniform references that are gradients of global configurations. If they do exist, the body is homogeneous and the theory becomes trivial.

Sections 7-9 contain an exposition of the mathematical prerequisites necessary to describe the local behavior of material uniformities and uniform references that possess a degree of smoothness. In the remainder of the paper, such smoothness is always assumed. A material uniformity is then equivalent to an affine connection, which is defined in Sect. 10 and called a material connection. The Cartan-torsion of this connection describes locally the deviation from homogeneity and is therefore called, in Sect. 11, the inhomogeneity of the given material uniformity.

Associated with each smooth uniform reference is also a Riemannian structure on the body, and the relation of this structure to the material connection is studied in Sect. 12. The difference between the Riemannian connection and the material connection determines what I call the contortion of the given uniform reference. Contortion and inhomogeneity determine one another.

Of particular interest is a special type of non-homogeneity called contorted aelotropy in Section 13. It generalizes the more familiar curvilinear aelotropy. In contorted aelotropy, the deviation from homogeneity is given by a distribution of rotations on a suitable global configuration, and the contortion describes the local behavior of this distribution. The curvature of the Riemannian structure mentioned before describes locally the deviation from contorted aelotropy.

Section 14 contains a number of results that apply when the response functions of the body have special properties, especially with respect to material symmetry.

The usual version of Cauchy's equation of balance (cf. [2], (16.6)) is very useful only when applied to bodies that are homogeneous. For applications to materially uniform but inhomogeneous bodies, a new version of Cauchy's equation, derived in the Sect. 15, is much more suitable than the usual one. This new version gives rise, for example, to a definite differential equation for the theory of inhomogeneous but materially uniform elastic bodies.

Unfortunately, there is no easily accessible exposition of the coordinate-free type of modern differential geometry that is the most appropriate for the applications made here. The monograph of Lang [3], although it explains some of the concepts used here, does not contain sufficient material and emphasizes matters not relevant in the present context. For this reason I develop here all the mathematical tools as they become needed, tailored to the requirements of the intended applications.

There is a large literature on theories of <u>continuous distributions of dislocations</u>, proposed in various forms by

KONDO, NYE, BILBY, BULLOUGH, SMITH, SEEGER, KRÖNER, GÜNTHER, and others.[1] Motivated by heuristic considerations, mostly concerning lattice defects in crystals, these authors lay down a priori certain geometric structures to describe distributions of dislocations. These geometric structures are formally of the same type as some of those occurring in the present paper. The conceptual status of the theory presented here, however, is very different. I show that once a constitutive assumption defining a materially uniform simple body is laid down, the geometric structures of the body are determined. The geometry is thus the natural outcome, not the first assumption, of the theory. Since the underlying constitutive assumption is very general, the real materials to which the theory can be expected to apply need neither be crystalline, nor elastic, nor solid.

2. Deformations.

We shall employ the concept of absolute physical space[2], as is customary in classical physics. This space $\mathcal{E}$, whose elements $\underset{\sim}{x}, \underset{\sim}{y}, \ldots$ we call spatial points, has the structure of a three-dimensional Euclidean point space[3]. The translation space of $\mathcal{E}$ is denoted by $\mathcal{V}$; it is a three-dimensional inner product space. The elements $\underset{\sim}{u}, \underset{\sim}{v}, \ldots$ of $\mathcal{V}$ are called spatial vectors.

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1) For details and references I refer to the expository articles [4] and [5].

2) The considerations of this paper can be adapted to the neo-classical space-time explained in [6]. When this is done, absolute space must be replaced by suitably defined "instantaneous spaces".

3) The exact meaning of this term is explained in [7], Sect. 4.

The translation which carries $\underset{\sim}{x} \in \mathcal{E}$ to $\underset{\sim}{y} \in \mathcal{E}$ is denoted by $\underset{\sim}{y} - \underset{\sim}{x} \in \mathcal{V}$, and $\underset{\sim}{x} + \underset{\sim}{u}$ denotes the point into which $\underset{\sim}{x} \in \mathcal{E}$ is carried by the translation $\underset{\sim}{u} \in \mathcal{V}$. The inner product of two spatial vectors $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$ is denoted by $\underset{\sim}{u} \cdot \underset{\sim}{v}$. Of course, $\underset{\sim}{u} \cdot \underset{\sim}{v} \in \mathcal{R}$, where $\mathcal{R}$ is the set of all real numbers.

The set of all linear transformations $\underset{\sim}{L}: \mathcal{V} \to \mathcal{V}$ of $\mathcal{V}$ into itself is denoted by $\mathcal{L}$. The composition of $\underset{\sim}{L} \in \mathcal{L}$ with $\underset{\sim}{M} \in \mathcal{L}$ is denoted by $\underset{\sim}{M}\underset{\sim}{L} \in \mathcal{L}$. The identity transformation on $\mathcal{V}$ is denoted by $\underset{\sim}{1} \in \mathcal{L}$. The transpose of $\underset{\sim}{L} \in \mathcal{L}$ is denoted by $\underset{\sim}{L}^T$, so that $\underset{\sim}{u} \cdot \underset{\sim}{L}\underset{\sim}{v} = \underset{\sim}{L}^T\underset{\sim}{u} \cdot \underset{\sim}{v}$ holds for all $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$. The trace and determiniant of $\underset{\sim}{L} \in \mathcal{L}$ are denoted by $tr\,\underset{\sim}{L}$ and $det\,\underset{\sim}{L}$, respectively. The set $\mathcal{L}$ of all linear transformations has the natural structure of a nine-dimensional algebra. It is also endowed with a natural inner product, whose values are given by $\underset{\sim}{L} \cdot \underset{\sim}{M} = tr(\underset{\sim}{L}\underset{\sim}{M}^T)$. A transformation $\underset{\sim}{L} \in \mathcal{L}$ is said to be <u>invertible</u> if it is a bijection (i.e., one-to-one and onto). In this case, there exists an <u>inverse</u> $\underset{\sim}{L}^{-1} \in \mathcal{L}$ so that $\underset{\sim}{L}\underset{\sim}{L}^{-1} = \underset{\sim}{L}^{-1}\underset{\sim}{L} = \underset{\sim}{1}$. The invertible members of $\mathcal{L}$ form a group $\ell \subset \mathcal{L}$ under composition; it is called the <u>linear group</u> of $\mathcal{V}$. Important subgroups of ℓ are the <u>unimodular group</u>

$$\mathcal{U} = \{\underset{\sim}{H} \in \ell \mid |det\,\underset{\sim}{H}| = 1\}$$

and the <u>orthogonal group</u>

$$\mathcal{O} = \{\underset{\sim}{Q} \in \ell \mid \underset{\sim}{Q}\underset{\sim}{Q}^T = \underset{\sim}{1}\}.$$

Of course, $\mathcal{O}$ is a subgroup of $\mathcal{U}$.

Consider a mapping $\varphi : \mathcal{G} \rightarrow \mathcal{E}'$ of an open subset $\mathcal{G} \subset \mathcal{E}$ into a point-space or vector-space $\mathcal{E}'$. Let $\mathcal{V}'$ be the translation space of $\mathcal{E}'$ ($\mathcal{V}' = \mathcal{E}'$ if $\mathcal{E}'$ is already a vectorspace) and let $\mathcal{L}(\mathcal{V}, \mathcal{V}')$ be the space of all linear transformations of $\mathcal{V}$ into $\mathcal{V}'$. We say that φ is <u>of class</u> C^1 if there is a continuous mapping $\nabla\varphi : \mathcal{G} \rightarrow \mathcal{L}(\mathcal{V}, \mathcal{V}')$ such that

$$\varphi(\underset{\sim}{x} + \underset{\sim}{u}) = \varphi(\underset{\sim}{x}) + (\nabla\varphi(\underset{\sim}{x}))\underset{\sim}{u} + \underset{\sim}{\sigma}(\underset{\sim}{x}, \underset{\sim}{u}),$$

where

$$\lim_{|\underset{\sim}{u}| \rightarrow 0} \frac{1}{|\underset{\sim}{u}|} \underset{\sim}{\sigma}(\underset{\sim}{x}, \underset{\sim}{u}) = \underset{\sim}{0}$$

holds for all $\underset{\sim}{x} \in \mathcal{G}$. The mapping $\nabla\varphi$, if it exists, is uniquely determined by φ and is called the <u>gradient</u> of φ. If $\nabla\varphi$ exists and is itself of class C^1, we say that φ is <u>of class</u> C^2. The gradient of $\nabla\varphi$ is denoted by $\nabla^{(2)}\varphi$ and is called the <u>second gradient</u> of φ. Continuing in this manner, we say that φ is <u>of class</u> C^r if it is of class C^{r-1} and if its $(r-1)$st gradient $\nabla^{(r-1)}\varphi$ is of class C^1. The gradient of $\nabla^{(r-1)}\varphi$ is denoted by $\nabla^{(r)}\varphi$. We say that φ is of class C^0 if it is merely continuous. If φ is of class C^2, its second gradient has the symmetry property $((\nabla^{(2)}\varphi)\underset{\sim}{u})\underset{\sim}{v} = ((\nabla^{(2)}\varphi)\underset{\sim}{v})\underset{\sim}{u}$, $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$.

The modifier "of class C^r" may apply, in particular, to a <u>scalar field</u>, i.e. a mapping $f : \mathcal{G} \rightarrow \mathcal{R}$, a <u>vector field</u>, i.e. a mapping $\underset{\sim}{h} : \mathcal{G} \rightarrow \mathcal{V}$, or a <u>tensor field</u>, i.e., a mapping

$\underset{\sim}{I} : \mathcal{G} \to \mathcal{L}$. A one-to-one mapping $\underset{\sim}{\lambda} : \mathcal{G} \to \mathcal{E}$ is called a <u>deformation of class C^r</u> $(r \geq 1)$ if it is not only of class C^r but if also the values of its gradient are invertible, i.e. if $\nabla \underset{\sim}{\lambda}(\underset{\sim}{x}) \in \mathcal{L}$ for all $\underset{\sim}{x} \in \mathcal{G}$.

The members of the linear group $\mathcal{L}$ are also called <u>local deformations,</u> so that a (global) deformation has a gradient whose values are local deformations.

3. Continuous bodies.

A physical object can often be described mathematically by the concept of a <u>body</u> $\mathcal{B}$, which is a set whose members $X, Y, \ldots$ are called <u>material points</u>, and which is endowed with a structure defined by a class $\underline{\underline{C}}$ of mappings $\underset{\sim}{\varkappa} : \mathcal{B} \to \mathcal{E}$. The mappings $\underset{\sim}{\varkappa} \in \underline{\underline{C}}$ are called the <u>configurations</u> of $\mathcal{B}$ (in the space $\mathcal{E}$). The spatial point $\underset{\sim}{\varkappa}(X) \in \mathcal{E}$ is called the <u>place</u> of the material point $X \in \mathcal{B}$ in the configuration $\underset{\sim}{\varkappa}$.

We say that $\mathcal{B}$ is a <u>continuous body of class</u> C^p $(p \geq 1)$ if the class $\underline{\underline{C}}$ of configurations satisfies the following axioms:

(C1) Every $\underset{\sim}{\varkappa} \in \underline{\underline{C}}$ is one-to-one and its range $\underset{\sim}{\varkappa}(\mathcal{B})$ is an open subset of $\mathcal{E}$, which is called the <u>region occupied by $\mathcal{B}$ in the configuration</u> $\underset{\sim}{\varkappa}$.

(C2) If $\underset{\sim}{\gamma}, \underset{\sim}{\varkappa} \in \underline{\underline{C}}$ then the composite[1] $\underset{\sim}{\lambda} = \underset{\sim}{\gamma} \circ \overset{-1}{\underset{\sim}{\varkappa}} : \underset{\sim}{\varkappa}(\mathcal{B}) \to \underset{\sim}{\gamma}(\mathcal{B})$ is a deformation of class C^p, which is called the

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[1] Composition of mappings other than linear mappings is denoted by $\circ$. The inverse of a one-to-one mapping $\underset{\sim}{\varkappa}$ is denoted by $\overset{-1}{\underset{\sim}{\varkappa}}$.

deformation of $\mathcal{B}$ from the configuration $\underset{\sim}{\varkappa}$ into the configuration $\underset{\sim}{\gamma}$.

(C3) If $\underset{\sim}{\varkappa} \in \underline{\underline{C}}$ and if $\underset{\sim}{\lambda} : \underset{\sim}{\varkappa}(\mathcal{B}) \to \mathcal{E}$ is a deformation of class C^p , then $\underset{\sim}{\lambda} \circ \underset{\sim}{\varkappa} \in \underline{\underline{C}}$. The mapping $\underset{\sim}{\lambda} \circ \underset{\sim}{\varkappa}$ is called the configuration obtained from the configuration $\underset{\sim}{\varkappa}$ by the deformation $\underset{\sim}{\lambda}$.

In the remainder of this paper we shall always assume that $\mathcal{B}$ is a continuous body of class C^p , $p \geqq 1$.

The axioms (C1)-(C3) ensure that the class $\underline{\underline{C}}$ endows the body $\mathcal{B}$ with the structure of a " C^p- manifold modelled on $\mathcal{E}$ " in the sense of S. Lang ([3], Ch. II, §1). Topologically, it is a very simple manifold because it can be mapped out with a single "chart" ("configuration" in our terminology).

Of central importance for the present paper is the concept of a local configuration[1] at a material point X . Two (global) configurations $\underset{\sim}{\varkappa}$ and $\underset{\sim}{\gamma}$ are said to be equivalent at X and we write[2]

$$\underset{\sim}{\varkappa} \sim_X \underset{\sim}{\gamma} \qquad \text{if} \qquad \nabla(\underset{\sim}{\varkappa} \circ \underset{\sim}{\gamma}^{-1})\Big|_{\underset{\sim}{\gamma}(X)} = \underset{\sim}{1} . \tag{3.1}$$

It is an immediate consequence of the chain rule for gradients that $\sim_X$ is an equivalence relation on $\underline{\underline{C}}$. The resulting partition of $\underline{\underline{C}}$ is denoted by $\mathcal{C}_X$ and its members $\underset{\sim}{K}_X, \underset{\sim}{G}_X, \ldots,$ i.e. the equivalence classes, are called local configurations at X . Instead of writing $\underset{\sim}{\varkappa} \in \underset{\sim}{K}_X$ when $\underset{\sim}{\varkappa}$ is a member of the class $\underset{\sim}{K}_X$ we often write

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1) The term "configuration gradient" was used and another meaning was assigned to the term "local configuration" in [1].

2) For better reading, we sometimes write $f|_x$ instead of $f(x)$ for the value of f at x .

$$\nabla \underset{\sim}{\varkappa}(\underset{\sim}{X}) = \underset{\sim}{K}_{\underset{\sim}{X}} \tag{3.2}$$

and say that the local configuration $\underset{\sim}{K}_{\underset{\sim}{X}}$ is the gradient at $\underset{\sim}{X}$ of the (global) configuration $\underset{\sim}{\varkappa}$.

Let $\underset{\sim}{K}_{\underset{\sim}{X}}, \underset{\sim}{G}_{\underset{\sim}{X}} \in \mathcal{C}_{\underset{\sim}{X}}$ be two local configurations and let $\underset{\sim}{\varkappa} \in \underset{\sim}{K}_{\underset{\sim}{X}}, \underset{\sim}{\gamma} \in \underset{\sim}{G}_{\underset{\sim}{X}}$. It is easily seen that the local deformation $\nabla(\underset{\sim}{\gamma} \circ \underset{\sim}{\varkappa}^{-1})\big|_{\underset{\sim}{\varkappa}(\underset{\sim}{X})} \in \ell$ depends only on $\underset{\sim}{K}_{\underset{\sim}{X}}$ and $\underset{\sim}{G}_{\underset{\sim}{X}}$, and not on the particular choices of $\underset{\sim}{\varkappa} \in \underset{\sim}{K}_{\underset{\sim}{X}}$ and $\underset{\sim}{\gamma} \in \underset{\sim}{G}_{\underset{\sim}{X}}$. We denote this local deformation by $\underset{\sim}{G}_{\underset{\sim}{X}} \underset{\sim}{K}_{\underset{\sim}{X}}^{-1}$ and call it the <u>local deformation from the local configuration</u> $\underset{\sim}{K}_{\underset{\sim}{X}}$ <u>into the local configuration</u> $\underset{\sim}{G}_{\underset{\sim}{X}}$. Using the notation (3.2) we then have

$$\nabla(\underset{\sim}{\gamma} \circ \underset{\sim}{\varkappa}^{-1})\big|_{\underset{\sim}{\varkappa}(\underset{\sim}{X})} = \nabla \underset{\sim}{\gamma}(\underset{\sim}{X}) \left[\nabla \underset{\sim}{\varkappa}(\underset{\sim}{X})\right]^{-1}. \tag{3.3}$$

If $\underset{\sim}{K}_{\underset{\sim}{X}} \in \mathcal{C}_{\underset{\sim}{X}}$ is a local configuration and $\underset{\sim}{L} \in \ell$ any local deformation, we can define a new local configuration $\underset{\sim}{L}\underset{\sim}{K}_{\underset{\sim}{X}} \in \mathcal{C}_{\underset{\sim}{X}}$ by

$$\underset{\sim}{L}\underset{\sim}{K}_{\underset{\sim}{X}} = \{ \underset{\sim}{\lambda} \circ \underset{\sim}{\varkappa} \mid \nabla \underset{\sim}{\lambda}\big|_{\varkappa(\underset{\sim}{X})} = \underset{\sim}{L} ,\ \nabla \underset{\sim}{\varkappa}(\underset{\sim}{X}) = \underset{\sim}{K}_{\underset{\sim}{X}} \}. \tag{3.4}$$

We call $\underset{\sim}{L}\underset{\sim}{K}_{\underset{\sim}{X}}$ the <u>local configuration obtained from the local configuration</u> $\underset{\sim}{K}_{\underset{\sim}{X}}$ <u>by the local deformation</u> $\underset{\sim}{L}$. Clearly, we have the rules

$$(\underset{\sim}{G}_{\underset{\sim}{X}} \underset{\sim}{K}_{\underset{\sim}{X}}^{-1}) \underset{\sim}{K}_{\underset{\sim}{X}} = \underset{\sim}{G}_{\underset{\sim}{X}} , \quad (\underset{\sim}{L}\underset{\sim}{K}_{\underset{\sim}{X}}) \underset{\sim}{K}_{\underset{\sim}{X}}^{-1} = \underset{\sim}{L}. \tag{3.5}$$

4. Tangent spaces.

Consider pairs $(\underset{\sim}{K}_{\underset{\sim}{X}}, \underset{\sim}{u})$, where $\underset{\sim}{K}_{\underset{\sim}{X}} \in \mathcal{C}_{\underset{\sim}{X}}$ is a local configuration at $\underset{\sim}{X}$ and $\underset{\sim}{u} \in \mathcal{V}$ a spatial vector. We say that two such pairs $(\underset{\sim}{K}_{\underset{\sim}{X}}, \underset{\sim}{u})$ and $(\underset{\sim}{G}_{\underset{\sim}{X}}, \underset{\sim}{v})$ are <u>equivalent</u> if

$$(\underset{\sim}{K}_X \underset{\sim}{G}_X^{-1})\underset{\sim}{v} = \underset{\sim}{u} \tag{4.1}$$

It follows from the rules (3.5) that (4.1) does indeed define an equivalence relation. The resulting equivalence classes are called <u>tangent vectors</u> $\check{u}_X, v_X, \ldots$ at X. The totality of all these tangent vectors is denoted by $\mathcal{T}_X$ and is called the <u>tangent space</u> at $X \in \mathcal{B}$. Let $\check{u}_X \in \mathcal{T}_X$ and $\underset{\sim}{K}_X \in \mathcal{C}_X$ be given and let $(\underset{\sim}{G}_X, \underset{\sim}{v})$ be any pair belonging to the class $\check{u}_X$. Now, if $(\underset{\sim}{K}_X, \underset{\sim}{u})$ is to belong to $\check{u}_X$ then (4.1) must hold. Therefore, we see that $\check{u}_X \in \mathcal{T}_X$ and $\underset{\sim}{K}_X \in \mathcal{C}$ determine a unique spatial vector $\underset{\sim}{u} \in \mathcal{V}$ such that $(\underset{\sim}{K}_X, \underset{\sim}{u}) \in \check{u}_X$. We can therefore use the notation

$$\underset{\sim}{u} = \underset{\sim}{K}_X \check{u}_X, \quad \check{u}_X = \underset{\sim}{K}_X^{-1} \underset{\sim}{u} \qquad \text{if} \qquad (\underset{\sim}{K}_X, \underset{\sim}{u}) \in \check{u}_X, \tag{4.2}$$

and we see that $\underset{\sim}{K}_X$ determines a one-to-one mapping of the tangent space $\mathcal{T}_X$ onto the space $\mathcal{V}$ of spatial vectors.

The tangent space $\mathcal{T}_X$ has the natural structure of a three-dimensional vector space, with addition defined by

$$\check{u}_X + v_X = \underset{\sim}{K}_X^{-1}(\underset{\sim}{u} + \underset{\sim}{v}) \qquad \text{if} \quad \check{u}_X = \underset{\sim}{K}_X^{-1}\underset{\sim}{u},\ v_X = \underset{\sim}{K}_X^{-1}\underset{\sim}{v} \tag{4.3}$$

and multiplication with scalars by

$$\alpha \check{u}_X = \underset{\sim}{K}_X^{-1}(\alpha \underset{\sim}{u}) \qquad \text{if} \quad \check{u}_X = \underset{\sim}{K}_X^{-1}\underset{\sim}{u}, \quad \alpha \in \mathcal{R}. \tag{4.4}$$

It is immediately seen that these definitions of $\check{u}_X + v_X$ and $\alpha \check{u}_X$ are legitimate because the results are independent of the choice of the local configuration $\underset{\sim}{K}_X$ used to represent $\check{u}_X$ and v_X in $\mathcal{V}$. The local configurations can be identified with the invertible linear transformations of $\mathcal{T}_X$ onto $\mathcal{V}$.

Given a local configuration $\underset{\sim}{K}_X \in \mathcal{C}_X$, we can define an inner product $\breve{u}_X * w_X$ of $\breve{u}_X, w_X \in \breve{\mathcal{T}}_X$ by

$$\breve{u}_X * w_X = (\underset{\sim}{K}_X^{-1} \breve{u}_X) \cdot (\underset{\sim}{K}_X^{-1} w_X). \tag{4.5}$$

However, we obtain different inner products on $\breve{\mathcal{T}}_X$ for different choices of $\underset{\sim}{K}_X$, and hence $\breve{\mathcal{T}}_X$ is not naturally an inner product space.

5. Simple bodies, material isomorphisms, intrinsic isotropy groups.

To describe mathematically the physical characteristics of a body $\mathcal{B}$ we must endow $\mathcal{B}$ with additional structure. Some of these characteristics, such as elasticity, viscosity, heat capacity, and electrical conductivity, are local, i.e., they are attached to the individual material points $X \in \mathcal{B}$ rather than to the body as a whole. Other characteristics, such as mutual gravitation and internal radiative heat transfer, involve more than one material point. We deal here only with local characteristics. The physical response of the body $\mathcal{B}$ at a particular material point $X \in \mathcal{B}$ and a particular time will depend on the configuration $\underset{\sim}{\varkappa}$ of $\mathcal{B}$ at that time. It may happen that only the local configuration $\nabla\underset{\sim}{\varkappa}(X)$ at X determined by $\underset{\sim}{\varkappa}$, and no other properties of $\underset{\sim}{\varkappa}$, has an influence on the response. If this is the case, we say that the <u>material at X is simple</u>. We say that the whole of $\mathcal{B}$ is simple or that $\mathcal{B}$ is a <u>simple body</u> if the material at X is simple for all $X \in \mathcal{B}$.

We assume that a possible physical response at a material point is given mathematically by specifying an element from a set

$\mathcal{R}$ of mathematical objects. The nature of $\mathcal{R}$ depends on the particular physical phenomena to be described. For example, in the theory of elasticity $\mathcal{R}$ consists of all possible 'stress tensors', i.e., of all symmetric linear transformations of $\mathcal{V}$ into $\mathcal{V}$. In the mechanical theory of simple materials with fading memory, $\mathcal{R}$ consists of "memory functionals" that relate relative deformations histories to stresses and are subject to certain smoothness requirements. In theories that include non-mechanical effects $\mathcal{R}$ consists of functions or functionals whose independent and dependent variables have interpretations as local temperatures, energy or entropy densities, heat fluxes, electric or magnetic field strengths, polarizations, magnetizations, electric currents, etc. For the purpose of the present paper, no specific assumptions about the nature of $\mathcal{R}$ need be made.

We can now make our definition of a simple body precise:

Definition 1: Let $\mathcal{R}$ be a set, whose elements we call **response descriptors**. A continuous body $\mathcal{B}$ of class C^p will be called a **simple body** with respect to $\mathcal{R}$ if it is endowed with a structure by a function $\underset{\sim}{\mathcal{G}}$ which assigns to each material point $X \in \mathcal{B}$ a mapping

$$\underset{\sim}{\mathcal{G}}_X : \mathcal{C}_X \to \mathcal{R}. \tag{5.1}$$

The value $\underset{\sim}{\mathcal{G}}_X(\underset{\sim}{G}_X)$ is the response descriptor of the material at X in any configuration $\underset{\sim}{\chi}$ of $\mathcal{B}$ such that $\nabla \underset{\sim}{\chi}(X) = \underset{\sim}{G}_X$.

The mappings $\underset{\sim}{\mathcal{G}}_X$ cannot be entirely arbitrary, they are subject to restrictions imposed by general physical principles such as the principle of frame-indifference and the principle of dissipation. These restrictions need not be made explicit here.

We would like to give now a precise meaning to the statement that the material at one point $X \in \mathcal{B}$ is the same as the material at another point $Y \in \mathcal{B}$. We cannot construe this statement to mean that $\underset{\sim}{\mathcal{G}}_X$ and $\underset{\sim}{\mathcal{G}}_Y$ are the same, for they have different domains and hence cannot be directly compared. However, we can connect the domains $\mathcal{C}_X$ and $\mathcal{C}_Y$ if an isomorphism $\Phi_{XY}: \mathcal{T}_Y \to \mathcal{T}_X$ of the tangent space at Y onto the tangent space at X is given. Recalling that a local configuration $\underset{\sim}{G}_X \in \mathcal{C}_X$ can be regarded as a mapping $\underset{\sim}{G}_X: \mathcal{T}_X \to \mathcal{V}$, we can let $\underset{\sim}{G}_X \in \mathcal{C}_X$ correspond to the composition $\underset{\sim}{G}_X \Phi_{XY} \in \mathcal{C}_Y$. We are thus led to the following definition:

<u>Definition 2:</u> <u>An invertible linear transformation</u> $\Phi_{XY}: \mathcal{T}_Y \to \mathcal{T}_X$ <u>is called a</u> *material isomorphism* <u>from</u> $\mathcal{T}_Y$ <u>onto</u> $\mathcal{T}_X$ <u>if</u>

$$\underset{\sim}{\mathcal{G}}_X(\underset{\sim}{G}_X) = \underset{\sim}{\mathcal{G}}_Y(\underset{\sim}{G}_X \Phi_{XY}) \tag{5.2}$$

<u>holds for all</u> $\underset{\sim}{G}_X \in \mathcal{C}_X$.

To say that the material at X is the same as the material at Y means that there exists a material isomorphism from $\mathcal{T}_X$ onto $\mathcal{T}_Y$.

It follows immediately from Definition 2 that if $\Phi_{XY}: \mathcal{T}_Y \to \mathcal{T}_X$ and $\Phi_{YZ}: \mathcal{T}_Z \to \mathcal{T}_Y$ are material isomorphisms, so is their composition $\Phi_{XY}\Phi_{YZ}: \mathcal{T}_Z \to \mathcal{T}_X$. Also, if $\Phi_{XY}: \mathcal{T}_Y \to \mathcal{T}_X$ is a material isomorphisms, so is its inverse $\Phi_{XY}^{-1}: \mathcal{T}_X \to \mathcal{T}_Y$. If we denote the set of all material isomorphisms from $\mathcal{T}_X$ onto

$\mathcal{T}_Y$ by $\mathcal{G}_{XY}$, these facts can be expressed by [1)]

$$\mathcal{G}_{ZY}\mathcal{G}_{YX} = \mathcal{G}_{ZY}\,, \qquad \mathcal{G}_{YX} = \mathcal{G}_{XY}^{-1} \tag{5.3}$$

It is clear that $\mathcal{G}_{XX}$, the set of all material isomorphisms of $\mathcal{T}_X$ onto itself, is a subgroup of the linear group $\mathcal{L}_X$ of $\mathcal{T}_X$, which consists of all invertible linear transformations of $\mathcal{T}_X$. We write

$$\mathcal{G}_X = \mathcal{G}_{XX} \tag{5.4}$$

and call $\mathcal{G}_X$ the <u>intrinsic isotropy group</u> of the material at X For any $\Phi_{XY} \in \mathcal{G}_{XY}$ one easily establishes the relations

$$\mathcal{G}_{XY} = \mathcal{G}_X \Phi_{XY} \mathcal{G}_Y\,, \qquad \mathcal{G}_X = \Phi_{XY}\mathcal{G}_Y\Phi_{XY}^{-1}\,. \tag{5.5}$$

It follows from $(5.5)_2$ that if a material isomorphism $\Phi_{XY}\ \mathcal{T}_Y \to \mathcal{T}_X$ exists, i.e. if the material at X is the same as the material at Y, then the intrinsic isotropy groups $\mathcal{G}_X$ and $\mathcal{G}_Y$ are isomorphic.

6. Material uniformity, uniform references, relative isotropy groups.

A simple body $\mathcal{B}$ is said to be <u>materially uniform</u> if the material at any two of its points is the same, i.e. if $\mathcal{G}_{XY}$ is never empty. From now on we assume that $\mathcal{B}$ is a materially uniform simple body. We select a member $\Phi'(X,Y)$ from each $\mathcal{G}_{XY}$

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1) If $\mathcal{G}$ and $\mathcal{H}$ are sets of linear transformations of any kind such that the composition $\mathbf{L}\mathbf{M}$ makes sense whenever $\mathbf{L} \in \mathcal{G}$, $\mathbf{M} \in \mathcal{H}$, we write $\mathcal{G}\mathcal{H} = \{\mathbf{L}\mathbf{M} \mid \mathbf{L} \in \mathcal{G}, \mathbf{M} \in \mathcal{H}\}$. If the $\mathbf{L} \in \mathcal{G}$ are invertible, we write $\mathcal{G}^{-1} = \{\mathbf{L}^{-1} \mid \mathbf{L} \in \mathcal{G}\}$. Also, we write $\mathbf{K}\mathcal{G} = \{\mathbf{K}\mathbf{L} \mid \mathbf{L} \in \mathcal{G}\}$ if $\mathbf{K}\mathbf{L}$ makes sense for all $\mathbf{L} \in \mathcal{G}$.

and thereby define a function Φ' which assigns to each pair (X,Y) of material points of $\mathcal{B}$ a material isomorphism from $\mathcal{T}_X$ onto $\mathcal{T}_Y$. Choose $X_0 \in \mathcal{B}$ arbitrarily and define Φ by

$$\Phi(X,Y) = \Phi'(X,X_0)\,\Phi'(Y,X_0)^{-1}. \tag{6.1}$$

It follows from (5.3) that $\Phi(X,Y) \in g_{XY}$. Moreover, we have

$$\Phi(Z,Y)\,\Phi(Y,X) = \Phi(Z,X), \qquad \Phi(X,X) = 1_X, \tag{6.2}$$

where 1_X is the identity transformation of $\mathcal{T}_X$.

<u>Definition 3</u>: <u>A function</u> Φ <u>which assigns to each pair</u> (X,Y) <u>of material points of</u> $\mathcal{B}$ <u>a material isomorphism</u> $\Phi(X,Y) \in g_{XY}$ <u>is called a</u> <u>material uniformity</u> <u>if</u> (6.2) <u>holds</u>.

The construction (6.1) shows that the materially uniform bodies are those that admit material uniformities. It follows from (5.5) and (6.2) that any two material uniformities Φ and $\hat{\Phi}$ are related by

$$\hat{\Phi}(X,Y) = P(X)\,\Phi(X,Y)\,P(Y)^{-1}, \tag{6.3}$$

where P is a function on $\mathcal{B}$ whose values $P(X)$ belong to the intrinsic isotropy groups g_X.

<u>Definition 4</u>: <u>A function</u> K <u>on</u> $\mathcal{B}$ <u>whose values</u> $K(X) \in \mathcal{C}_X$ <u>are local configurations is called a</u> <u>reference</u> <u>for</u> $\mathcal{B}$. <u>If, moreover,</u>

$$\Phi(X,Y) = K(X)^{-1}K(Y) \tag{6.4}$$

<u>is a material isomorphism of</u> $\mathcal{T}_Y$ <u>onto</u> $\mathcal{T}_X$ <u>for any</u> $X, Y \in \mathcal{B}$, <u>then</u> K <u>is called a</u> <u>uniform reference</u> for $\mathcal{B}$.

Actually, (6.2) holds if Φ is defined by (6.4), so that Φ is a material uniformity if $\underset{\sim}{K}$ is a uniform reference. Hence, every uniform reference $\underset{\sim}{K}$ determines a material uniformity Φ through (6.4). Conversely, if a material uniformity Φ and a local configuration $\underset{\sim}{K}_{X_0} \in \mathcal{C}_{X_0}$ for a particular material point $X_0 \in \mathcal{B}$ are given, then there exist a unique uniform reference $\underset{\sim}{K}$ such that (6.4) and $\underset{\sim}{K}(X_0) = \underset{\sim}{K}_{X_0}$ hold. In fact, $\underset{\sim}{K}$ is given by

$$\underset{\sim}{K}(X) = \underset{\sim}{K}_{X_0}\, \Phi(X_0, X). \tag{6.5}$$

Therefore, every material uniformity has representations (6.4) in terms of uniform references.

If $\underset{\sim}{\varkappa}$ is a (global) configuration, then $\nabla\underset{\sim}{\varkappa}$, which assigns to X the local configuration $\nabla\underset{\sim}{\varkappa}(X)$ at X, i.e., the equivalence class to which $\underset{\sim}{\varkappa}$ belongs, is a reference, called the <u>gradient</u> of the configuration $\underset{\sim}{\varkappa}$. We say that a body is <u>homogeneous</u> if it admits a gradient as a uniform reference. Of course, not every reference is a gradient and it may happen that none of the uniform references of a materially uniform body is a gradient.

Let $\underset{\sim}{K}$ be a uniform reference. Every local configuration $\underset{\sim}{G}_X \in \mathcal{C}_X$ can be characterized by the local deformation $\underset{\sim}{F} = \underset{\sim}{G}_X \underset{\sim}{K}(X)^{-1} \in \mathcal{L}$ from $\underset{\sim}{K}(X)$ into $\underset{\sim}{G}_X$, so that

$$\underset{\sim}{G}_X = \underset{\sim}{F}\, \underset{\sim}{K}(X). \tag{6.6}$$

Substituting (6.6) and (6.4) into (5.2) with the choice $\Phi_{XY} = \Phi(X, Y)$ we see that

$$\mathcal{G}_X(\mathbf{F}\mathbf{K}(X)) = \mathcal{G}_Y(\mathbf{F}\mathbf{K}(Y)) \tag{6.7}$$

must hold for all $\mathbf{F} \in \mathcal{L}$ and all $X, Y \in \mathcal{B}$. Conversely, if (6.7) holds for all $\mathbf{F} \in \mathcal{L}$ and all $X, Y \in \mathcal{B}$, then $\mathbf{K}$ is a uniform reference. This result may be formulated as follows:

Theorem 1: A reference $\mathbf{K}$ for $\mathcal{B}$ is uniform if and only if there is a function $\mathfrak{G}_{\mathbf{K}} : \mathcal{L} \longrightarrow \mathbb{R}$ which satisfies

$$\mathfrak{G}_{\mathbf{K}}(\mathbf{F}) = \mathcal{G}_X(\mathbf{F}\mathbf{K}(X)) \tag{6.8}$$

for all $X \in \mathcal{B}$ and all $\mathbf{F} \in \mathcal{L}$.

The function $\mathfrak{G}_{\mathbf{K}}$, which assigns to each local deformation a response descriptor, will be called the response function of the body relative to the uniform reference $\mathbf{K}$.

Let $\mathbf{K}$ be uniform reference. If we substitute (6.4) for Φ_{XY} in (5.5) we see that

$$\mathbf{K}(X)\, g_X\, \mathbf{K}(X)^{-1} = \mathbf{K}(Y)\, g_Y\, \mathbf{K}(Y)^{-1}, \tag{6.9}$$

i.e. that

$$g_{\mathbf{K}} = \mathbf{K}(X)\, g_X\, \mathbf{K}(X)^{-1} \tag{6.10}$$

is independent of X. The group $g_{\mathbf{K}}$ is a subgroup of the linear group $\mathcal{L}$. We call $g_{\mathbf{K}}$ the isotropy group of the body $\mathcal{B}$ relative to the uniform reference $\mathbf{K}$. In view of (6.10), all the intrinsic isotropy groups g_X, $X \in \mathcal{B}$, are isomorphic to the relative isotropy group $g_{\mathbf{K}}$. It is easily seen that $g_{\mathbf{K}}$ is given in terms of the response function $\mathfrak{G}_{\mathbf{K}}$ by

$$g_{\mathbf{K}} = \left\{ \mathbf{P} \in \mathcal{L} \;\middle|\; \mathfrak{G}_{\mathbf{K}}(\mathbf{F}) = \mathfrak{G}_{\mathbf{K}}(\mathbf{F}\mathbf{P}) \text{ for all } \mathbf{F} \in \mathcal{L} \right\} \tag{6.11}$$

The relation between two uniform references and the corresponding response functions and isotropy groups is described by the following theorem:

Theorem 2: Any two uniform references $\underset{\sim}{K}$ and $\hat{\underset{\sim}{K}}$ are related by

$$\hat{\underset{\sim}{K}}(X) = \underset{\sim}{L}\,\underset{\sim}{P}(X)\,\underset{\sim}{K}(X), \tag{6.12}$$

where $\underset{\sim}{L} \in \mathcal{l}$ and where $\underset{\sim}{P}$ is a function on $\mathcal{B}$ with values in $g_{\underset{\sim}{K}}$.

The isotropy groups $g_{\underset{\sim}{K}}$ and $g_{\hat{\underset{\sim}{K}}}$ relative to $\underset{\sim}{K}$ and $\hat{\underset{\sim}{K}}$ are conjugate:

$$g_{\hat{\underset{\sim}{K}}} = \underset{\sim}{L}\, g_{\underset{\sim}{K}}\, \underset{\sim}{L}^{-1}. \tag{6.13}$$

The response functions $\underset{\sim}{\mathfrak{f}}_{\hat{\underset{\sim}{K}}}$ and $\underset{\sim}{\mathfrak{f}}_{\underset{\sim}{K}}$ are related by the identity

$$\underset{\sim}{\mathfrak{f}}_{\hat{\underset{\sim}{K}}}(\underset{\sim}{F}) = \underset{\sim}{\mathfrak{f}}_{\underset{\sim}{K}}(\underset{\sim}{F}\underset{\sim}{L}) \qquad \text{for all } \underset{\sim}{F} \in \mathcal{l}. \tag{6.14}$$

Proof: The two material uniformities Φ and $\hat{\Phi}$ given by

$$\Phi(X,Y) = \underset{\sim}{K}(X)^{-1}\underset{\sim}{K}(Y), \quad \hat{\Phi}(X,Y) = \hat{\underset{\sim}{K}}(X)^{-1}\hat{\underset{\sim}{K}}(Y)$$

must be related by (6.3). It follows that

$$\hat{\underset{\sim}{K}}(Y)\,\mathcal{P}(Y)\,\underset{\sim}{K}(Y)^{-1} = \hat{\underset{\sim}{K}}(X)\,\mathcal{P}(X)\,\underset{\sim}{K}(X)^{-1} = \underset{\sim}{L} \in \mathcal{l}$$

is independent of $X \in \mathcal{B}$. Hence (6.12) holds with the choice

$$\underset{\sim}{P}(X) = \underset{\sim}{K}(X)\,\mathcal{P}(X)\,\underset{\sim}{K}(X)^{-1}. \tag{6.15}$$

It follows from (6.10) that $\underset{\sim}{P}(X) \in g_{\underset{\sim}{K}}$ for all $X \in \mathcal{B}$, which proves the first assertion of the theorem. If we write (6.10) with $\underset{\sim}{K}$ replaced by $\hat{\underset{\sim}{K}}$ and substitute (6.12) we obtain

$$\mathcal{g}_{\hat{K}} = L\,P(X)\,K(X)\,\mathcal{g}_X\,K(X)^{-1}P(X)^{-1}L^{-1}$$
$$= L\,P(X)\,\mathcal{g}_K\,P(X)^{-1}L^{-1}.$$

Since $P(X) \in \mathcal{g}_K$ we have $P(X)\mathcal{g}_K P(X)^{-1} = \mathcal{g}_K$ and hence (6.13). The identity (6.14) is derived by writing (6.8) with K replaced by $\hat{K}$, then substituting (6.12) and observing (6.11). Q.E.D.

The theory of isotropy groups relative to a local reference configuration at a single material point[1)] extends without change to isotropy groups relative to a uniform reference K of a whole materially uniform body. In particular, we say that the uniform reference K is <u>undistorted</u> if $\mathcal{g}_K$ is comparable, with respect to inclusion, to the orthogonal group $\mathcal{O}$, i.e., if either $\mathcal{g}_K \subset \mathcal{O}$ or $\mathcal{O} \subset \mathcal{g}_K$. If there are uniform references K such that $\mathcal{g}_K \supset \mathcal{O}$, we say that $\mathcal{B}$ is a <u>uniform isotropic body</u>; if there are uniform references K such that $\mathcal{g}_K \subset \mathcal{O}$, we say that $\mathcal{B}$ is a <u>uniform solid body</u>. It is possible that a uniform simple body has no undistorted uniform references at all; such a body would be neither a solid nor isotropic.

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1) This theory was initiated in [1], §§ 19-21. An exposition is given in [2], §§ 31-33.

7. Vector and tensor fields.

As before, we assume that $\mathcal{B}$ is a continuous body of class C^p, $p \geqq 1$.

A mapping $\psi: \mathcal{B} \to \mathcal{E}'$ of $\mathcal{B}$ into some point-space or vector-space $\mathcal{E}'$ is said to be <u>of class</u> C^r, $0 \leqq r \leqq p$ if for every configuration $\underset{\sim}{\varkappa} \in \underline{\underline{C}}$, the mapping $\psi \circ \underset{\sim}{\varkappa}^{-1}: \underset{\sim}{\varkappa}(\mathcal{B}) \to \mathcal{E}'$ is of class C^r. In view of the axioms for $\mathcal{B}$ it is clear that $\psi \circ \underset{\sim}{\varkappa}^{-1}$ is of class C^p for <u>every</u> $\underset{\sim}{\varkappa} \in \underline{\underline{C}}$ if it is of class C^p for <u>some</u> $\underset{\sim}{\varkappa} \in \underline{\underline{C}}$. These definitions apply, in particular, to <u>functions</u> (<u>scalar fields</u>) on $\mathcal{B}$, i.e. mappings $f: \mathcal{B} \to \mathcal{R}$, to <u>vector fields</u> on $\mathcal{B}$, i.e. mappings $\underset{\sim}{h}: \mathcal{B} \to \mathcal{V}$, and to <u>tensor fields</u> on $\mathcal{B}$, i.e. mappings $\underset{\sim}{T}: \mathcal{B} \to \mathcal{L}$.

A mapping $\mathfrak{f}$ which assigns to each material point $X \in \mathcal{B}$ a tangent vector $\mathfrak{f}(X) \in \mathcal{T}_X$ is called a <u>tangent vector field</u>. We say that such a tangent vector field $\mathfrak{f}$ is of class C^r, $0 \leqq r \leqq p-1$, if the vector field $(\nabla \underset{\sim}{\varkappa})\mathfrak{f}$ on $\mathcal{B}$ defined by

$$(\nabla \underset{\sim}{\varkappa})\mathfrak{f}\Big|_X = (\nabla \underset{\sim}{\varkappa}(X))\,\mathfrak{f}(X) \tag{7.1}$$

is of class C^r for some — and hence every — configuration $\underset{\sim}{\varkappa} \in \underline{\underline{C}}$.

The algebra of all linear transformations of the tangent space $\mathcal{T}_X$ into itself will be denoted by $\mathcal{I}_X$. A mapping $\mathfrak{T}$ which assigns to each material point $X \in \mathcal{B}$ a linear transformation $\mathfrak{T}(X) \in \mathcal{I}_X$ is called an <u>intrinsic tensor field.</u> We say that $\mathfrak{T}$ is of class C^r, $0 \leqq r \leqq p-1$, if the tensor field $(\nabla \underset{\sim}{\varkappa})\,\mathfrak{T}\,(\nabla \underset{\sim}{\varkappa})^{-1}$ on $\mathcal{B}$ defined by

$$(\nabla \underset{\sim}{\varkappa})\,\mathfrak{T}\,(\nabla \underset{\sim}{\varkappa})^{-1}\Big|_X = \nabla \underset{\sim}{\varkappa}(X)\,\mathfrak{T}(X)\big(\nabla \underset{\sim}{\varkappa}(X)\big)^{-1} \tag{7.2}$$

is of class C^r for some--and hence every--configuration $\underset{\sim}{\varkappa} \in \underline{\underline{C}}$.

We shall use the term <u>field</u> on $\mathcal{B}$ for any mapping that assigns to every $X \in \mathcal{B}$ an element of some vector space (which may consist of linear or multilinear transformations).

We shall employ the following scheme of notation:

$\mathcal{F}_{\mathcal{B}}^r$ = set of all functions (scalar fields) of class C^r on $\mathcal{B}$.
$\mathcal{V}_{\mathcal{B}}^r$ = set of all vector fields of class C^r on $\mathcal{B}$.
$\mathcal{T}_{\mathcal{B}}^r$ = set of all tangent vector fields of class C^r on $\mathcal{B}$.
$\mathcal{L}_{\mathcal{B}}^r$ = set of all tensor fields of class C^r on $\mathcal{B}$.
$\mathcal{Y}_{\mathcal{B}}^r$ = set of all intrinsic tensor fields of class C^r on $\mathcal{B}$.

The set $\mathcal{F}_{\mathcal{B}}^r$ is a commutative algebra under pointwise addition and multiplication. The sets $\mathcal{V}_{\mathcal{B}}^s$, $\mathcal{T}_{\mathcal{B}}^s$, $\mathcal{L}_{\mathcal{B}}^s$, and $\mathcal{Y}_{\mathcal{B}}^s$ can be made modules with respect to any of the algebras $\mathcal{F}_{\mathcal{B}}^r$, $s \leqq r \leqq p-1$, by defining addition and scalar multiplication with functions in $\mathcal{F}_{\mathcal{B}}^r$ pointwise. For example, if $\boldsymbol{f}, \boldsymbol{h} \in \mathcal{T}_{\mathcal{B}}^r$ and $f \in \mathcal{F}_{\mathcal{B}}^r$ we define $\boldsymbol{f}+\boldsymbol{h} \in \mathcal{T}_{\mathcal{B}}^r$ and $f\boldsymbol{f} \in \mathcal{T}_{\mathcal{B}}^r$ by

$$(\boldsymbol{f}+\boldsymbol{h})\big|_X = \boldsymbol{f}(X)+\boldsymbol{h}(X), \quad (f\boldsymbol{f})\big|_X = f(X)\boldsymbol{f}(X), \; X \in \mathcal{B}. \tag{7.3}$$

The sets $\mathcal{L}_{\mathcal{B}}^r$ and $\mathcal{Y}_{\mathcal{B}}^r$ become associative (but not commutative) algebras over $\mathcal{F}_{\mathcal{B}}^r$ if multiplication is defined pointwise.

It is evident that we have $\mathcal{F}_{\mathcal{B}}^r \subset \mathcal{F}_{\mathcal{B}}^s$ if $s \leqq r$ and similar inclusions for the other sets in the list given above. Actually, $\mathcal{F}_{\mathcal{B}}^r$ is a subalgebra of $\mathcal{F}_{\mathcal{B}}^s$. Also, $\mathcal{V}_{\mathcal{B}}^r$ is not only a $\mathcal{F}_{\mathcal{B}}^r$-module, but also a submodule of $\mathcal{V}_{\mathcal{B}}^s$, regarded as a $\mathcal{F}_{\mathcal{B}}^r$-module. Analogous observations apply to the other modules and algebras of the list above.

If $\underset{\sim}{T} \in \mathcal{L}_{\mathcal{B}}^{s}$ and $\underset{\sim}{h} \in \mathcal{V}_{\mathcal{B}}^{r}$ or $\mathcal{Z} \in \mathcal{Y}_{\mathcal{B}}^{s}$ and $f \in \mathcal{T}_{\mathcal{B}}^{r}$ we define $\underset{\sim}{T}\underset{\sim}{h}$ or $\mathcal{Z}f$ pointwise, i.e. by

$$\underset{\sim}{T}\underset{\sim}{h}\Big|_{X} = \underset{\sim}{T}(X)\,\underset{\sim}{h}(X)\,, \quad \mathcal{Z}f\Big|_{X} = \mathcal{Z}(X)f(X). \tag{7.4}$$

When $s \leqq r$, one can see that $\underset{\sim}{T}\underset{\sim}{h} \in \mathcal{V}_{\mathcal{B}}^{s}$, $\mathcal{Z}f \in \mathcal{T}_{\mathcal{B}}^{s}$. It is evident from $(7.4)_1$ that the rules

$$\underset{\sim}{T}(\underset{\sim}{h}+\underset{\sim}{k}) = \underset{\sim}{T}\underset{\sim}{h} + \underset{\sim}{T}\underset{\sim}{k}\,, \quad \underset{\sim}{T}(f\underset{\sim}{h}) = f\underset{\sim}{T}\underset{\sim}{h} \tag{7.5}$$

are valid. Hence every $\underset{\sim}{T} \in \mathcal{L}_{\mathcal{B}}^{s}$ gives rise to a mapping

$$\underset{\sim}{T}: \mathcal{V}_{\mathcal{B}}^{r} \to \mathcal{V}_{\mathcal{B}}^{s} \tag{7.6}$$

which satisfies the rules (7.5) for $\underset{\sim}{h}, \underset{\sim}{k} \in \mathcal{V}_{\mathcal{B}}^{r}$, $f \in \mathcal{F}_{\mathcal{B}}^{r}$. Mappings of the type (7.6) satisfying the rules (7.5) are homomorphism with respect to the $\mathcal{F}_{\mathcal{B}}^{r}$-module structures of $\mathcal{V}_{\mathcal{B}}^{r}$ and $\mathcal{V}_{\mathcal{B}}^{s}$. We also call them <u>$\mathcal{F}$-linear mappings</u>. Thus, every $\underset{\sim}{T} \in \mathcal{L}_{\mathcal{B}}^{s}$ gives rise to an $\mathcal{F}$-linear mapping (7.6). It is remarkable that the converse is also true, i.e. that every $\mathcal{F}$-linear mapping of the type (7.6) arises from a tensor field of class C^{s} on $\mathcal{B}$:

<u>Proposition 1</u>: <u>If</u> $\underset{\sim}{T}: \mathcal{V}_{\mathcal{B}}^{r} \to \mathcal{V}_{\mathcal{B}}^{s}\ (s \leq r)$ <u>is</u> <u>$\mathcal{F}$-linear, then there exist a unique tensor field</u> $\overline{T} \in \mathcal{Y}_{\mathcal{B}}^{s}$ <u>such that</u> $\underset{\sim}{T}\underset{\sim}{h} = \overline{\underset{\sim}{T}}\underset{\sim}{h}$ <u>holds for all</u> $\underset{\sim}{h} \in \mathcal{T}_{\mathcal{B}}^{r}$.

<u>Proof</u>: Let $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ be a basis of $\mathcal{V}$. The vectors $\underset{\sim}{e}_i$ can be regarded as constant vector fields on $\mathcal{B}$, so that $\underset{\sim}{e}_i \in \mathcal{V}_{\mathcal{B}}^{p-1} \subset \mathcal{V}_{\mathcal{B}}^{r}$. Every $\underset{\sim}{h} \in \mathcal{V}_{\mathcal{B}}^{r}$ has a unique component representation

$$\underset{\sim}{h} = \sum_i h^i \underset{\sim}{e}_i\,, \quad h^i \in \mathcal{F}_{\mathcal{B}}^{r}. \tag{7.7}$$

Applying the given $\mathcal{F}$-linear mapping $\underset{\sim}{T}$ to (7.7) we obtain

$$\underset{\sim}{T}\,\underset{\sim}{h} = \sum_i h^i \underset{\sim}{T}\,\underset{\sim}{e}_i \,. \tag{7.8}$$

Now, if there is a tensor field $\overline{T}$ such that $\underset{\sim}{T}\,\underset{\sim}{h} = \overline{T}\,\underset{\sim}{h}$ for all $\underset{\sim}{h} \in \mathcal{V}_{\mathcal{B}}^{r}$, we must have, in particular, $\underset{\sim}{T}\,\underset{\sim}{e}_i = \overline{T}\,\underset{\sim}{e}_i$, i.e.

$$\overline{\underset{\sim}{T}}(X)\,\underset{\sim}{e}_i = (\underset{\sim}{T}\,\underset{\sim}{e}_i)\big|_X \tag{7.9}$$

for all $X \in \mathcal{B}$. But since $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ is a basis of $\mathcal{V}$, we can find, for each $X \in \mathcal{B}$, exactly one $\overline{\underset{\sim}{T}}(X) \in \mathcal{L}$ such that (7.9) holds. Since the vector fields $\underset{\sim}{T}\,\underset{\sim}{e}_i$ are of class C^s, it is easily seen that the tensor field $\overline{T}$ obtained in this way is also of class C^s. Moreover, in view of (7.8), (7.9) and the $\mathcal{F}$-linearity of $\underset{\sim}{T}$ we have

$$\overline{T}\,\underset{\sim}{h} = \sum_i h^i\, \overline{T}\,\underset{\sim}{e}_i = \sum_i h^i\, \underset{\sim}{T}\,\underset{\sim}{e}_i = \underset{\sim}{T}\Big(\sum_i h^i \underset{\sim}{e}_i\Big) = \underset{\sim}{T}\,\underset{\sim}{h}$$

for all $\underset{\sim}{h}$. Q.E.D.

Proposition 1 enables us to identify the set of all $\mathcal{F}$-linear mappings of the type (7.6) with the set $\mathcal{L}_{\mathcal{B}}^{s}$ of all tensor fields of class C^s on $\mathcal{B}$. Similarly, we can identify the set of all $\mathcal{F}$-linear mappings of the type

$$\mathcal{T} : \mathcal{T}_{\mathcal{B}}^{r} \longrightarrow \mathcal{T}_{\mathcal{B}}^{s} \qquad (s \leqq r)$$

with the set $\mathcal{Y}_{\mathcal{B}}^{s}$ of all intrinsic tensor fields of class C^s on $\mathcal{B}$. The proof of this fact follows from Proposition 1 by choosing a configuration $\underset{\sim}{\varkappa}$ of $\mathcal{B}$ and letting $\mathcal{T}$ correspond to $\overline{T} = \nabla\varkappa\, \mathcal{T}\, (\nabla\varkappa)^{-1} : \mathcal{V}_{\mathcal{B}}^{r} \to \mathcal{V}_{\mathcal{B}}^{s}$. The result just stated is a special case of a general proposition referring to

$\mathcal{F}$-multilinear mappings. For later application we state another special case:

Proposition 2: If

$$\mathcal{T} : \mathcal{T}_{\mathcal{B}}^{r} \times \mathcal{T}_{\mathcal{B}}^{r} \longrightarrow \mathcal{T}_{\mathcal{B}}^{s} \quad (\text{or } \mathcal{Y}_{\mathcal{B}}^{s}) \tag{7.10}$$

is $\mathcal{F}$-bilinear (i.e. $\mathcal{F}$-linear in each of the two variables), then there exist a unique field $\overline{\mathcal{T}}$ on $\mathcal{B}$ whose values $\overline{\mathcal{T}}(X)$ are bilinear mappings

$$\overline{\mathcal{T}}(X) : \mathcal{T}_{X} \times \mathcal{T}_{X} \rightarrow \mathcal{T}_{X} \quad (\text{or } \mathcal{Y}_{X}) \tag{7.11}$$

such that

$$\overline{\mathcal{T}}(X)(f(X), k(X)) = \mathcal{T}(f, k)\big|_{X} \tag{7.12}$$

holds for all $f, k \in \mathcal{T}_{\mathcal{B}}^{r}$ and all $X \in \mathcal{B}$. The function $\overline{\mathcal{T}}$ is of class C^{s} (in the obvious sense).

8. Relative gradients, brackets.

From now on we assume that $\mathcal{B}$ is a continuous body of class C^{p} with $p \geqq 2$.

Let $\psi : \mathcal{B} \rightarrow \mathcal{E}'$ be a mapping of class C^{r}, $1 \leqq r \leqq p$, where $\mathcal{E}'$ is some point space or vector space. Given a configuration $\underset{\sim}{\kappa}$ of $\mathcal{B}$, we can then define

$$\nabla_{\underset{\sim}{\kappa}} \psi : \mathcal{B} \rightarrow \mathcal{L}(\mathcal{V}, \mathcal{V}') \tag{8.1}$$

where $\mathcal{V}'$ is the translation space of $\mathcal{E}'$, by $\nabla_{\underset{\sim}{\kappa}} \psi = \nabla(\psi \circ \underset{\sim}{\kappa}^{-1}) \circ \underset{\sim}{\kappa}$, i.e.

$$\nabla_{\underset{\sim}{\kappa}} \psi\big|_{X} = \nabla(\psi \circ \underset{\sim}{\kappa}^{-1})\big|_{\underset{\sim}{\kappa}(X)}, \quad X \in \mathcal{B}. \tag{8.2}$$

We call $\nabla_{\underset{\sim}{\kappa}} \psi$ the <u>gradient of</u> ψ <u>relative to the configuration</u> $\underset{\sim}{\kappa}$. It is clear that $\nabla_{\underset{\sim}{\kappa}} \psi$ is of class C^{r-1}.

Let $\underset{\sim}{\kappa}, \underset{\sim}{\gamma} \in \underline{\underline{C}}$ be two configurations. Taking the gradient of $\psi \circ \underset{\sim}{\kappa}^{-1} = (\psi \circ \underset{\sim}{\gamma}^{-1}) \circ (\underset{\sim}{\gamma} \circ \underset{\sim}{\kappa}^{-1})$ and using the chain rule, we see with the help of (3.3) that the gradients of ψ relative to $\underset{\sim}{\kappa}$ and $\underset{\sim}{\gamma}$ are related by

$$\nabla_{\underset{\sim}{\kappa}} \psi \big|_X = \nabla_{\underset{\sim}{\gamma}} \psi \big|_X \circ \left[\nabla \underset{\sim}{\gamma}(X) (\nabla \underset{\sim}{\kappa}(X))^{-1} \right]. \tag{8.3}$$

Let $\underset{\sim}{K}$ and $\underset{\sim}{G}$ be two references for $\mathcal{B}$ (see Definition 4). We define $\underset{\sim}{K}\underset{\sim}{G}^{-1}$ pointwise, i.e. by

$$\underset{\sim}{K}\underset{\sim}{G}^{-1}\big|_X = \underset{\sim}{K}(X)\, \underset{\sim}{G}(X)^{-1}. \tag{8.4}$$

Recalling that the configuration gradients $\nabla\underset{\sim}{\kappa}$ and $\nabla\underset{\sim}{\gamma}$ are references, we see that (8.3) can then be written as

$$\nabla_{\underset{\sim}{\kappa}} \psi = \nabla_{\underset{\sim}{\gamma}} \psi \circ \left(\nabla \underset{\sim}{\gamma} (\nabla \underset{\sim}{\kappa})^{-1} \right). \tag{8.5}$$

We say that a reference $\underset{\sim}{K}$ is <u>of class</u> C^r, $r \leq p-1$, if for some--and hence every configuration $\underset{\sim}{\kappa} \in \underline{\underline{C}}$ the tensor field $(\nabla \underset{\sim}{\kappa}) \underset{\sim}{K}^{-1}$ is of class C^r, i.e. belongs to $\mathcal{L}^r_{\mathcal{B}}$. It is clear that every gradient reference $\nabla\underset{\sim}{\kappa}$ is of class C^{p-1}.

Let a local configuration $\underset{\sim}{K}_X \in \mathcal{C}_X$ be given. If $\underset{\sim}{\kappa}$ and $\underset{\sim}{\gamma}$ both belong to the equivalence class that defines $\underset{\sim}{K}_X$, which means that $\nabla\underset{\sim}{\kappa}(X) = \nabla\underset{\sim}{\gamma}(X) = \underset{\sim}{K}_X$, we have, by (8.3), $\nabla_{\underset{\sim}{\kappa}} \psi(X) = \nabla_{\underset{\sim}{\gamma}} \psi(X)$. Hence, $\nabla_{\underset{\sim}{\kappa}} \psi(X)$ depends on $\underset{\sim}{\kappa}$ only, through the equivalence class $\underset{\sim}{K}_X \in \mathcal{C}_X$ to which $\underset{\sim}{\kappa}$ belongs and it is legitimate to define

$$\nabla_{\underset{\sim}{K}_X} \psi(X) = \nabla_{\underset{\sim}{\kappa}} \psi(X) \qquad \text{if} \quad \underset{\sim}{\kappa} \in \underset{\sim}{K}_X. \tag{8.6}$$

If $\underset{\sim}{K}$ is a reference, we define the <u>gradient of</u> Ψ <u>relative to the reference</u> $\underset{\sim}{K}$ by

$$\nabla_{\underset{\sim}{K}} \Psi \big|_X = \nabla_{\underset{\sim}{K}(X)} \Psi(X) \;, \quad X \in \mathcal{B}. \tag{8.7}$$

If $\underset{\sim}{K}$ and $\underset{\sim}{G}$ are any two references for $\mathcal{B}$, we see that (8.3) and the definitions (8.6) and (8.7) yield the formula

$$\nabla_{\underset{\sim}{K}} \Psi = \nabla_{\underset{\sim}{G}} \Psi \circ (\underset{\sim}{G}\underset{\sim}{K}^{-1}), \tag{8.8}$$

which generalizes (8.5). By writing (8.8) with $\underset{\sim}{G} = \nabla \underset{\sim}{\gamma}$, where $\underset{\sim}{\gamma} \in \underset{\approx}{C}$, we infer that $\nabla_{\underset{\sim}{K}} \Psi$ is of class C^{r-1} if Ψ is of class C^r and $\underset{\sim}{K}$ of class C^{r-1}.

When the range of Ψ coincides with the set $\mathcal{R}$ of real numbers, in which case we write f instead of Ψ, we can identify $\nabla_{\varkappa} f$ with a vector field on $\mathcal{B}$. Thus, if $f \in \mathcal{F}_{\mathcal{B}}^r$ then $\nabla_{\varkappa} f \in \mathcal{V}_{\mathcal{B}}^{r-1}$. The formula (8.5) becomes

$$\nabla_{\underset{\sim}{\varkappa}} f = (\nabla \underset{\sim}{\gamma} (\nabla \underset{\sim}{\varkappa})^{-1})^T \nabla_{\underset{\sim}{\gamma}} f. \tag{8.9}$$

Let $\mathcal{f} \in \mathcal{T}_{\mathcal{B}}^{r-1}$ and $f \in \mathcal{F}_{\mathcal{B}}^r$. The function $\mathcal{f}(f)$ on $\mathcal{B}$ defined by

$$\mathcal{f}(f) = \nabla_{\underset{\sim}{\varkappa}} f \cdot (\nabla \underset{\sim}{\varkappa}) \mathcal{f}, \tag{8.10}$$

where the inner product is defined pointwise, does not depend on the choice of the configuration $\underset{\sim}{\varkappa} \in \underset{\approx}{C}$, as is easily seen with the help of (8.9). Moreover, $\mathcal{f}(f)$ is of class C^{r-1}. Therefore, every $\mathcal{f} \in \mathcal{T}_{\mathcal{B}}^{r-1}$ gives rise to a mapping

$$\mathcal{f} : \mathcal{F}_{\mathcal{B}}^r \to \mathcal{F}_{\mathcal{B}}^{r-1}. \tag{8.11}$$

Actually, every $\mathcal{f} \in \mathcal{T}_{\mathcal{B}}^{r-1}$ can be <u>identified</u> with a mapping of the type (8.11), because it is easily seen from (8.10) that $\mathcal{f}_1(f) = \mathcal{f}_2(f)$ cannot hold for all $f \in \mathcal{F}_{\mathcal{B}}^{r}$ unless $\mathcal{f}_1 = \mathcal{f}_2$. The mapping (8.11) defined by (8.10) has the following basic property, which follows immediately from the chain rule.

Proposition 3: <u>If</u> $\mathcal{f} \in \mathcal{T}_{\mathcal{B}}^{r-1}$, <u>if</u> H <u>is a real-valued function of class</u> C^r <u>of any number</u> m <u>of real variables, and if</u> $f_1, f_2, \ldots, f_m \in \mathcal{F}_{\mathcal{B}}^{r}$, <u>then</u>

$$\mathcal{f}\,(H(f_1, f_2, \ldots, f_m)) = \sum_{k=1}^{m} H_{,k}(f_1, f_2, \ldots, f_m)\,\mathcal{f}(f_k), \tag{8.12}$$

<u>where</u> $H_{,k}$ <u>denotes the derivative of</u> H <u>with respect to its</u> k<u>'th variable</u>.

Actually, the property described in Proposition 1 characterizes the tangent vector fields of class C^{r-1} and hence could have been used for their definition; but we shall neither use nor prove this fact.

Applying (8.12) to the cases when $H(\xi_1, \xi_2) = \xi_1 + \xi_2$ and $H(\xi_1, \xi_2) = \xi_1 \xi_2$ we obtain

$$\mathcal{f}(f+g) = \mathcal{f}(f) + \mathcal{f}(g), \quad \mathcal{f}(fg) = f\,\mathcal{f}(g) + g\,\mathcal{f}(f). \tag{8.13}$$

Let $\mathcal{f}, \mathcal{k} \in \mathcal{T}_{\mathcal{B}}^{r-1}$ with $r \geq 2$. Since $\mathcal{T}_{\mathcal{B}}^{r-1} \subset \mathcal{T}_{\mathcal{B}}^{r-2}$ and hence also $\mathcal{f}, \mathcal{k} \in \mathcal{T}_{\mathcal{B}}^{r-2}$ we can identify $\mathcal{f}$ and $\mathcal{k}$ not only with mappings from $\mathcal{F}_{\mathcal{B}}^{r}$ into $\mathcal{F}_{\mathcal{B}}^{r-1}$, but also with mappings from $\mathcal{F}_{\mathcal{B}}^{r-1}$ into $\mathcal{F}_{\mathcal{B}}^{r-2}$. Therefore, we can form the compositions $\mathcal{f} \circ \mathcal{k}$ and $\mathcal{k} \circ \mathcal{f}$ as mappings from $\mathcal{F}_{\mathcal{B}}^{r}$ into $\mathcal{F}_{\mathcal{B}}^{r-2}$. By themselves, these compositions do not correspond to tangent vector fields, but it is remarkable that the difference

$$[\![\mathscr{f}, \mathscr{k}]\!] = \mathscr{f} \circ \mathscr{k} - \mathscr{k} \circ \mathscr{f} \; : \; \mathscr{F}_{\mathcal{B}}^{r} \to \mathscr{F}_{\mathcal{B}}^{r-2}, \tag{8.14}$$

called the <u>bracket</u> of $\mathscr{f}$ and $\mathscr{k}$, has values that belong to $\mathscr{F}_{\mathcal{B}}^{r-1}$ $(\subset \mathscr{F}_{\mathcal{B}}^{r-2})$ and does correspond to a tangent field:

<u>Proposition 4:</u> <u>The bracket of two tangent vector fields $\mathscr{f}, \mathscr{k} \in \mathscr{T}_{\mathcal{B}}^{r-1}$ $(r \geq 2)$ can be identified with the tangent vector field of class C^{r-2} given by</u>

$$[\![\mathscr{f}, \mathscr{k}]\!] = (\nabla \chi)^{-1} [(\nabla_{\chi} \mathbf{k}) \mathbf{h} - (\nabla_{\chi} \mathbf{h}) \mathbf{k}] \;, \quad \mathbf{h} = (\nabla \chi) \mathscr{f}, \; \mathbf{k} = (\nabla \chi) \mathscr{k} \tag{8.15}$$

<u>where χ is an arbitrary configuration</u>.

<u>Proof</u>: We denote the tangent vector field of class C^{r-2} defined by the right-hand side of (8.15) by $\mathscr{b}$, so that

$$(\nabla \chi) \mathscr{b} = [(\nabla_{\chi} \mathbf{k}) \mathbf{h} - (\nabla_{\chi} \mathbf{h}) \mathbf{k}]. \tag{8.16}$$

Now let $f \in \mathscr{F}_{\mathcal{B}}^{r}$. In view of (8.10), it follows from (8.16) that

$$\mathscr{b}(f) = \nabla_{\chi} f \cdot [(\nabla_{\chi} \mathbf{k}) \mathbf{h} - (\nabla_{\chi} \mathbf{h}) \mathbf{k}] \tag{8.17}$$

and from (8.10) that

$$(\mathscr{f} \circ \mathscr{k})(f) = \mathscr{f}(\mathscr{k}(f)) = \nabla_{\chi} (\nabla_{\chi} f \cdot \mathbf{k}) \cdot \mathbf{h}. \tag{8.18}$$

The rules of ordinary differential calculus yield $\nabla_{\chi} (\nabla_{\chi} f \cdot \mathbf{k}) \cdot \mathbf{h} = \mathbf{h} \cdot (\nabla_{\chi}^{(2)} f) \mathbf{k} + \nabla_{\chi} f \cdot (\nabla_{\chi} \mathbf{k}) \mathbf{h}$. Hence, since $\nabla_{\chi}^{(2)} f$ is symmetric, if we write (8.18) with $\mathscr{f}$ and $\mathscr{k}$ interchanged, take the difference, and then compare with (8.17), we obtain

$$\not{h}(f) = (\not{f} \circ \not{k})(f) - (\not{k} \circ \not{f})(f) = [\![\not{f}, \not{k}]\!](f)$$

i.e. the desired result $\not{h} = [\![\not{f}, \not{k}]\!]$. Q.E.D.

The bracket $[\![\not{f}, \not{k}]\!]$ depends linearly (but not $\mathcal{F}$-linearly) on $\not{f}$ and $\not{k}$ and satisfies for $\not{f}, \not{k}, \not{h} \in \mathcal{T}_{\mathcal{B}}^{r-1}$; $f \in \mathcal{F}_{\mathcal{B}}^{r}$, $2 \leqq r$, the identities

$$[\![\not{f}, \not{k}]\!] = - [\![\not{k}, \not{f}]\!], \tag{8.19}$$

$$[\![\not{f}, f \not{k}]\!] = f [\![\not{f}, \not{k}]\!] + \not{f}(f) \not{k}, \tag{8.20}$$

and for $\not{f}, \not{k}, \not{\ell} \in \mathcal{T}_{\mathcal{B}}^{r}$, $2 \leqq r \leqq p-1$, the Jacobi-identity

$$\sum_{cyclic} [\![\not{f}, [\![\not{k}, \not{\ell}]\!]]\!] = 0, \tag{8.21}$$

where the sum is taken of all terms obtained from the one written by cyclic permutation of $\not{f}, \not{k}, \not{\ell}$. The identity (8.19) is obvious from (8.14), and (8.21) is the result of a trivial calculation. The identity (8.20) follows from (8.14) and (8.13).

It would have been possible to define the bracket $[\![\not{f}, \not{k}]\!] \in \mathcal{T}_{\mathcal{B}}^{r-1}$ for $\not{f}, \not{k} \in \mathcal{T}_{\mathcal{B}}^{r}$, $1 \leqq r \leqq p-1$, directly by (8.15), for it is easy to see that the right-hand side of (8.15) does not depend on the choice of the configuration $\underset{\sim}{\varkappa}$.

9. Affine connections, torsion, curvature.

From now on we assume that $\mathcal{B}$ is a continuous body of class C^p, $p \geqq 3$.

A mapping

$$\Gamma : \mathcal{T}_{\mathcal{B}}^{r} \to \mathcal{Y}_{\mathcal{B}}^{r-1} \tag{9.1}$$

is called an <u>affine connection</u> of class C^{r-1} $(1 \leqq r \leqq p-1)$ on

$\mathcal{B}$ if

$$\Gamma(\mathfrak{f}+\mathfrak{k}) = \Gamma\mathfrak{f} + \Gamma\mathfrak{k} \tag{9.2}$$

holds for all $\mathfrak{f}, \mathfrak{k} \in \mathcal{T}_{\mathcal{B}}^{r}$ and

$$\Gamma(f\mathfrak{f})\mathfrak{k} = f(\Gamma\mathfrak{f})\mathfrak{k} + \mathfrak{k}(f)\mathfrak{f} \tag{9.3}$$

holds for all $\mathfrak{f} \in \mathcal{T}_{\mathcal{B}}^{r}$, $f \in \mathcal{F}_{\mathcal{B}}^{r}$ and all $\mathfrak{k} \in \mathcal{T}_{\mathcal{B}}^{r-1}$.

If a is a real constant then $\mathfrak{k}(a) = 0$ by the definition (8.10). Hence (9.3) reduces to $\Gamma(a\mathfrak{f}) = a\,\Gamma\mathfrak{f}$ when $a \in \mathcal{R}$, $\mathfrak{f} \in \mathcal{T}_{\mathcal{B}}^{r}$. Thus, Γ is a linear mapping, but it is never $\mathcal{F}$-linear. The rule (9.3) resembles one of the product rules for gradient operators.

A triple $(\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3)$ of tangent vector fields of class C^r, $r \leq p-1$, is called a <u>frame</u> of class C^r if the values $\mathfrak{v}_i(X)$ form a basis of the tangent space $\mathcal{T}_X$ for each $X \in \mathcal{B}$. Frames of class C^{p-1} (and hence of class C^r, $r \leq p-1$) exist. For example, if $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ is a basis of $\mathcal{V}$ and $\underset{\sim}{\varkappa}$ a configuration, then $\mathfrak{v}_i = (\nabla\underset{\sim}{\varkappa})^{-1}\underset{\sim}{e}_i$ defines a frame of class C^{p-1} . Every tangent vector field $\mathfrak{f} \in \mathcal{T}_{\mathcal{B}}^{r}$ has a component representation

$$\mathfrak{f} = \sum_i h^i \mathfrak{v}_i \tag{9.4}$$

with respect to a given frame $(\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3)$ of class C^r such that the component functions h^i belong to $\mathcal{F}_{\mathcal{B}}^{r}$.

Now let Γ be a connection of class C^{r-1} . Substituting (9.4) into $(\Gamma\mathfrak{f})\mathfrak{v}_j$ and using the rules (9.2) and (9.3) we obtain

$$(\Gamma f)\mathfrak{r}_j = \sum_i [h^i(\Gamma \mathfrak{r}_i)\mathfrak{r}_j + \mathfrak{r}_j(h^i)\mathfrak{r}_i]. \tag{9.5}$$

The components Γ_{ij}^k of the three intrinsic tensor fields $\Gamma \mathfrak{r}_i$ with respect to the frame $(\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3)$ are defined by

$$(\Gamma \mathfrak{r}_i)\mathfrak{r}_j = \sum_k \Gamma_{ij}^k \mathfrak{r}_k \tag{9.6}$$

These components Γ_{ij}^k belong to $\mathcal{F}_{\mathcal{B}}^{r-1}$ and are called the <u>components of the connection</u> Γ with respect to the frame $(\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3)$. If we prescribe a frame $(\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3)$ of class C^r and 27 functions $\Gamma_{ij}^k \in \mathcal{F}_{\mathcal{B}}^{r-1}$ on $\mathcal{B}$ arbitrarily, then (9.5) and (9.6) determine a unique affine connection of class C^{r-1}.

Let Γ be a connection of class C^{r-1} having components $\Gamma_{ij}^k \in \mathcal{F}_{\mathcal{B}}^{r-1}$ with respect to the frame $(\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3)$ of class C^r. When $1 \leqq s \leqq r$ then $(\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3)$ is also of class C^s and $\Gamma_{ij}^k \in \mathcal{F}_{\mathcal{B}}^{s-1} \supset \mathcal{F}_{\mathcal{B}}^{r-1}$. Hence, (9.5) and (9.6) define an affine connection of class C^{s-1}. Therefore, the mapping $\Gamma : \mathcal{T}_{\mathcal{B}}^r \to \mathcal{Y}_{\mathcal{B}}^{r-1}$ has a unique extension to $\mathcal{T}_{\mathcal{B}}^s$ that is an affine connection of class C^{s-1}. We denote this extension by the same symbol Γ. With this convention, we can say that every affine connection of class C^{r-1} is also of class C^{s-1} when $1 \leqq s \leqq r \leqq p-1$

Let Γ be a connection of class C^{r-1}, and hence also of class C^{s-1} when $1 \leqq s \leqq r$. Using the notation

$$\Gamma_f \mathfrak{k} = (\Gamma \mathfrak{k}) f, \tag{9.7}$$

we can identify $\Gamma_{\mathfrak{f}}$ with a mapping

$$\Gamma_{\mathfrak{f}} : \mathcal{T}_{\mathcal{B}}^{s} \to \mathcal{T}_{\mathcal{B}}^{s-1} \tag{9.8}$$

for any choice of s, $1 \leqq s \leqq r$, and any choice of $\mathfrak{f} \in \mathcal{T}_{\mathcal{B}}^{s-1}$.

In terms of $\Gamma_{\mathfrak{f}}$ the rule (9.3) reads

$$\Gamma_{\mathfrak{f}}(f\mathfrak{k}) = f(\Gamma_{\mathfrak{f}}\mathfrak{k}) + \mathfrak{f}(f)\mathfrak{k}. \tag{9.9}$$

Moreover, $\Gamma_{\mathfrak{f}}$ depends $\mathcal{F}$-linearly on $\mathfrak{f}$.

The <u>Cartan-torsion</u> (or simply <u>torsion</u>) of the connection Γ is the mapping

$$\mathcal{V} : \mathcal{T}_{\mathcal{B}}^{r} \times \mathcal{T}_{\mathcal{B}}^{r} \longrightarrow \mathcal{T}_{\mathcal{B}}^{r-1} \tag{9.10}$$

defined by

$$\mathcal{V}(\mathfrak{f},\mathfrak{k}) = \Gamma_{\mathfrak{f}}\mathfrak{k} - \Gamma_{\mathfrak{k}}\mathfrak{f} - [\![\mathfrak{f},\mathfrak{k}]\!]. \tag{9.11}$$

In view of (8.19) it is obvious that $\mathcal{V}$ is skew in the sense that

$$\mathcal{V}(\mathfrak{f},\mathfrak{k}) = -\mathcal{V}(\mathfrak{k},\mathfrak{f}). \tag{9.12}$$

It is an almost immediate consequence of (9.9) and the rule (8.20) that $\mathcal{V}$ is $\mathcal{F}$-bilinear. Hence, by Proposition 2, (Sect. 7), the torsion $\mathcal{V}$ can be identified with a field on $\mathcal{B}$ of class C^{r-1} whose value $\mathcal{V}(\mathfrak{X})$ at $\mathfrak{X} \in \mathcal{B}$ is a bilinear mapping from $\mathcal{T}_{\mathfrak{X}} \times \mathcal{T}_{\mathfrak{X}}$ into $\mathcal{T}_{\mathfrak{X}}$. It follows that $\mathcal{V}(\mathfrak{f},\mathfrak{k})$ remains meaningful for any, even discontinuous, tangent vector fields $\mathfrak{f}$, $\mathfrak{k}$, and that $\mathcal{V}(\mathfrak{f},\mathfrak{k})|_{\mathfrak{X}} = \mathcal{V}(\mathfrak{X})(\mathfrak{f}(\mathfrak{X}),\mathfrak{k}(\mathfrak{X}))$ depends on $\mathfrak{f}$ and $\mathfrak{k}$ only through their values at $\mathfrak{X}$.

Let $\mathfrak{k}, \mathfrak{f} \in \mathcal{T}_{\mathcal{B}}^{r-1}$, $2 \leq r \leq p-1$. In view of (9.8) we can regard $\Gamma_{\mathfrak{f}}$ and $\Gamma_{\mathfrak{k}}$ as mappings from $\mathcal{F}_{\mathcal{B}}^{r}$

into $\mathcal{F}_{\mathcal{B}}^{r-1}$ and also as mappings from $\mathcal{F}_{\mathcal{B}}^{r-1}$ into $\mathcal{F}_{\mathcal{B}}^{r-2}$. Hence we can form the compositions $\Gamma_{\mathbf{f}} \circ \Gamma_{\mathbf{k}}$ and $\Gamma_{\mathbf{k}} \circ \Gamma_{\mathbf{f}}$ and the bracket

$$[\![\Gamma_{\mathbf{f}}, \Gamma_{\mathbf{k}}]\!] = \Gamma_{\mathbf{f}} \circ \Gamma_{\mathbf{k}} - \Gamma_{\mathbf{k}} \circ \Gamma_{\mathbf{f}} \tag{9.13}$$

as mappings from $\mathcal{T}_{\mathcal{B}}^{r}$ into $\mathcal{T}_{\mathcal{B}}^{r-2}$. Since $[\![\mathbf{f},\mathbf{k}]\!] \in \mathcal{T}_{\mathcal{B}}^{r-1}$, we can regard $\Gamma_{[\![\mathbf{f},\mathbf{k}]\!]}$ as a mapping from $\mathcal{T}_{\mathcal{B}}^{r} (\subset \mathcal{T}_{\mathcal{B}}^{r-1})$ into $\mathcal{T}_{\mathcal{B}}^{r-2}$. Hence we can define

$$\mathcal{R}(\mathbf{f},\mathbf{k}) : \mathcal{T}_{\mathcal{B}}^{r} \to \mathcal{T}_{\mathcal{B}}^{r-2} \tag{9.14}$$

by

$$\mathcal{R}(\mathbf{f},\mathbf{k}) = [\![\Gamma_{\mathbf{f}}, \Gamma_{\mathbf{k}}]\!] - \Gamma_{[\![\mathbf{f},\mathbf{k}]\!]} . \tag{9.15}$$

An easy calculation, based on the definition (9.13) and the rules (9.9) and (8.20), shows that the mapping (9.14) is $\mathcal{F}$-linear. Hence, by an analogue of Proposition 1 (Sect. 7) for intrinsic tensor fields, $\mathcal{R}(\mathbf{f},\mathbf{k})$ can be identified with an element of $\mathcal{Y}_{\mathcal{B}}^{r-2}$ and $\mathcal{R}$ can be regarded as a mapping

$$\mathcal{R} : \mathcal{T}_{\mathcal{B}}^{r-1} \times \mathcal{T}_{\mathcal{B}}^{r-1} \longrightarrow \mathcal{Y}_{\mathcal{B}}^{r-2}, \tag{9.16}$$

which is called the <u>Riemann-curvature</u> (or simply <u>curvature</u>) of the connection Γ. It is obvious that $\mathcal{R}$ is skew in the sense that

$$\mathcal{R}(\mathbf{f},\mathbf{k}) = -\mathcal{R}(\mathbf{k},\mathbf{f}). \tag{9.17}$$

A short calculation shows that the mapping (9.16) is $\mathcal{F}$-bilinear. Therefore, by Proposition 2 (Sect. 7), we can identify the curvature $\mathcal{R}$ with a field on $\mathcal{B}$ of class C^{r-2} whose value $\mathcal{R}(X)$ at $X \in \mathcal{B}$ is a bilinear transformation from $\mathcal{T}_X \times \mathcal{T}_X$ into $\mathcal{T}_X$. We have $\mathcal{R}(f,k)|_X = \mathcal{R}(X)(f(X), k(X))$, which shows that $\mathcal{R}(f,k)$ is meaningful for any tangent vector fields f, k.

There is an important relation between the torsion and the curvature of an affine connection:

<u>Proposition 5</u>: <u>Let</u> Γ <u>be an affine connection of class</u> C^{r-1} <u>on</u> $\mathcal{B}$, $2 \leqq r \leqq p-1$. <u>The torsion</u> $\mathcal{T}$ <u>and the curvature</u> $\mathcal{R}$ <u>of</u> Γ <u>satisfy</u>

$$\sum_{\text{cyclic}}{}' \left\{ \Gamma_{\ell}(\mathcal{T}(f,k)) + \mathcal{T}(k, [\![\ell, f]\!]) - \mathcal{R}(\ell, f)k \right\} = 0 \qquad (9.18)$$

<u>for all</u> $f, k, \ell \in \mathcal{T}^r_{\mathcal{B}}$. <u>The sum is to be taken over all terms obtained from the one written by cyclic permutation of</u> f, k, ℓ.

The identity (9.18) is often called the <u>First Bianchi Identity</u>.

<u>Proof</u>: Operating with Γ_{ℓ} on (9.11) gives

$$\Gamma_{\ell}(\mathcal{T}(f,k)) - (\Gamma_{\ell} \circ \Gamma_{f})k + (\Gamma_{\ell} \circ \Gamma_{k})f + \Gamma_{\ell}[\![f,k]\!] = 0.$$

The cyclic sum of the left side of this equation remains unchanged if the third term is changed by one and the fourth by two cyclic permutations of f, k, ℓ. Hence we have

$$\sum_{\text{cyclic}}{}' \left\{ \Gamma_{\ell}(\mathcal{T}(f,k)) - (\Gamma_{\ell} \circ \Gamma_{f})k + (\Gamma_{f} \circ \Gamma_{\ell})k + \Gamma_{k}[\![\ell, f]\!] \right\} =$$

$$= \sum_{\text{cyclic}}{}' \left\{ \Gamma_{\ell}(\mathcal{T}(f,k)) - [\![\Gamma_{\ell}, \Gamma_{f}]\!]k + \Gamma_{k}[\![\ell, f]\!] \right\} = 0.$$

Using the definitions (9.15) and (9.11) we obtain

$$\sum_{cyclic}{}' \left\{ \Gamma_{\ell}(\mathcal{T}(f,k)) - \mathcal{R}(\ell,f)k + \mathcal{T}(k,[\![\ell,f]\!]) + [\![k,[\![\ell,f]\!]]\!] \right\} = 0.$$

In view of the Jacobi-identity (8.21) the last term gives no contribution and (9.18) results. Q.E.D.

Let Γ and $\overset{*}{\Gamma}$ be two connections of class C^{r-1} on $\mathcal{B}$. Using the notation (9.7) and observing the rule (9.9), we see that, for each s, $1 \leq s \leq r$, and each $f \in \mathcal{T}_{\mathcal{B}}^{s-1}$, the difference

$$\vartheta_f = \Gamma_f - \overset{*}{\Gamma}_f : \mathcal{T}_{\mathcal{B}}^{s} \rightarrow \mathcal{T}_{\mathcal{B}}^{s-1} \tag{9.19}$$

is actually $\mathcal{F}$-linear. Hence ϑ_f can be identified with an intrinsic tensor field of class C^{s-1}, i.e. $\vartheta_f \in \mathcal{Y}_{\mathcal{B}}^{s-1}$. Since ϑ_f depends $\mathcal{F}$-linearly on f, ϑ can be regarded as an $\mathcal{F}$-linear mapping

$$\vartheta : \mathcal{T}_{\mathcal{B}}^{s-1} \longrightarrow \mathcal{Y}_{\mathcal{B}}^{s-1}, \quad 1 \leq s \leq r, \tag{9.20}$$

and hence can be identified with a field of class C^{r-1} whose values $\vartheta(X)$ are linear transformations from $\mathcal{T}_X$ into $\mathcal{Y}_X$. The possibility of identifying ϑ_f and ϑ with fields on $\mathcal{B}$ follows from analogues of Proposition 1 (Sect. 7).

Let $\mathcal{T}$ and $\overset{*}{\mathcal{T}}$ denote the torsions and $\mathcal{R}$ and $\overset{*}{\mathcal{R}}$ the curvatures of Γ and $\overset{*}{\Gamma}$, respectively. If we write the definition (9.11) of the torsion for both Γ and $\overset{*}{\Gamma}$ and take the difference, we obtain,

$$\mathcal{T}(f,k) - \overset{*}{\mathcal{T}}(f,k) = \vartheta_f k - \vartheta_k f. \tag{9.21}$$

If we write the definition (9.15) of the curvature for $\overset{*}{\Gamma}$ and substitute $\overset{*}{\Gamma}_f = \Gamma_f - \vartheta_f$, we find

$$\overset{*}{\mathcal{R}}(f,k) = \mathcal{R}(f,k) - [\![\Gamma_f, \vartheta_k]\!] - [\![\vartheta_f, \Gamma_k]\!] + [\![\vartheta_f, \vartheta_k]\!] + \vartheta_{[\![f,k]\!]}. \quad (9.22)$$

10. Material connections.

Let Φ be a material uniformity for the simple body $\mathcal{B}$ of class C^p. (See Definition 3, Sect. 6.) We say that a tangent vector field $\mathbf{t}$ is <u>materially constant</u> if

$$\mathbf{t}(X) = \Phi(X,Y)\,\mathbf{t}(Y) \quad (10.1)$$

holds for all $X, Y \in \mathcal{B}$. If $X_0 \in \mathcal{B}$ is fixed and $\mathbf{u}_{X_0} \in \mathcal{T}_{X_0}$ is prescribed arbitrarily, then

$$\mathbf{t}(X) = \Phi(X, X_0)\,\mathbf{u}_{X_0} \quad (10.2)$$

is easily seen to define a materially constant field $\mathbf{t}$ such that $\mathbf{t}(X_0) = \mathbf{u}_{X_0}$. Moreover, every materially constant field $\mathbf{t}$ can be obtained in this fashion. Thus (10.2) describes a one-to-one correspondence between $\mathcal{T}_{X_0}$ and the set $\mathcal{T}_\Phi$ of all materially constant vector fields. This correspondence is actually a vector-space isomorphism, showing that $\mathcal{T}_\Phi$ is a three-dimensional vectorspace when addition and multiplication with scalars in $\mathcal{T}_\Phi$ are defined pointwise.

Let $\underset{\sim}{K}$ be a uniform reference (see Definition 4, Sect. 6). and $\mathbf{t} \in \mathcal{T}_\Phi$. Then it follows from (10.1) and (6.4) that $\underset{\sim}{K}(X)\,\mathbf{t}(X) = \underset{\sim}{K}(Y)\,\mathbf{t}(Y)$ for all $X, Y \in \mathcal{B}$, i.e. that

$$\underset{\sim}{K}\,\mathbf{t} = \underset{\sim}{c} = \text{constant}. \quad (10.3)$$

Conversely, if $\underset{\sim}{c} \in \mathcal{V}$, then $\mathbf{t} = \underset{\sim}{K}^{-1}\underset{\sim}{c}$ is easily seen to be materially constant. Thus $\mathcal{T}_{\Phi}$ is exactly the set of all tangent vector fields with the property (10.3).

A material uniformity Φ is said to be <u>of class</u> C^r, $r \leq p-1$, if $\mathcal{T}_{\Phi} \subset \mathcal{T}_{\mathcal{B}}^r$, i.e., if all tangent vector fields materially constant with respect to Φ are of class C^r. For the remainder of this paper we lay down the following:

<u>Smoothness assumption</u>:[1] <u>$\mathcal{B}$ is a materially uniform continuous body of class C^p, $p \geqq 3$,[2] which admits a material uniformity Φ of class C^{p-1}.</u>

Let Φ be a material uniformity of class C^{p-1} and let $\underset{\sim}{K}$ be a uniform reference such that (6.4) holds. Then for $\underset{\sim}{c} \in \mathcal{V}$, $\underset{\sim}{\varkappa} \in \underline{\underline{C}}$, and $\underset{\sim}{v} = \underset{\sim}{K}^{-1}\underset{\sim}{c} \in \mathcal{T}_{\Phi}$, the vector field $((\nabla \underset{\sim}{\varkappa})\underset{\sim}{K}^{-1})\underset{\sim}{c} = (\nabla \underset{\sim}{\varkappa})\underset{\sim}{v}$ is of class C^{p-1} because $\underset{\sim}{v}$ is of class C^{p-1}. This is possible for all $\underset{\sim}{c} \in \mathcal{V}$ only if $(\nabla \underset{\sim}{\varkappa})\underset{\sim}{K}$ and hence $\underset{\sim}{K}$ is of class C^{p-1}. Thus, if $\underset{\sim}{K}$ is a uniform reference such that

$$\Phi(X, Y) = \underset{\sim}{K}(X)\underset{\sim}{K}(Y) \qquad X, Y \in \mathcal{B} \tag{10.4}$$

then $\underset{\sim}{K}$ is of class C^{p-1}.

<u>Theorem 3</u>: <u>Given a material uniformity Φ of class</u> C^{p-1},

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[1] C. C. Wang [8] has recently shown that the theory given here can be extended to the case when each point has a neighborhood that admits a smooth materially uniformity. This can happen even when all material uniformities for the whole body are discontinuous.

[2] For all considerations not referring to curvature, $p \geqq 2$ is actually sufficient.

<u>there is a unique affine connection Γ such that $\Gamma \mathbf{v} = \mathbf{0}$ holds for all material constant tangent vector fields $\mathbf{v} \in \mathcal{T}_{\Phi}$. In terms of any uniform reference $\underset{\sim}{K}$ satisfying (10.4), Γ is given by</u>

$$\Gamma \mathbf{f} = \underset{\sim}{K}^{-1} \nabla_{\underset{\sim}{K}} (\underset{\sim}{K} \mathbf{f}) \underset{\sim}{K} \quad , \quad \mathbf{f} \in \mathcal{T}_{\mathcal{B}}^{1} . \tag{10.5}$$

<u>Also, Γ is of class</u> C^{p-2}.

<u>Proof</u>: To prove the uniqueness, assume that Γ and $\overline{\Gamma}$ are connections such that $\Gamma \mathbf{v} = \overline{\Gamma} \mathbf{v}$ for all $\mathbf{v} \in \mathcal{T}_{\Phi}$. Putting $\vartheta_{\mathbf{f}} = \Gamma_{\mathbf{f}} - \overline{\Gamma}_{\mathbf{f}}$, we then have $\vartheta_{\mathbf{f}} \mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in \mathcal{T}_{\Phi}$. We have seen at the end of the previous section that $\vartheta_{\mathbf{f}}$ can be identified with an intrinsic tensor field in $\mathcal{Y}_{\mathcal{B}}^{0}$ when $\mathbf{f} \in \mathcal{T}_{\mathcal{B}}^{0}$. Hence $\vartheta_{\mathbf{f}}(X_0)\mathbf{v}(X_0) = \mathbf{0}$ for all $X_0 \in \mathcal{B}$ and all $\mathbf{v} \in \mathcal{T}_{\Phi}$. Since for any prescribed $\mathbf{u}_{X_0} \in \mathcal{T}_{X_0}$ the $\mathbf{v} \in \mathcal{T}_{\Phi}$ given by (10.2) has the property $\mathbf{v}(X_0) = \mathbf{u}_{X_0}$, it follows that $\vartheta_{\mathbf{f}}(X_0)\mathbf{u}_{X_0} = \mathbf{0}$ for all $\mathbf{u}_{X_0} \in \mathcal{T}_{X_0}$, i.e., that $\vartheta_{\mathbf{f}}(X_0) = \mathbf{0}$. Since $X_0 \in \mathcal{B}$ is arbitrary, we infer that $\vartheta_{\mathbf{f}} = \mathbf{0}$, i.e. that $\Gamma = \overline{\Gamma}$.

To prove the existence of Γ we choose a uniform reference $\underset{\sim}{K}$ with the property (10.4), define Γ by (10.5), and show that it has all the necessary properties. It is clear that $\Gamma \mathbf{v} = \mathbf{0}$ when $\mathbf{v} \in \mathcal{T}_{\Phi}$ because, by (10.3), $\nabla_{\underset{\sim}{K}} (\underset{\sim}{K} \mathbf{v}) = \mathbf{0}$ when $\mathbf{v} \in \mathcal{T}_{\Phi}$. Since $\underset{\sim}{K}$ is of class C^{p-1} it follows that $\Gamma \mathbf{f}$ is of class C^{p-1} when $\mathbf{f} \in \mathcal{T}_{\mathcal{B}}^{p-1}$. The validity of the rules (9.2) and (9.3) follows from the validity of the analogous rules for the relative gradient $\nabla_{\underset{\sim}{K}}$. Hence

Γ is indeed an affine connection of class C^{p-2} . Q.E.D.

Definition 5: The affine connection (of class C^{p-2}) with the property $\Gamma \mathbf{t} = 0$ for all $\mathbf{t} \in \mathcal{T}_{\Phi}$ is called the material connection for the material uniformity Φ (of class C^{p-1}).

Theorem 4: Material connections have zero Riemann-curvature.

Proof: Let $X \in \mathcal{B}$ and $\mathbf{u}_X \in \mathcal{T}_X$ be given. We can determine $\mathbf{t} \in \mathcal{T}_{\Phi} \subset \mathcal{T}_{\mathcal{B}}^{p-1}$ such that $\mathbf{t}(X) = \mathbf{u}_X$. If Γ is the material connection for Φ we have $\Gamma_{\mathbf{f}} \mathbf{t} = 0$ for all $\mathbf{f} \in \mathcal{T}_{\mathcal{B}}^{0}$. Hence the definition (9.15) shows that $\mathcal{R}(\mathbf{f},\mathbf{k})\mathbf{t} = 0$ for all $\mathbf{f},\mathbf{k} \in \mathcal{T}_{\mathcal{B}}^{p-1}$ Since $\mathcal{R}(\mathbf{f},\mathbf{k})$ can be identified with an intrinsic tensor field it follows that

$$\mathcal{R}(\mathbf{f},\mathbf{k})\mathbf{t}\Big|_X = \mathcal{R}(\mathbf{f},\mathbf{k})\Big|_X \mathbf{t}(X) = \mathcal{R}(\mathbf{f},\mathbf{k})\Big|_X \mathbf{u}_X = 0 .$$

This can be valid for all $X \in \mathcal{B}$ and all $\mathbf{u}_X \in \mathcal{T}_X$ only if $\mathcal{R}(\mathbf{f},\mathbf{k}) = 0$. Hence, since $\mathbf{f}, \mathbf{k} \in \mathcal{T}_{\mathcal{B}}^{p-1}$ are arbitrary, we must have $\mathcal{R} = 0$. Q.E.D.

11. Inhomogenity.

Let Φ be a material uniformity of class C^{p-1} , let Γ be the associated material connection (of class C^{p-2}) with torsion $\mathcal{T}$, and let $\mathbf{K}$ be a uniform reference (of class C^{p-1}) such that (10.4) holds. We can define a field $\mathbf{S}$ of class C^{p-2} with values $\mathbf{S}(X) : \mathcal{V} \to \mathcal{L}$ by the condition

$$(\mathbf{S}\mathbf{u})\mathbf{v} = \mathbf{K}\,\mathcal{T}(\mathbf{K}^{-1}\mathbf{u}, \mathbf{K}^{-1}\mathbf{v}) \tag{11.1}$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$. In view of the linearity of the values

$\underset{\sim}{S}(X)$, $\mathcal{T}(X)$, and $\underset{\sim}{K}(X)$, (11.1) continues to hold if the fixed vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ in (11.1) are replaced by vector fields $\underset{\sim}{h}$ and $\underset{\sim}{k}$. The following theorem shows how $\underset{\sim}{S}$ and hence $\mathcal{T}$ can be expressed directly in terms of $\underset{\sim}{K}$:

<u>Theorem 5</u>: <u>Let $\underset{\sim}{\gamma}$ be an arbitrary configuration of $\mathcal{B}$ and</u>

$$\underset{\sim}{F} = (\nabla \underset{\sim}{\gamma})\underset{\sim}{K}^{-1} \in \mathcal{L}_{\mathcal{B}}^{r}. \tag{11.2}$$

<u>Then $\underset{\sim}{S}$ is given by</u>

$$(\underset{\sim}{S}\underset{\sim}{u})\underset{\sim}{v} = \underset{\sim}{F}^{-1}[((\nabla_{\underset{\sim}{K}}\underset{\sim}{F})\underset{\sim}{v})\underset{\sim}{u} - ((\nabla_{\underset{\sim}{K}}\underset{\sim}{F})\underset{\sim}{u})\underset{\sim}{v}], \tag{11.3}$$

$$\underset{\sim}{S}\underset{\sim}{u} = \underset{\sim}{F}^{-1}[\nabla_{\underset{\sim}{K}}(\underset{\sim}{F}\underset{\sim}{u}) - (\nabla_{\underset{\sim}{K}}\underset{\sim}{F})\underset{\sim}{u}], \tag{11.4}$$

<u>or</u>

$$(\underset{\sim}{S}\underset{\sim}{u})\underset{\sim}{v} = ((\nabla_{\underset{\sim}{\gamma}}\underset{\sim}{F}^{-1})\underset{\sim}{h})\underset{\sim}{k} - ((\nabla_{\underset{\sim}{\gamma}}\underset{\sim}{F}^{-1})\underset{\sim}{k})\underset{\sim}{h}, \tag{11.5}$$

<u>where</u> $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$ <u>and</u> $\underset{\sim}{h} = \underset{\sim}{F}\underset{\sim}{u}$, $\underset{\sim}{k} = \underset{\sim}{F}\underset{\sim}{v}$.

<u>Proof</u>: The tangent vector fields $\underset{\sim}{K}^{-1}\underset{\sim}{u}$ and $\underset{\sim}{K}^{-1}\underset{\sim}{v}$ are materially constant and hence are annihilated by Γ. Hence, the definitions (11.1) and (9.11) of $\underset{\sim}{S}$ and $\mathcal{T}$ yield

$$(\underset{\sim}{S}\underset{\sim}{u})\underset{\sim}{v} = -\underset{\sim}{K}[\![\underset{\sim}{K}^{-1}\underset{\sim}{u}, \underset{\sim}{K}^{-1}\underset{\sim}{v}]\!] = -\underset{\sim}{K}[\![(\nabla\underset{\sim}{\gamma})^{-1}\underset{\sim}{h}, (\nabla\underset{\sim}{\gamma})^{-1}\underset{\sim}{k}]\!]. \tag{11.6}$$

Using Proposition 4, (8.15), we find

$$(\underset{\sim}{S}\underset{\sim}{u})\underset{\sim}{v} = \underset{\sim}{F}^{-1}[(\nabla_{\underset{\sim}{\gamma}}\underset{\sim}{h})\underset{\sim}{k} - (\nabla_{\underset{\sim}{\gamma}}\underset{\sim}{k})\underset{\sim}{h}]. \tag{11.7}$$

The formula (8.8), with the choices $\psi = \underset{\sim}{h}$ and $\underset{\sim}{G} = \nabla\underset{\sim}{\gamma}$, gives

$$(\nabla_{\underset{\sim}{\gamma}}\underset{\sim}{h})\underset{\sim}{k} = (\nabla_{\underset{\sim}{\gamma}}\underset{\sim}{h})\underset{\sim}{F}\underset{\sim}{v} = (\nabla_{\underset{\sim}{K}}\underset{\sim}{h})\underset{\sim}{v} = \nabla_{\underset{\sim}{K}}(\underset{\sim}{F}\underset{\sim}{u})\underset{\sim}{v} = ((\nabla_{\underset{\sim}{K}}\underset{\sim}{F})\underset{\sim}{v})\underset{\sim}{u}. \tag{11.8}$$

Substituting (11.8) and the formula obtained from (11.8) by interchanging $\underset{\sim}{h}$ and $\underset{\sim}{k}$ into (11.7), we obtain (11.3) and (11.4).

To prove (11.5) we note that $\nabla_{\underset{\sim}{\gamma}} \underset{\sim}{u} = 0$ for constant $\underset{\sim}{u} \in \mathcal{V}^{4}$. Using one of the product rules for gradient operators we find

$$\underset{\sim}{0} = (\nabla_{\underset{\sim}{\gamma}} \underset{\sim}{u})\underset{\sim}{k} = \nabla_{\underset{\sim}{\gamma}} (\underset{\sim}{F}^{-1}\underset{\sim}{h})\underset{\sim}{k} = ((\nabla_{\underset{\sim}{\gamma}} \underset{\sim}{F}^{-1})\underset{\sim}{k})\underset{\sim}{h} + \underset{\sim}{F}^{-1}(\nabla_{\underset{\sim}{\gamma}} \underset{\sim}{h})\underset{\sim}{k}. \quad (11.9)$$

Of course, (11.9) remains valid if we interchange $\underset{\sim}{h}$ and $\underset{\sim}{k}$. The formula (11.5) follows from (11.7) and (11.9). Q.E.D.

Recall that the body $\mathcal{B}$ is homogeneous if it admits a configuration gradient $\underset{\sim}{K} = \nabla \underset{\sim}{\kappa}$ as a uniform reference. We can then choose $\underset{\sim}{\gamma} = \underset{\sim}{\kappa}$ in (11.2), obtaining $\underset{\sim}{F} = \underset{\sim}{1}$, which is constant and hence has gradient zero. Thus, Theorem 5 shows that $\underset{\sim}{S} = \underset{\sim}{0}$ and hence $\mathcal{T} = 0$ for suitable uniform references if the body is homogeneous. The converse of this result is not true, but it becomes true if "homogeneous" is replaced by "locally homogeneous" in a sense we shall now make precise.

Let $\mathcal{N}$ be an open subset of a simple body $\mathcal{B}$ of class C^{p}. We can give $\mathcal{N}$ the structure of a continuous body of class C^{p} by letting $\underset{\sim}{\gamma} : \mathcal{N} \to \mathcal{E}$ be a configuration of $\mathcal{N}$ if $\underset{\sim}{\kappa} \circ \underset{\sim}{\gamma}^{-1} : \underset{\sim}{\gamma}(\mathcal{N}) \to \mathcal{E}$ is of class C^{p} for all configurations $\underset{\sim}{\kappa} \in \underline{\underline{C}}$ of $\mathcal{B}$. We denote the set of all configurations of $\mathcal{N}$ by $\underline{\underline{C}}_{\mathcal{N}}$. If $\underset{\sim}{\kappa} \in \underline{\underline{C}}$, then the restriction of $\underset{\sim}{\kappa}$ to $\mathcal{N}$ belongs to $\underline{\underline{C}}_{\mathcal{N}}$. However, not all configurations $\underset{\sim}{\gamma} \in \underline{\underline{C}}_{\mathcal{N}}$ of $\mathcal{N}$ can be obtained in this manner. Still, given any $X \in \mathcal{N}$ and any configuration $\underset{\sim}{\gamma}$ of $\mathcal{N}$, one can easily construct a configuration $\underset{\sim}{\kappa}$ of $\mathcal{B}$ such that

(3.1) holds. Therefore, an equivalence class $\underset{\sim}{K}_X$ which defines a local configuration at X relative to $\mathcal{N}$ can be made to correspond to the non-empty set $\{\underset{\sim}{\varkappa} \in \underset{=}{C} \mid \nabla(\varkappa \circ \gamma^{-1})\big|_{\gamma(X)} = 1$ for all $\gamma \in \underset{\sim}{K}_X\}$, which is a local configuration at X relative to $\mathcal{B}$. This correspondence is one-to-one and can be used to identify local configurations at X relative to $\mathcal{N}$ with local configurations at X relative to $\mathcal{B}$. Using this identification, we can endow $\mathcal{N}$ with the structure of a simple body by using the restriction to $\mathcal{N}$ of the function $\underset{\sim}{\mathcal{G}}$ which defines the simple body structure on $\mathcal{B}$ according to Definition 1 (Sect. 5). Thus, every open subset $\mathcal{N}$ of $\mathcal{B}$ has a natural structure of a simple body of class C^p, i.e., every open subset $\mathcal{N}$ of $\mathcal{B}$ can be regarded as a simple body of class C^p. Such a subset is called a neighborhood of a material point if it contains that point.

A simple body $\mathcal{B}$ is called locally homogeneous if every $X \in \mathcal{B}$ has a neighborhood $\mathcal{N}$ that is homogeneous. A body can be locally homogeneous without being homogeneous, even if it is simply connected.

Definition 6: The Cartan-torsion $\mathcal{T}$ of the material connection Γ associated with a material uniformity $\underset{-}{\Phi}$ of class C^{p-1} is called the inhomogeneity of $\underset{-}{\Phi}$.

The field $\underset{\sim}{S}$ defined by (11.1) is called inhomogeneity[1] relative to the reference $\underset{\sim}{K}$.

This definition finds its motivation in the result already mentioned:

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[1]It corresponds to what is called "dislocation density" in the theory of continuous distributions of dislocations (cf. [4]).

Theorem 6:[1] If $\mathcal{B}$ is homogeneous then it admits a material uniformity of class C^{p-1} with zero inhomogeneity. If $\mathcal{B}$ admits a material uniformity of class C^{p-1} with zero inhomogeneity, then it is locally homogenous.

Proof: Only the second part of the theorem remains to be proved. Assume, therefore, that Φ is a material uniformity of class C^{p-1} with zero inhomogeneity. Using the same notation as before, we then have $\mathcal{T} = 0$ and hence $\underset{\sim}{S} = \underset{\sim}{0}$. Theorem 5, (11.5), shows that if $\gamma \in \underset{\sim}{C}$ is arbitrary and $\underset{\sim}{F}$ defined by (11.2), $\nabla_{\gamma} \underset{\sim}{F}^{-1}$ has the symmetry property

$$((\nabla_{\gamma} \underset{\sim}{F}^{-1})\underset{\sim}{u})\underset{\sim}{v} = ((\nabla_{\gamma} \underset{\sim}{F}^{-1})\underset{\sim}{v})\underset{\sim}{u} \tag{11.10}$$

for all $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$. Let $X \in \mathcal{B}$ be given and let $\mathcal{N}'$ be a simply connected neighborhood of $\mathcal{B}$. By a classical theorem of analysis, the symmetry (11.10) implies the existence of a mapping $\underset{\sim}{\lambda} : \underset{\sim}{\gamma}(\mathcal{N}') \to \mathcal{E}$ such that $\underset{\sim}{F}^{-1} = (\nabla \underset{\sim}{\lambda}) \circ \underset{\sim}{\gamma}^{-1}$ holds in $\mathcal{N}'$. Moreover, $\underset{\sim}{K}$, ∇_{γ} and hence $\underset{\sim}{F}$ and $\underset{\sim}{F}^{-1}$ being of class C^{p-1}, $\underset{\sim}{\lambda}$ is of class C^{p}. Since $\nabla \underset{\sim}{\lambda} = \underset{\sim}{F}^{-1} \circ \underset{\sim}{\gamma}$ is invertible, it follows by the inverse function theorem that $\underset{\sim}{\lambda}$ is locally (but not necessarily globally) invertible, i.e. that X has a neighborhood $\mathcal{N} \subset \mathcal{N}'$ on which $\underset{\sim}{\lambda}$ is invertible. The mapping $\underset{\sim}{\varkappa} = \underset{\sim}{\lambda} \circ \underset{\sim}{\gamma}$, when restricted to $\mathcal{N}$, is therefore a configuration of $\mathcal{N}$ with gradient $\nabla \underset{\sim}{\varkappa} = ((\nabla \underset{\sim}{\lambda}) \circ \underset{\sim}{\gamma}^{-1}) = \underset{\sim}{F}^{-1} \nabla_{\underset{\sim}{\gamma}} = \underset{\sim}{K}$ on $\mathcal{N}$. Hence the uniform reference $\underset{\sim}{K}$ on $\mathcal{N}$ is the gradient of the configuration $\underset{\sim}{\varkappa}$ of $\mathcal{N}$, i.e., $\mathcal{N}$ is homogeneous. Q.E.D.

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[1] The theorem stated in the middle of p. 90 in reference [2] is incorrect and should be replaced by Theorem 6.

The first Bianchi identity gives rise to the following identity for the relative inhomogeneity $\underset{\sim}{S}$ and its gradient $\nabla_{\underset{\sim}{K}} \underset{\sim}{S}$ relative to $\underset{\sim}{K}$:

$$\sum \left[(((\nabla_{\underset{\sim}{K}} \underset{\sim}{S})\underset{\sim}{u})\underset{\sim}{v})\underset{\sim}{w}) - (\underset{\sim}{S}\underset{\sim}{u})(\underset{\sim}{S}\underset{\sim}{v})\underset{\sim}{w} \right] = \underset{\sim}{0} \qquad (11.11)$$

To prove (11.11), substitute $\mathbf{\ell} = \underset{\sim}{K}^{-1}\underset{\sim}{u}$, $\mathbf{\omega} = \underset{\sim}{K}^{-1}\underset{\sim}{w}$, $\mathbf{f} = \underset{\sim}{K}^{-1}\underset{\sim}{v}$ into (9.16), observe that $\mathcal{R} = \mathcal{O}$ (Theorem 4, Sect. 10), and make use of $(11.6)_1$.

12. Relative Riemanian structures, contortion.

Let $\underset{\sim}{K}$ be a uniform reference of class C^{p-1}. If we choose $\underset{\sim}{K}_X = \underset{\sim}{K}(X)$ in (4.5), then this equation defines an inner product $*$ on each of the tangent spaces $\mathcal{T}_X$, $X \in \mathcal{B}$. The structure on $\mathcal{B}$ defined by these inner products will be called the Riemannian structure of $\mathcal{B}$ relative to the uniform reference $\underset{\sim}{K}$. If f and k are tangent vector fields, we define $f * k$ pointwise. For such fields, (4.5) then yields

$$f * k = (\underset{\sim}{K} f) \cdot (\underset{\sim}{K} k). \tag{12.1}$$

It is clear that $f * k \in \mathcal{F}_{\mathcal{B}}^{r}$ if $f, k \in \mathcal{T}_{\mathcal{B}}^{r}$ for $0 \leqq r \leqq p-1$. This fact is expressed by saying that the Riemannian structure relative to $\underset{\sim}{K}$ is of class C^{p-1}.

Although the following proposition is one of the basic facts of Riemannian geometry, we shall give an independent proof:

Proposition 6: There is a unique affine connection $\overset{*}{\Gamma}$ of class C^{p-2} with the following properties:

(a) The torsion $\overset{*}{\mathcal{T}}$ of $\overset{*}{\Gamma}$ vanishes,

(b) For any $f, k, \ell \in \mathcal{T}_{\mathcal{B}}^{p-1}$ the relation

$$f(k * \ell) = k * \overset{*}{\Gamma}_f \ell + \ell * \overset{*}{\Gamma}_f k \tag{12.2}$$

is valid.

Proof: First we assume the existence of $\overset{*}{\Gamma}$. Let Γ be the material connection associated with $\underset{\sim}{K}$ and consider the difference

$$\vartheta_f = \Gamma_f - \overset{*}{\Gamma}_f, \qquad f \in \mathcal{T}_{\mathcal{B}}^{p-1}. \tag{12.3}$$

According to the results given at the end of Sect. 9, ϑ can be identified with a field of class C^{p-2} on $\mathcal{B}$ whose values are linear transformations from $\mathcal{T}_X$ into $\mathcal{Y}_X$. Therefore, we can define a field $\underset{\sim}{D}$ on $\mathcal{B}$ of class C^{p-2} with values $\underset{\sim}{D}(X) : \mathcal{V} \to \mathcal{L}$ by the condition

$$\underset{\sim}{D}\underset{\sim}{u} = \underset{\sim}{K}\, \vartheta_{\underset{\sim}{K}^{-1}\underset{\sim}{u}}\, \underset{\sim}{K}^{-1} \tag{12.4}$$

for all $\underset{\sim}{u} \in \mathcal{V}$. Since $\overset{*}{\gamma} = \mathcal{O}$, the relation (9.21) and the definitions (12.4) and (11.1) give

$$(\underset{\sim}{S}\underset{\sim}{u})\underset{\sim}{v} = (\underset{\sim}{D}\underset{\sim}{u})\underset{\sim}{v} - (\underset{\sim}{D}\underset{\sim}{v})\underset{\sim}{u} \;, \qquad \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}. \tag{12.5}$$

By the Definition 5 (Sect. 10), we have $\Gamma_{\mathfrak{f}}\, \mathfrak{t} = \mathcal{O}$ whenever $\mathfrak{t}$ is materially constant. Hence, since $\mathfrak{t} = \underset{\sim}{K}^{-1}\underset{\sim}{c}$ is materially constant when $\underset{\sim}{c} \in \mathcal{V}$, we infer from (12.3) and (12.4) that

$$\underset{\sim}{K}\, \overset{*}{\Gamma}_{\mathfrak{f}}(\underset{\sim}{K}^{-1}\underset{\sim}{c}) = -(\underset{\sim}{D}\underset{\sim}{K}\mathfrak{f})\underset{\sim}{c} \tag{12.6}$$

when $\underset{\sim}{c} \in \mathcal{V}$. Now let $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in \mathcal{V}$. If we substitute $\mathfrak{f} = \underset{\sim}{K}^{-1}\underset{\sim}{u}$, $\mathfrak{k} = \underset{\sim}{K}^{-1}\underset{\sim}{v}$, and $\mathfrak{l} = \underset{\sim}{K}^{-1}\underset{\sim}{w}$ into (12.2) and observe (12.6) and (12.1) we obtain

$$\mathfrak{f}\,(\underset{\sim}{v}\cdot\underset{\sim}{w}) = -\underset{\sim}{v}\cdot(\underset{\sim}{D}\underset{\sim}{u})\underset{\sim}{w} + \underset{\sim}{w}\cdot(\underset{\sim}{D}\underset{\sim}{u})\underset{\sim}{v}\,. \tag{12.7}$$

Since $\underset{\sim}{v}\cdot\underset{\sim}{w}$ is constant, we have $\mathfrak{f}(\underset{\sim}{v}\cdot\underset{\sim}{w}) = 0$ (see the definition (8.10)). Therefore (12.7) states that $\underset{\sim}{D}\underset{\sim}{u} \in \mathcal{L}$ is skew for all $\underset{\sim}{u} \in \mathcal{V}$:

$$\underset{\sim}{D}\underset{\sim}{u} = -(\underset{\sim}{D}\underset{\sim}{u})^T \;, \qquad \underset{\sim}{u} \in \mathcal{V}. \tag{12.8}$$

The equations (12.5) and (12.8) enable us to express $\underset{\sim}{D}$ in terms of $\underset{\sim}{S}$. Indeed, if we take the inner product of (12.5) with $\underset{\sim}{w} \in \mathcal{V}$, subtract from the resulting equation the two equations obtained from it by cyclic permutations of $\mathbf{u}, \mathbf{v}, \mathbf{w}$, and observe (12.8), we find

$$2\,\mathbf{u}\cdot(\mathbf{D}\mathbf{w})\mathbf{v} = \mathbf{w}\cdot(\mathbf{S}\mathbf{u})\mathbf{v} - \mathbf{u}\cdot(\mathbf{S}\mathbf{v})\mathbf{w} - \mathbf{v}\cdot(\mathbf{S}\mathbf{w})\mathbf{u}. \tag{12.9}$$

Since $(\mathbf{S}\mathbf{u})\mathbf{v} = -(\mathbf{S}\mathbf{v})\mathbf{u}$, (12.9) is equivalent to

$$(\mathbf{D}\mathbf{u})\mathbf{v} = \frac{1}{2}\left\{[(\mathbf{S}\mathbf{u}) - (\mathbf{S}\mathbf{u})^T]\mathbf{v} - (\mathbf{S}\mathbf{v})^T\mathbf{w}\right\}. \tag{12.10}$$

Now, since $\mathbf{S}$ is determined by the uniform reference $\mathbf{K}$, it follows from (12.10), (12.4), and (12.3) that $\overset{*}{\Gamma}$ is uniquely determined by $\mathbf{K}$.

To prove the existence of a connection $\overset{*}{\Gamma}$ with the properties (a) and (b), one can define $\overset{*}{\Gamma}$ by (12.3), (12.4), and (12.10) and verify that it has all the required properties. Q.E.D.

Definition 7: The connection $\overset{*}{\Gamma}$ determined by the conditions (a) and (b) of Proposition 6 is called the Riemannian connection relative to the uniform reference $\mathbf{K}$. The field $\mathbf{D}$ determined by (12.3) and (12.) or (12.5) and (12.8) is called the contortion[1] of $\mathbf{K}$.

[1] It corresponds to what is called "Cosserat structure curvature" in the theory of continuous distributions of dislocations (cf. [4]).

The term "contortion" will be motivated in Section 13.

In general, the curvature $\overset{*}{\mathcal{R}}$ of the Riemannian connection is not zero. We can define a field $\overset{*}{\mathbf{R}}$ with values $\overset{*}{\mathbf{R}}(X): \mathcal{V}\times\mathcal{V}\to\mathcal{L}$ by the condition

$$\overset{*}{\mathbf{R}}(\mathbf{u},\mathbf{v}) = \mathbf{K}\,\overset{*}{\mathcal{R}}(\mathbf{K}^{-1}\mathbf{u}, \mathbf{K}^{-1}\mathbf{v})\mathbf{K}^{-1} \tag{12.11}$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$ and put $\mathbf{f} = \mathbf{K}^{-1}\mathbf{u}$, $\mathbf{k} = \mathbf{K}^{-1}\mathbf{v}$, $\mathbf{l} = \mathbf{K}^{-1}\mathbf{w}$. The fields $\mathbf{f}, \mathbf{k}, \mathbf{l}$ are then materially constant and hence are annihilated by Γ. Recalling that the curvature $\mathcal{R}$ of Γ vanishes (Theorem 4, Sect. 10), we then infer from (9.22), (10.5), $(11.6)_1$, (12.4), and (12.11) that

$$\overset{*}{\underset{\sim}{R}}(\underset{\sim}{u},\underset{\sim}{v})\underset{\sim}{w} = -(\nabla_{\underset{\sim}{K}}(\underset{\sim}{D}\underset{\sim}{v})\underset{\sim}{u})\underset{\sim}{w} + (\nabla_{\underset{\sim}{K}}(\underset{\sim}{D}\underset{\sim}{u})\underset{\sim}{v})\underset{\sim}{w}$$

$$+ (\underset{\sim}{D}\underset{\sim}{u})(\underset{\sim}{D}\underset{\sim}{v})\underset{\sim}{w} - (\underset{\sim}{D}\underset{\sim}{v})(\underset{\sim}{D}\underset{\sim}{u})\underset{\sim}{w} - \underset{\sim}{D}((\underset{\sim}{S}\underset{\sim}{u})\underset{\sim}{v})\underset{\sim}{w}$$

and hence

$$\overset{*}{\underset{\sim}{R}}(\underset{\sim}{u},\underset{\sim}{v}) = ((\nabla_{\underset{\sim}{K}}\underset{\sim}{D})\underset{\sim}{v})\underset{\sim}{u} - ((\nabla_{\underset{\sim}{K}}\underset{\sim}{D})\underset{\sim}{u})\underset{\sim}{v} + (\underset{\sim}{D}\underset{\sim}{u})(\underset{\sim}{D}\underset{\sim}{v}) - (\underset{\sim}{D}\underset{\sim}{v})(\underset{\sim}{D}\underset{\sim}{u}) - \underset{\sim}{D}((\underset{\sim}{S}\underset{\sim}{u})\underset{\sim}{v}). \qquad (12.12)$$

In view of (12.5) and (12.10), equation (12.12) shows that $\overset{*}{\underset{\sim}{R}}$ can be expressed in terms of the contortion $\underset{\sim}{D}$ and its gradient relative to $\underset{\sim}{K}$ or in terms of the inhomogeneity $\underset{\sim}{S}$ and its gradient relative to $\underset{\sim}{K}$.

13. Contorted aeolotropy.

Definition 8: A uniform reference $\mathbf{K}$ of class C^{p-1} is called a *state of contorted aeolotropy* if there exists a configuration $\boldsymbol{\varkappa}$ such that the tensor field

$$\mathbf{Q} = (\nabla \boldsymbol{\varkappa})\mathbf{K}^{-1} \in \mathcal{L}_{\mathcal{B}}^{p-1} \tag{13.1}$$

has orthogonal values ($\mathbf{Q}(X) \in \mathcal{O}$ for all $X \in \mathcal{B}$).

Assume that K is such a state of contorted aelotropy. Since the inner product in $\mathcal{V}$ is preserved under orthogonal transformations, the Riemannian structure (12.1) relative to $\mathbf{K}$ satisfies

$$f * h = \mathbf{Q}(\mathbf{K}f)\cdot\mathbf{Q}(\mathbf{K}h) = (\nabla\boldsymbol{\varkappa})f\cdot(\nabla\boldsymbol{\varkappa})h \tag{13.2}$$

for all tangent vector fields f and h. It follows from (13.2) that the Riemannian connection $\overset{*}{\Gamma}$ relative to $\mathbf{K}$ is obtained by transporting the gradient operator ∇ from $\boldsymbol{\varkappa}(\mathcal{B})$ into $\mathcal{B}$ via $(\nabla\boldsymbol{\varkappa})^{-1}$, so that

$$\overset{*}{\Gamma} h = (\nabla\boldsymbol{\varkappa})^{-1}\nabla_{\boldsymbol{\varkappa}}\big((\nabla\boldsymbol{\varkappa})h\big)(\nabla\boldsymbol{\varkappa}) \;, \quad h \in \mathcal{T}_{\mathcal{B}}^{p-1} . \tag{13.3}$$

Indeed, if $\overset{*}{\Gamma}$ is defined by (13.3), condition (a) of Proposition 6 follows from the symmetry of the second gradient and condition (b) from the rule for the differentiation of inner products. By virtue of (13.1), (13.3) is equivalent to

$$\mathbf{K}\overset{*}{\Gamma}_{f} h = \mathbf{Q}^{T}\nabla_{\boldsymbol{\varkappa}}(\mathbf{Q}\mathbf{K}h)\mathbf{Q}\mathbf{K}f \;, \quad f, h \in \mathcal{T}_{\mathcal{B}}^{p-1} \tag{13.4}$$

Now let $\mathbf{u}, \mathbf{v} \in \mathcal{V}$. If we substitute $f = \mathbf{K}^{-1}\mathbf{u}$, $h = \mathbf{K}^{-1}\mathbf{v}$ into (13.4) and observe (12.6) we obtain

$$\underset{\sim}{Q}^T \nabla_{\underset{\sim}{x}} (\underset{\sim}{Q}\underset{\sim}{v}) \underset{\sim}{Q}\underset{\sim}{u} = -(\underset{\sim}{D}\underset{\sim}{u})\underset{\sim}{v},$$

which, by (13.1) and (8.8), is equivalent to

$$\underset{\sim}{D}\underset{\sim}{u} = -\underset{\sim}{Q}^T(\nabla_{\underset{\sim}{K}}\underset{\sim}{Q})\underset{\sim}{u} \quad , \quad \underset{\sim}{u} \in \mathcal{V}. \tag{13.5}$$

This equation shows that the skew transformation $-\underset{\sim}{D}\underset{\sim}{u}\big|_{X} \in \mathcal{L}$ is the instantaneous rate of change of $\underset{\sim}{Q}$ at X in the direction of $\underset{\sim}{u}$, if viewed in any configuration belonging to $\underset{\sim}{K}(\underset{\sim}{x})$. In other words, $\underset{\sim}{D}$ describes the local behavior of the rotation field $\underset{\sim}{Q}$, which changes the given state of contorted aeolotropy $\underset{\sim}{K}$ into the gradient of a global configuration $\underset{\sim}{x}$. It is this property that the term "contortion" for $\underset{\sim}{D}$ is meant to express.

Theorem 7: If $\underset{\sim}{K}$ is a state of contorted aelotropy then the curvature of the Riemannian connection relative to $\underset{\sim}{K}$ vanishes. Conversely, if the curvature of the Riemannian connection relative to $\underset{\sim}{K}$ vanishes, then $\underset{\sim}{K}$ is locally a state of contorted aeolotropy (i.e., every point in $\mathcal{B}$ has a neighborhood $\mathcal{N}$ such that the restriction of $\underset{\sim}{K}$ to $\mathcal{N}$ is a state of contorted aeolotropy for $\mathcal{B}$.)

Proof: Assume first that (13.1) holds. It follows from (13.3) that $\overset{*}{\Gamma}\mathcal{k} = 0$ if and only if $(\nabla\underset{\sim}{x})\mathcal{k}$ is constant. Hence we could give a simple direct proof of $\overset{*}{\mathcal{R}} = 0$ by using the same argument as we used in the proof of Theorem 4 (Sect. 10). Another proof can be obtained on the basis of (13.5) as follows:

If $\underset{\sim}{\gamma}$ is an arbitrary configuration and $\underset{\sim}{F} = (\nabla\underset{\sim}{\gamma})\underset{\sim}{K}^{-1}$, then (13.5) is equivalent to

$$(\nabla_{\underset{\sim}{\gamma}}\underset{\sim}{Q})\underset{\sim}{h} = -\underset{\sim}{Q}\underset{\sim}{D}(\underset{\sim}{F}^{-1}\underset{\sim}{h}) \quad , \quad \underset{\sim}{h} \in \mathcal{V}. \tag{13.6}$$

If we take the gradient ∇_{γ} of (13.6) in the direction of $k \in \mathcal{V}$, we find

$$((\nabla_{\gamma}^{(2)} Q)k)h = -Q\{((\nabla_{K} D)v)u - (Dv)(Du) + D(((\nabla_{\gamma} F^{-1})k)h)\}, \tag{13.7}$$

where $u = F^{-1}h$, $v = F^{-1}k$. Of course, because of the linearity of the values of the fields D, $\nabla_{K} D$, $\nabla_{\gamma}^{(2)} Q$, etc., (13.6) and (13.7) remain valid if h and k are not fixed vectors but vector fields. In particular, they remain valid when u and v are fixed. If we interchange u and v and hence h and k in (13.7) and subtract the resulting formula from (13.7), we obtain, after observing (11.5) and (12.12),

$$((\nabla_{\gamma}^{(2)} Q)k)h - ((\nabla_{\gamma}^{(2)} Q)k)h = -Q\overset{*}{R}(u, v). \tag{13.8}$$

Thus, $\overset{*}{R} = 0$ and hence $\overset{*}{\mathcal{R}} = 0$ follows also from the symmetry of the second gradient $\nabla_{\gamma}^{(2)} Q$.

Assume now that K is a uniform reference such that $\overset{*}{R} = 0$. Let γ be an arbitrary configuration and put $F = (\nabla\gamma)K^{-1}$, as before. We can then regard (13.6) as a differential equation for the determination of Q. As we have seen, $\overset{*}{R} = 0$ is an integrability condition necessary for the existence of a solution. According to a classical theorem, $\overset{*}{R} = 0$ is also sufficient for the existence of a solution that is valid in a simply connected neighborhood $\mathcal{N}'$ of a given point $X_0 \in \mathcal{B}$. The solution can be chosen so that for $X_0 \in \mathcal{B}$ $Q(X_0)$ has a prescribed value, which we take to be the identity 1. Since Du is skew for all $u \in \mathcal{V}$, it follows from (13.5) that QQ^T has gradient zero and hence must be equal to 1 everywhere in $\mathcal{N}'$. Hence Q has orthogonal values. To summarize: If $\overset{*}{R} = 0$, every point in $\mathcal{B}$ has a neighborhood $\mathcal{N}'$ on which we can find an orthogonal-valued tensor field Q (of class C^{p-1}) such that (13.5) holds.

Assume, then, that (13.5) holds on $\mathcal{N}'$. Combining (12.6) with (13.5) we obtain

$$\underset{\sim}{K}\overset{*}{\Gamma}_{f}\mathfrak{t} = \underset{\sim}{Q}^{T}((\nabla_{\underset{\sim}{K}}\underset{\sim}{Q})\underset{\sim}{K}f)\underset{\sim}{c} = \underset{\sim}{Q}^{T}\nabla_{\underset{\sim}{K}}(\underset{\sim}{Q}\underset{\sim}{c})\underset{\sim}{K}f, \tag{13.9}$$

which is valid when $\mathfrak{t} = \underset{\sim}{K}^{-1}\underset{\sim}{c}$ is materially constant. Consider the affine connection $\overline{\Gamma}$ of class C^{p-1} defined by

$$\overline{\Gamma}k = \overline{\underset{\sim}{K}}^{-1}\nabla_{\overline{\underset{\sim}{K}}}(\overline{\underset{\sim}{K}}k)\overline{\underset{\sim}{K}} \quad , \qquad k \in \mathcal{T}_{\mathcal{B}}^{1}, \tag{13.10}$$

where

$$\overline{\underset{\sim}{K}} = \underset{\sim}{Q}\underset{\sim}{K}. \tag{13.11}$$

It is easily seen that (13.9) is equivalent to the statement that

$$\overset{*}{\Gamma}\mathfrak{t} = \overline{\Gamma}\mathfrak{t} \tag{13.12}$$

holds for all materially constant tangent vector fields $\mathfrak{t}$. Using the same argument as in the uniqueness proof of Theorem 3 (Sect. 10) we conclude that $\overset{*}{\Gamma} = \overline{\Gamma}$. Since $\overset{*}{\Gamma}$ has zero torsion, an analogue of Theorem 6 (Sect. 10) shows that every point in $\mathcal{N}'$ must have a neighborhood $\mathcal{N}$ such that $\overline{\underset{\sim}{K}} = \nabla\underset{\sim}{\varkappa}$ for some configuration $\underset{\sim}{\varkappa}$ of $\mathcal{N}$. Hence, by (13.11), we have $\underset{\sim}{Q} = (\nabla\underset{\sim}{\varkappa})\underset{\sim}{K}^{-1}$ on $\mathcal{N}$, i.e., $\underset{\sim}{K}$ is a state of contorted aeolotropy on $\mathcal{N}$. Q.E.D.

A Special case of contorted aeolotropy is <u>curvilinear aelotropy</u>. It corresponds to the case when there exists an orthogonal coordinate system on $\underset{\sim}{\varkappa}(\mathcal{B}) \subset \mathcal{E}$ with the following property: If $(\underset{\sim}{e}_1(X), \underset{\sim}{e}_2(X), \underset{\sim}{e}_3(X))$ is the orthonormal basis which consists of the unit vectors that point in the direction of the coordinate lines at $\underset{\sim}{\varkappa}(X)$, then $\underset{\sim}{Q}(X)\underset{\sim}{e}_i(X)$ does not depend on $X \in \mathcal{B}$.

14. Special types of materially uniform bodies.

We consider first the case when the isotropy group $\mathcal{g}_{\underset{\sim}{K}}$ of $\mathcal{B}$ relative to some—and hence every—uniform re erence $\underset{\sim}{K}$ is discrete. Suppose that $\underset{\sim}{K}$ and $\hat{\underset{\sim}{K}}$ are two continuous uniform references. They must be related by (6.12), where $\underset{\sim}{P}(\underset{\sim}{X}) \in \mathcal{g}_{\underset{\sim}{K}}$ must depend continuously on $\underset{\sim}{X}$. Since $\mathcal{g}_{\underset{\sim}{K}}$ is discrete, this is possible only when $\underset{\sim}{P}$ is constant. Thus we can absorb $\underset{\sim}{P}$ into $\underset{\sim}{L}$ and (6.12) becomes $\hat{\underset{\sim}{K}} = \underset{\sim}{L}\,\underset{\sim}{K}$, with $\underset{\sim}{L}$ = const. If we write (6.4) for both $\underset{\sim}{K}$ and $\hat{\underset{\sim}{K}}$, we see that they correspond to the same continuous material uniformity. Since every continuous material uniformity must be of the form (6.4), where $\underset{\sim}{K}$ is a continuous uniform reference , we have the following result:

Theorem 8: If the isotropy groups of a materially uniform simple body $\mathcal{B}$ are discrete, then $\mathcal{B}$ has at most one continuous material uniformity $\underset{\sim}{\Phi}$. Any two continuous uniform references $\underset{\sim}{K}$, $\hat{\underset{\sim}{K}}$ are related by

$$\hat{\underset{\sim}{K}} = \underset{\sim}{L}\,\underset{\sim}{K}\,, \qquad \underset{\sim}{L} = \mathit{const.} \in \mathcal{L}. \tag{14.1}$$

Since material connections are only associated with differentiable uniformities $\underset{\sim}{\Phi}$ and not with discontinuous ones, it follows from the uniqueness assertion of Theorem 8 that in the case when the isotropy groups are discrete, the inhomogeneity $\mathcal{T}$ is a characteristic of the body. If the isotropy groups are non-discrete Lie-groups, however, and if there are any material uniformities of class C^{p-1} at all, there will be many, and hence also many inhomogeneities $\mathcal{T}$ for one and the same body. This is the case, in particular, for uniform isotropic bodies.

Next, we consider a uniform isotropic body $\mathcal{B}$ with an undistorted uniform reference $\underset{\sim}{K}$ of class C^{p-1}, so that $\mathcal{O} \subset \mathcal{g}_{\underset{\sim}{K}}$. If $\underset{\sim}{K}$ is a state of contorted aelotropy, so that (13.1) holds, it follows from Theorem 2, (6.12), that $\nabla \underset{\sim}{\varkappa}$ is again a uniform reference and that $\mathcal{g}_{\underset{\sim}{K}}$ is also the isotropy group relative to $\nabla \underset{\sim}{\varkappa}$:

Theorem 9: If a uniform isotropic body has an undistorted state of contorted aeolotropy it is homogeneous.

The conclusion of the theorem becomes false when the qualifier "undistorted" is omitted; i.e., there are inhomogeneous isotropic bodies with distorted states of contorted aeolotropy.

Finally, suppose there is a natural way to single out, among all uniform references for $\mathcal{B}$, a particular class $\underline{\underline{U}}$ with the following property (P): All members of $\underline{\underline{U}}$ are of class C^{p-1} and differ from one another by a field of similarity transformations with constant ratio, so that $\underset{\sim}{K}, \underset{\sim}{K}' \in \underline{\underline{U}}$ implies

$$\underset{\sim}{K}' = a \underset{\sim}{Q} \underset{\sim}{K}, \tag{14.2}$$

where a is a real constant and $\underset{\sim}{Q}$ an orthogonal valued tensor field on $\mathcal{B}$. For example, if $\mathcal{B}$ is a uniform solid body that is either isotropic or has cubic symmetry then the class $\underline{\underline{U}}$ of all undistorted references has the property (P). This follows from results proved in reference [9]. Other examples are obtained by letting $\underline{\underline{U}}$ be the class of all uniform references $\underset{\sim}{K}$ such that the corresponding response functions $\underset{\sim}{\mathfrak{g}}_{\underset{\sim}{K}}$ satisfy a certain special condition such as $\underset{\sim}{\mathfrak{g}}_{\underset{\sim}{K}}(\underset{\sim}{1}) = \underset{\sim}{0}$. Such references are often called natural references. The nature of the response function is often such that the class $\underline{\underline{U}}$ of natural references has the property (P).

If (14.2) holds, it follows from (12.1) and the fact that orthogonal transformations preserve inner products that the two Riemannian inner products corresponding to $\underset{\sim}{K}'$ and $\underset{\sim}{K}$ differ from one another only by the constant factor a^2. Therefore, Proposition 6 shows that the Riemannian connections relative to $\underset{\sim}{K}$ and $\underset{\sim}{K}'$ are the same, and we have the following result:

Theorem 10: If $\mathcal{B}$ is a uniform simple body with a distinguished class $\underline{\underline{U}}$ of uniform references with the property (P), then the Riemannian connection $\overset{*}{\Gamma}$ and its curvature $\overset{*}{\mathcal{R}}$ are characteristics of the body.

The assertion of Theorem 10 applies in particular, to uniform isotropic solid bodies, for which the curvature, $\overset{*}{\mathcal{R}}$, defined by the class of undistorted uniform references, is an intrinsic measure of deviation from homogeneity.

15. Cauchy's equations of balance.

We now derive a new version of Cauchy's equation of balance, which expresses the fact that the forces acting on every part of a given body $\mathcal{B}$ must add to zero. In order to do so, we first derive a lemma, Proposition 7 below.

Let $\underset{\sim}{h}$ be a vector field and $\underset{\sim}{T}$ a tensor field of class C^1 on $\mathcal{B}$. We define the divergence of these fields relative to some reference $\underset{\sim}{K}$ by

$$\operatorname{div}_{\underset{\sim}{K}} \underset{\sim}{h} = \operatorname{tr}(\nabla_{\underset{\sim}{K}} \underset{\sim}{h}) \quad \text{and} \quad (\operatorname{div}_{\underset{\sim}{K}} \underset{\sim}{T})\cdot \underset{\sim}{u} = \operatorname{div}_{\underset{\sim}{K}}(\underset{\sim}{T}^T \underset{\sim}{u}), \; \underset{\sim}{u} \in \mathcal{V}, \tag{15.1}$$

respectively. The following product rules are valid for $f \in \mathcal{F}_{\mathcal{B}}^1$, $\underset{\sim}{h} \in \mathcal{V}_{\mathcal{B}}^1$, $\underset{\sim}{T} \in \mathcal{L}_{\mathcal{B}}^1$:

$$\left.\begin{aligned} \operatorname{div}_{\underset{\sim}{K}}(f\underset{\sim}{h}) &= (\nabla_{\underset{\sim}{K}} f)\cdot \underset{\sim}{h} + f \operatorname{div}_{\underset{\sim}{K}} \underset{\sim}{h}, \\ \operatorname{div}_{\underset{\sim}{K}}(\underset{\sim}{T}\underset{\sim}{h}) &= (\operatorname{div}_{\underset{\sim}{K}} \underset{\sim}{T}^T)\cdot \underset{\sim}{h} + \operatorname{tr}(\underset{\sim}{T}\nabla_{\underset{\sim}{K}} \underset{\sim}{h}). \end{aligned}\right\} \tag{15.2}$$

If $\underset{\sim}{K} = \nabla \underset{\sim}{\varkappa}$ is a configuration gradient, we write $\mathrm{div}_{\underset{\sim}{\varkappa}}$ instead of $\mathrm{div}_{\underset{\sim}{K}}$.

Suppose that a uniform reference $\underset{\sim}{K}$ of class C^{p-1} for $\mathcal{B}$ and a tensor field $\underset{\sim}{T}$ of class C^1 on $\mathcal{B}$ are given. For any configuration $\underset{\sim}{\gamma}$ of $\mathcal{B}$ we then define another tensor field $\underset{\sim}{T}_{\underset{\sim}{\gamma}}$ of class C^1 by

$$\underset{\sim}{T}_{\underset{\sim}{\gamma}} = \frac{1}{J} \underset{\sim}{T} \underset{\sim}{E}^T, \tag{15.3}$$

where

$$\underset{\sim}{E} = (\nabla \underset{\sim}{\gamma}) \underset{\sim}{K}^{-1}, \qquad J = |\det \underset{\sim}{E}|. \tag{15.4}$$

<u>Proposition 7</u>: <u>If</u>

$$\underset{\sim}{K}(X) = \nabla \underset{\sim}{\varkappa}(X) \tag{15.5}$$

<u>for some</u> $X \in \mathcal{B}$, <u>then the divergences at</u> X <u>relative to</u> $\underset{\sim}{K}$ <u>of the tensor field</u> $\underset{\sim}{T}$ <u>and of the tensor field</u> $\underset{\sim}{T}_{\underset{\sim}{\varkappa}}$ <u>defined by</u> (15.3) <u>and</u> (15.4) <u>are related by</u>

$$\mathrm{div}_{\underset{\sim}{K}} \underset{\sim}{T}_{\underset{\sim}{\varkappa}} \Big|_X = \left(\mathrm{div}_{\underset{\sim}{K}} \underset{\sim}{T} + \underset{\sim}{T} \underset{\sim}{s} \right)\Big|_X, \tag{15.6}$$

<u>where the vector field</u> $\underset{\sim}{s}$ <u>is defined, in terms of the inhomogeneity</u> $\underset{\sim}{S}$ <u>relative to</u> $\underset{\sim}{K}$, <u>by</u>

$$\underset{\sim}{s} \cdot \underset{\sim}{u} = \mathrm{tr}\,(\underset{\sim}{S}\,\underset{\sim}{u}), \qquad \underset{\sim}{u} \in \mathcal{V}. \tag{15.7}$$

<u>Proof</u>: We make use of (15.3) and (15.4) with $\underset{\sim}{\gamma}$ replaced by $\underset{\sim}{\varkappa}$. We then have

$$J\,(\underset{\sim}{T}_{\underset{\sim}{\varkappa}}^T \underset{\sim}{u}) = \underset{\sim}{E}\,(\underset{\sim}{T}^T \underset{\sim}{u}), \qquad \underset{\sim}{u} \in \mathcal{V} \tag{15.8}$$

Using the rule

$$\underset{\sim}{u} \cdot \nabla_{\underset{\sim}{K}} (\det \underset{\sim}{E}) = (\det \underset{\sim}{E})\, \mathrm{tr}\left[\underset{\sim}{E}^{-1}((\nabla_{\underset{\sim}{K}} \underset{\sim}{E})\underset{\sim}{u})\right], \quad \underset{\sim}{u} \in \mathcal{V}, \tag{15.9}$$

for the differentiation of a determinant, the product rules (15.2), and the definition $(15.1)_2$, we see that taking $div_{\underset{\sim}{K}}$ of (15.8) yields

$$J\, tr[\underset{\sim}{F}^{-1}((\nabla_{\underset{\sim}{K}}\underset{\sim}{F})(\underset{\sim}{T}_{\varkappa}^{T}\underset{\sim}{u}))] + J\,(div_{\underset{\sim}{K}}\underset{\sim}{T}_{\varkappa})\cdot\underset{\sim}{u} =$$
$$= (div_{\underset{\sim}{K}}\underset{\sim}{F}^{T})\cdot\underset{\sim}{T}^{T}\underset{\sim}{u} + tr[\underset{\sim}{F}\nabla_{\underset{\sim}{K}}(\underset{\sim}{T}^{T}\underset{\sim}{u})]. \tag{15.10}$$

Since $\underset{\sim}{F}(\underset{\sim}{X})=\underset{\sim}{1}$, $J(\underset{\sim}{X})=1$, and $\underset{\sim}{T}_{\varkappa}(\underset{\sim}{X})=\underset{\sim}{T}(\underset{\sim}{X})$ by (15.5), (15.4), and (15.3), evaluation of (15.10) at $\underset{\sim}{X}\in\mathcal{B}$ gives

$$\left\{tr[(\nabla_{\underset{\sim}{K}}\underset{\sim}{F})(\underset{\sim}{T}^{T}\underset{\sim}{u})] + (div_{\underset{\sim}{K}}\underset{\sim}{T}_{\varkappa})\cdot\underset{\sim}{u} - (div_{\underset{\sim}{K}}\underset{\sim}{F}^{T})\cdot\underset{\sim}{T}^{T}\underset{\sim}{u} - (div_{\underset{\sim}{K}}\underset{\sim}{T})\cdot\underset{\sim}{u}\right\}_{\underset{\sim}{X}} = 0. \tag{15.11}$$

Using the rule $tr[(\nabla_{\underset{\sim}{K}}\underset{\sim}{F})\underset{\sim}{v}] = \nabla_{\underset{\sim}{K}}(tr\underset{\sim}{F})\cdot\underset{\sim}{v}$, we see that (15.11) can hold for all $\underset{\sim}{u}\in\mathcal{V}$ only if

$$\left\{\underset{\sim}{T}\,[\nabla_{\underset{\sim}{K}}(tr\underset{\sim}{F}) - div_{\underset{\sim}{K}}\underset{\sim}{F}^{T}] + div_{\underset{\sim}{K}}\underset{\sim}{T}_{\varkappa} - div_{\underset{\sim}{K}}\underset{\sim}{T}\right\}_{\underset{\sim}{X}} = \underset{\sim}{0}. \tag{15.12}$$

On the other hand, if we evaluate (11.4) at $\underset{\sim}{X}\in\mathcal{B}$ and take the trace, we obtain

$$tr(\underset{\sim}{S}\,\underset{\sim}{u})\Big|_{\underset{\sim}{X}} = \left\{(div_{\underset{\sim}{K}}\underset{\sim}{F}^{T})\cdot\underset{\sim}{u} - \nabla_{\underset{\sim}{K}}(tr\underset{\sim}{F})\cdot\underset{\sim}{u}\right\}_{\underset{\sim}{X}},$$

which, in view of (15.7), is equivalent to

$$\underset{\sim}{S}\Big|_{\underset{\sim}{X}} = \left\{div_{\underset{\sim}{K}}\underset{\sim}{F}^{T} - \nabla_{\underset{\sim}{K}}(tr\underset{\sim}{F})\right\}_{\underset{\sim}{X}}. \tag{15.13}$$

The desired result (15.6) follows from (15.12) and (15.13). Q.E.D.

Let us assume now that the body $\mathcal{B}$ is subject to internal contact forces and external body forces[1]. If the forces acting on every part of $\mathcal{B}$ are

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[1] Inertial forces should be regarded as part of the body forces.

balanced and if suitable regularity assumptions are satisfied one can prove the following results (cf. [10]):

(i) With every configuration $\underset{\sim}{\varkappa}$ of $\mathcal{B}$ one can associate a stress tensor field $\underset{\sim}{T}_{\underset{\sim}{\varkappa}}$ of class C^1 and a body force field $\underset{\sim}{b}_{\varkappa}$ of class C^0 such that the force $\underset{\sim}{f}$ exerted on a part $\mathcal{P}$ of $\mathcal{B}$ by the combined action of a separate part $\mathcal{P}'$ of $\mathcal{B}$ and the external world is given by

$$\underset{\sim}{f} = \int_{\underset{\sim}{\varkappa}(\mathcal{P})} \underset{\sim}{b}_{\varkappa}\, dV + \int_{\mathcal{C}} \underset{\sim}{T}_{\underset{\sim}{\varkappa}} \underset{\sim}{n}\, dS, \tag{15.14}$$

where $\mathcal{C}$ is the surface of contact between $\mathcal{P}$ and $\mathcal{P}'$ in the configuration $\underset{\sim}{\varkappa}$ and where $\underset{\sim}{n}$ is the unit normal to $\mathcal{C}$ directed away from $\underset{\sim}{\varkappa}(\mathcal{P})$.

(ii) Cauchy's equation of balance

$$\operatorname{div}_{\underset{\sim}{\varkappa}} \underset{\sim}{T}_{\underset{\sim}{\varkappa}} + \underset{\sim}{b}_{\underset{\sim}{\varkappa}} = \underset{\sim}{0} \tag{15.15}$$

is valid on $\mathcal{B}$ for every configuration $\underset{\sim}{\varkappa}$.

(iii) If $\underset{\sim}{\varkappa}$ and $\underset{\sim}{\gamma}$ are two configurations of $\mathcal{B}$, then the stress fields $\underset{\sim}{T}_{\varkappa}$ and $\underset{\sim}{T}_{\underset{\sim}{\gamma}}$ and the body force fields $\underset{\sim}{b}_{\varkappa}$ and $\underset{\sim}{b}_{\underset{\sim}{\gamma}}$ are related by

$$\underset{\sim}{T}_{\underset{\sim}{\gamma}} = \frac{1}{J} \underset{\sim}{T}_{\underset{\sim}{\varkappa}} \underset{\sim}{F}^{T}, \qquad \underset{\sim}{b}_{\underset{\sim}{\gamma}} = \frac{1}{J} \underset{\sim}{b}_{\varkappa}, \tag{15.16}$$

where

$$\underset{\sim}{F} = (\nabla \underset{\sim}{\gamma})(\nabla \underset{\sim}{\varkappa})^{-1}, \qquad J = |\det \underset{\sim}{F}|. \tag{15.17}$$

(cf. equation (43A.3) of reference [2]).

Let $\underset{\sim}{K}$ be a uniform reference of class C^{p-1} for $\mathcal{B}$. Let a particular point $X \in \mathcal{B}$ be given. It is clear from (15.16) and (15.17) that $\underset{\sim}{T}_{\underset{\sim}{\gamma}}(X) = \underset{\sim}{T}_{\varkappa}(X)$

and $\underset{\sim}{b}_{\underset{\sim}{\gamma}}(X) = \underset{\sim}{b}_{\underset{\sim}{\chi}}(X)$ hold whenever both $\underset{\sim}{\chi}$ and $\underset{\sim}{\gamma}$ belong to the equivalence class by which the local configuration $\underset{\sim}{K}(X)$ is defined. Thus, we can define fields $\underset{\sim}{T}_{\underset{\sim}{K}}$ and $\underset{\sim}{b}_{\underset{\sim}{K}}$ by the condition that for each $X \in \mathcal{B}$,

$$\underset{\sim}{T}_{\underset{\sim}{K}}(X) = \underset{\sim}{T}_{\underset{\sim}{\chi}}(X) \quad , \qquad \underset{\sim}{b}_{\underset{\sim}{K}}(X) = \underset{\sim}{b}_{\underset{\sim}{\chi}}(X) \tag{15.18}$$

hold whenever $\nabla\underset{\sim}{\chi}(X) = \underset{\sim}{K}(X)$. We call $\underset{\sim}{T}_{\underset{\sim}{K}}$ and $\underset{\sim}{b}_{\underset{\sim}{K}}$ the <u>stress tensor field</u> and <u>body force field relative to the reference</u> $\underset{\sim}{K}$. It is clear from (15.16) and (15.17) that

$$\underset{\sim}{T}_{\underset{\sim}{\gamma}} = \frac{1}{J}\underset{\sim}{T}_{\underset{\sim}{K}}\underset{\sim}{F}^{T} \quad , \qquad \underset{\sim}{b}_{\underset{\sim}{\gamma}} = \frac{1}{J}\underset{\sim}{b}_{\underset{\sim}{K}} \tag{15.19}$$

when $\underset{\sim}{F}$ and J are given by (15.4), where $\underset{\sim}{\gamma}$ is an arbitrary configuration.

Since $\operatorname{div}_{\underset{\sim}{\chi}}\underset{\sim}{T}\big|_X = \operatorname{div}_{\underset{\sim}{K}}\underset{\sim}{T}\big|_X$ whenever $\nabla\underset{\sim}{\chi}(X) = \underset{\sim}{K}(X)$, it follows from (15.15) that

$$\left[\operatorname{div}_{\underset{\sim}{K}}\underset{\sim}{T}_{\chi} + \underset{\sim}{b}_{\underset{\sim}{K}}\right]_X = \underset{\sim}{0} \tag{15.20}$$

holds whenever $\nabla\underset{\sim}{\chi}(X) = \underset{\sim}{K}(X)$. On the other hand, Proposition 7 applies when we choose $\underset{\sim}{T} = \underset{\sim}{T}_{\underset{\sim}{K}}$. Thus, by substituting (15.6) with the choice $\underset{\sim}{T} = \underset{\sim}{T}_{\underset{\sim}{K}}$ into (15.20) we obtain the following result:

<u>Theorem 11</u>: <u>The stress tensor field</u> $\underset{\sim}{T}_{\underset{\sim}{K}}$ <u>and the body force field</u> $\underset{\sim}{b}_{\underset{\sim}{K}}$ <u>relative to a uniform reference</u> $\underset{\sim}{K}$ <u>satisfy the modified equation of balance</u>

$$\operatorname{div}_{\underset{\sim}{K}}\underset{\sim}{T}_{\underset{\sim}{K}} + \underset{\sim}{T}_{\underset{\sim}{K}}\underset{\sim}{s} + \underset{\sim}{b}_{\underset{\sim}{K}} = \underset{\sim}{0}, \tag{15.21}$$

<u>where</u> $\underset{\sim}{s}$ <u>is defined in terms of the inhomogeneity</u> $\underset{\sim}{S}$ <u>relative to</u> $\underset{\sim}{K}$ <u>by</u> (15.7).

The equation (15.21) is much more useful than (15.15) for dealing with inhomogeneous materially uniform bodies. Consider, for example, an elastic body $\mathcal{B}$, for which the set of response descriptors is the set $\mathcal{S} \subset \mathcal{L}$ of symmetric linear transformations. According to Theorem 1 (Sect. 6) we can associate with a given uniform reference $\underset{\sim}{K}$ a relative elastic response function $\underset{\sim}{g}_{\underset{\sim}{K}} : \ell \to \mathcal{S}$. In order that a configuration $\underset{\sim}{\gamma}$ be compatible with a given force system, the constitutive equation

$$\underset{\sim}{g}_{\underset{\sim}{K}}(\underset{\sim}{F}) = \underset{\sim}{T}_{\underset{\sim}{\gamma}} \quad , \quad \underset{\sim}{F} = (\nabla \underset{\sim}{\gamma})\underset{\sim}{K}^{-1} \tag{15.22}$$

must be satisfied on $\mathcal{B}$, where $\underset{\sim}{T}_{\underset{\sim}{\gamma}}$ is the stress tensor field for $\underset{\sim}{\gamma}$. In view of $(15.19)_1$, (15.22) is equivalent to

$$\underset{\sim}{f}_{\underset{\sim}{K}}(\underset{\sim}{F}) = \underset{\sim}{T}_{\underset{\sim}{K}} \quad , \quad \underset{\sim}{F} = (\nabla \underset{\sim}{\gamma})\underset{\sim}{K}^{-1}, \tag{15.23}$$

where $\underset{\sim}{f}_{\underset{\sim}{K}} : \ell \to \mathcal{L}$ is defined by

$$\underset{\sim}{f}_{\underset{\sim}{K}}(\underset{\sim}{F}) = |\det \underset{\sim}{F}|\, \underset{\sim}{g}_{\underset{\sim}{K}}(\underset{\sim}{F})\underset{\sim}{F}^{T^{-1}} \quad , \quad \underset{\sim}{F} \in \ell. \tag{15.24}$$

Assume that $\underset{\sim}{f}_{\underset{\sim}{K}}$ is of class C^1 and denote its gradient by $\mathbb{H}_{\underset{\sim}{K}}$. For each $\underset{\sim}{F} \in \ell$ the value $\mathbb{H}_{\underset{\sim}{K}}(\underset{\sim}{F})$ is then a linear transformation from $\mathcal{L}$ into $\mathcal{L}$. If we take the gradient of (15.23) relative to $\underset{\sim}{K}$, the chain rule yields

$$(\nabla_{\underset{\sim}{K}} \underset{\sim}{T}_{\underset{\sim}{K}})\underset{\sim}{u} = \mathbb{H}_{\underset{\sim}{K}}(\underset{\sim}{F})[(\nabla_{\underset{\sim}{K}} \underset{\sim}{F})\underset{\sim}{u}] \quad , \quad \underset{\sim}{u} \in \mathcal{V}. \tag{15.25}$$

It follows that

$$\operatorname{div}_{\underset{\sim}{K}} \underset{\sim}{T}_{\underset{\sim}{K}} = \mathbb{A}_{\underset{\sim}{K}}(\underset{\sim}{F})[(\nabla_{\underset{\sim}{K}} \underset{\sim}{F})], \tag{15.26}$$

where $\mathbb{A}_{\underset{\sim}{K}}$ is that function on ℓ whose values $\mathbb{A}_{\underset{\sim}{K}}(\underset{\sim}{F}) : \mathcal{L}(\mathcal{V},\mathcal{L}) \to \mathcal{V}$ are determined by the property that $\mathbb{A}_{\underset{\sim}{K}}(\underset{\sim}{F})[\underset{\sim}{Z}] \cdot \underset{\sim}{w}$ is the trace of the linear transformation $\underset{\sim}{u} \to \{\mathbb{H}_{\underset{\sim}{K}}(\underset{\sim}{F})[\underset{\sim}{Z}\underset{\sim}{u}]\}^T \underset{\sim}{w}$ for all $\underset{\sim}{w} \in \mathcal{V}$ and all $\underset{\sim}{Z} \in \mathcal{L}(\mathcal{V},\mathcal{L})$. Of course, $\mathbb{A}_{\underset{\sim}{K}}$ is determined by the response function $\underset{\sim}{f}_{\underset{\sim}{K}}$. If we substitute (15.26) and (15.23) into (15.21), we obtain[1)]

$$\mathbb{A}_{\underset{\sim}{K}}(\underset{\sim}{F})[\nabla_{\underset{\sim}{K}}\underset{\sim}{F}] + \underset{\sim}{f}_{\underset{\sim}{K}}(\underset{\sim}{F})\underset{\sim}{s} + \underset{\sim}{b}_{\underset{\sim}{K}} = 0\,, \quad \underset{\sim}{F} = (\nabla_{\underset{\sim}{\gamma}})^{-1}\underset{\sim}{K}\,, \tag{15.27}$$

1) This result, in terms of coordinates, was already announced in reference [2] as equation (44.7).

which is the differential equation for the determination of configurations $\underset{\sim}{\gamma}$ possible in a materially uniform elastic body. If the body is homogeneous we can choose $\underset{\sim}{K} = \nabla_{\underset{\sim}{\kappa}}$. Then $\underset{\sim}{s}$ vanishes and (15.27) reduces to the classical differential equation of finite elasticity.

Finally, we given another application of Proposition 7. Using the fact that $\mathcal{V}$ is three-dimensional (which was irrelevant up to now), we choose an orientation in $\mathcal{V}$ and consider the associated cross product $\times$. The curl of a vector field $\underset{\sim}{h}$ and a tensor field $\underset{\sim}{T}$ on $\mathcal{B}$ relative to some configuration $\underset{\sim}{\gamma}$ are defined by

$$\begin{aligned} \underset{\sim}{u}\cdot(\nabla_{\underset{\sim}{\gamma}}\underset{\sim}{h})\underset{\sim}{v} - \underset{\sim}{v}\cdot(\nabla_{\underset{\sim}{\gamma}}\underset{\sim}{h})\underset{\sim}{u} &= (\operatorname{curl}_{\underset{\sim}{\gamma}}\underset{\sim}{h})\cdot(\underset{\sim}{u}\times\underset{\sim}{v}),\\ (\operatorname{curl}_{\underset{\sim}{\gamma}}\underset{\sim}{T})^T\underset{\sim}{u} &= \operatorname{curl}_{\underset{\sim}{\gamma}}(\underset{\sim}{T}\underset{\sim}{u}), \end{aligned} \tag{15.28}$$

where $\underset{\sim}{u},\underset{\sim}{v} \in \mathcal{V}$. It follows from (15.28) that

$$(\operatorname{curl}_{\underset{\sim}{\gamma}}\underset{\sim}{T}^T)(\underset{\sim}{u}\times\underset{\sim}{v}) = ((\nabla_{\underset{\sim}{\gamma}}\underset{\sim}{T})\underset{\sim}{v})\underset{\sim}{u} - ((\nabla_{\underset{\sim}{\gamma}}\underset{\sim}{T})\underset{\sim}{u})\underset{\sim}{v}. \tag{15.29}$$

Also, we have the rule

$$\operatorname{div}_{\underset{\sim}{\gamma}}\operatorname{curl}_{\underset{\sim}{\gamma}}\underset{\sim}{T} = \underset{\sim}{0}. \tag{15.30}$$

The inhomogeneity $\underset{\sim}{S}$ relative to a uniform reference $\underset{\sim}{K}$ has the skew symmetry $(\underset{\sim}{S}\underset{\sim}{u})\underset{\sim}{v} = -(\underset{\sim}{S}\underset{\sim}{v})\underset{\sim}{u}$, $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$. Therefore, $\underset{\sim}{S}$ determines and is determined by tensor field $\underset{\sim}{A}$ on $\mathcal{B}$ such that[1]

$$(\underset{\sim}{S}\underset{\sim}{u})\underset{\sim}{v} = \underset{\sim}{A}(\underset{\sim}{u} \times \underset{\sim}{v}) , \quad \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}. \tag{15.31}$$

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[1] The field $\underset{\sim}{A}$ here corresponds to what has been denoted by $\underset{\sim}{A}^T$ in [2], Sect. 34.

If we substitute (15.31) into (11.5) and observe the rule (15.29), we see that

$$\underset{\sim}{A}(\underset{\sim}{u} \times \underset{\sim}{v}) = [\mathrm{curl}_{\underset{\sim}{\gamma}}(\underset{\sim}{F}^{T^{-1}})](\underset{\sim}{F}\underset{\sim}{v} \times \underset{\sim}{F}\underset{\sim}{u}). \tag{15.32}$$

Hence, since $\underset{\sim}{F}^T(\underset{\sim}{F}\underset{\sim}{v} \times \underset{\sim}{F}\underset{\sim}{u}) = (\det \underset{\sim}{F})(\underset{\sim}{v} \times \underset{\sim}{u})$ and since $\underset{\sim}{u} \times \underset{\sim}{v}$ is arbitrary, (15.32) yields

$$-\mathrm{curl}_{\underset{\sim}{\gamma}}(\underset{\sim}{F}^{T^{-1}}) = \frac{1}{J}\underset{\sim}{A}\underset{\sim}{F}^T , \quad J = \det \underset{\sim}{F}. \tag{15.33}$$

Thus, the tensor field $-\mathrm{curl}_{\underset{\sim}{\gamma}}(\underset{\sim}{F}^{T^{-1}})$ is obtained from $\underset{\sim}{A}$ by the rules (15.3), (15.4), except that the absolute value signs are omitted in the definition of J , which does not affect the validity of Proposition 7. Since by rule (15.30) we have

$$[\mathrm{div}_{\underset{\sim}{K}}\, \mathrm{curl}_x(\underset{\sim}{F}^{T^{-1}})]_X = [\mathrm{div}_x\, \mathrm{curl}_x(\underset{\sim}{F}^{T^{-1}})]_X = \underset{\sim}{0},$$

(15.6) yields

$$\mathrm{div}_{\underset{\sim}{K}}\underset{\sim}{A} + \underset{\sim}{A}\underset{\sim}{s} = \underset{\sim}{0}, \tag{15.34}$$

where $\underset{\sim}{s}$ is determined by $\underset{\sim}{A}$ through

$$(\underset{\sim}{A} - \underset{\sim}{A}^T)\underset{\sim}{u} = \underset{\sim}{s} \times \underset{\sim}{u} , \quad \underset{\sim}{u} \in \mathcal{V}, \tag{15.35}$$

Thus, (15.34) is a differential identity for $\underset{\sim}{A}$ and $\underset{\sim}{K}$. One can show that it is equivalent to the Bianchi identity (11.11).

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