ORDINAL INVARIANTS FOR TOPOLOGICAL SPACES

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Report 67-27

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0. INTRODUCTION. Cardinal invariants such as weight, density, dimension, etc. have been widely used in the classification of topological spaces. More rarely (see for example Maurice [10]) ordinal invariants have been employed. In this paper we introduce two related ordinal invariants, σ and κ, first in the categories of sequential and k-spaces (section 1) and later for arbitrary spaces (section 6). Our main result is the existence, for each $\alpha \leq \omega_1$ of a countable, zero-dimensional, Hausdorff space X with $\sigma(X) = \kappa(X) = \alpha$. (Theorems 4.1 and 5.1).

1. PRELIMINARIES. We begin by recalling some definitions. A topological space X is a k-space (see Arhangel'skiĭ [2], and Cohen [4]) if a subset F of X is closed whenever its intersection with each bicompact subset K of X is closed in K. For each subset A of X we will write $x \in A^{\sim}$ if and only if for some bicompact subset K of x, $x \in cl_K(A \cap K)$. Now let $A^0 = A$, and for each non-limit ordinal $\alpha = \beta + 1$, $A^{\alpha} = (A^{\beta})^{\sim}$. If α is a limit ordinal, let $A^{\alpha} = \cup\{A^{\beta} | \beta < \alpha\}$. For an arbitrary space X let us denote by $\kappa(X)$ the infemum of the ordinals α such that for each subset A of X, $A^{\alpha} = cl_X A$. A straight-forward argument, involving only cardinality in one direction and the fact that a single point may be added to a bicompact set without destroying bicompactness, shows that

1.1 PROPOSITION. X *is a k-space if and only if* $\kappa(X)$ *exists*.

Since the definition was given in terms of closure only, one sees immediately that

1.2 PROPOSITION. κ _is a topological invariant in the category of k-spaces._

$\kappa(X) = 0$ if and only if X is discrete, and $\kappa(X) \leq 1$ is just the definition of the k'-spaces (see Arhangel'skiĭ [2]).

We now restrict attention to a special case. A subset U of a topological space X is _sequentially open_ if each sequence which converges to a point in U is eventually in U. X is sequential if each sequentially open subset of X is open (see Franklin [8], [9]). For each subset A of X we will write $A^{\wedge}$ for the set of all limits of sequences in A. Now let $A^{0} = A$, and for each ordinal $\alpha = \beta + 1$, let $A^{\alpha} = (A^{\beta})^{\wedge}$. If α is a limit ordinal, let $A^{\alpha} = \cup\{A^{\beta} | \beta < \alpha\}$. (Whether A^{α} refers to the sequential closure $\wedge$ or the k-closure $\sim$ will always be clear from the context.) Denote by $\sigma(X)$ the infemum of those ordinals α such that $A^{\alpha} = cl_X A$ for all $A \subseteq X$. It is a folk theorem that

1.3 PROPOSITION. X _is sequential if and only if_ $\sigma(X)$ _exists. In this case_ $\sigma(X) \leq \omega_1$ (where ω_1 is the first uncountable ordinal).

For a proof of the second assertion see, for example, Dolcher [5] (22) or Vaidyanathaswamy [12] p. 278. Similarly we have

1.4 PROPOSITION. σ _is a topological invariant in the category of sequential spaces._

Again $\sigma(X) = 0$ if and only if X is discrete, and $\sigma(X) \leq 1$ is just the definition of the Frechét spaces (see [3], [8], [9]).

1.5 PROPOSITION. _Every sequential space is a k-space. Conversely, every countable Hausdorff k-space_ X _is sequential and_ $\sigma(X) = \kappa(X)$.

Proof. The first assertion follows from the bicompactness of a convergent sequence and one of its limit points. For the first part of the second, if a subset A of X is not closed, there is by

[2]10 a bicompact set $K \subseteq X$ and a point $p \in cl(K \cap A) \setminus (K \cap A)$. Since K is first countable, there is a sequence in $K \cap A$ which converges to $p \notin A$. Hence A is not sequentially closed (i.e., its complement is not sequentially open). Thus X is sequential. To complete the proof we need only note that $\kappa(X) \leq \sigma(X)$ is always true and again use the fact that countable bicompact Hausdorff spaces are first countable.

The one-point compactification of $M \setminus \mathbb{N}$, where M is the space of example 5.1 of [9] is a countable, T_1, k-space which is not sequential. (This is in fact the one-point compactification of S_2 without the level one points, as described in section 3.) There is a sequential bicompact Hausdorff space Ψ^* which is not Fréchet ([9],7.1). Hence $\kappa(\Psi^*) = 1 < \sigma(\Psi^*)$ (=2 as it happens). Also there are countable, bicompact, sequential T_1-spaces which are not Fréchet ([9]5.3). Hence the cardinality and seperation hypotheses of 1.5 are actually needed. For future reference we note that

1.6 PROPOSITION. _If_ X _is the disjoint topological sum of a family_ $\{X_\alpha\}$ _of_ k-_spaces_ (_or sequential spaces_), _then_ $\kappa(X) = \sup \kappa(X_\alpha)$ $(\sigma(X) = \sup \sigma(X_\alpha))$.

2. THE SEQUENTIAL SUM. Let $S = \{0\} \cup \{1/n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ have the relative topology, i.e., S is a convergent sequence with its limit point. For each $0 < i < \omega_0$ let $\langle X_i, 0_i \rangle$ be T_1-spaces with base points. We define their sequential sum $\Sigma\langle X_i, 0_i \rangle$ as follows. Let X be the disjoint topological sum of the X_i and $A = \{0_i \mid i < \omega_0\}$. Then A is a closed subspace of X and the function $f: A \to S$ defined by $f(0_i) = 1/i$ is continuous. Let $\Sigma\langle X_i, 0_i \rangle$ be the adjunction space $X \cup_f S$. The pertinent facts about the sequential sum are as follows.

2.1 PROPOSITION. *If each* X_i *is a* k-*space* (*or a sequential space*), *then so is* $\Sigma \langle X_i, O_i \rangle$ *for any choice of* O_i. *If for each* X_i, $\kappa_i = \kappa(X_i)$ ($\sigma_i = \sigma(X_i)$) *is a non-limit ordinal, there are* O_i *such that* $\kappa(X) = (\sup \kappa_i) + 1$ ($\sigma(X) = (\sup \sigma_i) + 1$).

Proof. The first assertion follows in the sequential case from [8] Propositions 1.2 and 1.6, and the k-space case is proved similarly. Since $X \setminus \{O\}$ is the disjoint topological sum of the X_i, $\kappa(X \setminus \{O\}) = \sup \kappa_i$ by Proposition 1.6. From the fact that $\{O\} \cup \{O_i\}$ is bicompact, it follows immediately that $\kappa(X) \leq (\sup \kappa_i) + 1$. We shall construct a subset M of $X \setminus \{O\}$; such that $O \in M^{(\sup\ {}_i) + 1} \setminus M^{\sup\ {}_i}$. Choose a subsequence $\{\kappa_j\}$ of the $\{\kappa_i\}$ which converges in the order topology monotonicly upwards to $\sup \kappa_i$. For each j, let $\theta_j + 1 = \kappa_j$. Then there exists $O_j \in X$, and $M_j \subseteq X_j$ so that $O_j \in (M_j)^{\kappa_j} \setminus (M_j)^{\theta_j}$. (the remaining O_i may be chosen arbitrarily.) Let $M = \cup M_j$. Let β be the least ordinal such that $O \in M^{\beta+1} \setminus M^{\beta}$. Then for some bicompact $B, O \in cl_X(B \cap M^{\beta})$. Letting $K = \{O_j\}$, this implies that $K \cap B \cap M^{\beta} = \cup(K \cap B \cap (M_i)^{\beta})$ is infinite. But $K \cap (M_j)^{\beta} \neq \emptyset$ only if $\beta \geq \kappa_j$. Hence $\beta \geq \sup \kappa_i$ and thus $O \notin M^{\sup \kappa_i}$. Hence $\kappa(X) = (\sup \kappa_i) + 1$.

An even simpler proof may be given in the sequential case.

In addition, one may easily verify that

2.2 PROPOSITION. *If each* X_i *is zero-dimensional so is their sequential sum*.

3. CONSTRUCTION OF THE S_n. In this section we shall construct (in two distinct ways) for each $n < \omega_o$, a countable space S_n (enjoying all 'nice' topological properties except local bicompactness) with $\kappa(S_n) = \sigma(S_n) = n$, which is minimal in a sense to be made explicit in Proposition 3.1.

Let $S_o = \{0\}$ and, having already defined S_{n-1} with base point 0, let S_n be the sequential sum of countably many copies of $\langle S_{n-1}, 0\rangle$, choosing 0 again as base point. Thus S_n is defined recursively for each $n < \omega_o$. Clearly $S_1 = S$ and S_2 is the space of Arens (see [1] and [9] Example 5.1.)

We now define the level $l_n(x)$ for points $x \in S_n$. For $n = 0$, let $l_o(0) = 0$. Having defined the level of each point in S_{n-1}, choose $x \in S_n$. If $x = 0$, let $l_n(x) = 0$. If not, $x \in S_{n-1}$ and we let $l_n(x) = l_{n-1}(x) + 1$.

Now for each level n point x of S_n take a copy S_x of S and let X be their disjoint topological sum. Let $A = \{0_x \in S_x \mid l_n(x) = n\}$ and define $f: A \longrightarrow S_n$ by $f(0_x) = x$. Then the adjunction space $X \cup_f S_n$ is homeomorphic to S_{n+1} and we have the second construction.

Suppose we have defined for each $k < n$, a partial order $\leq_k$ on S_k with 0 as maximal element. Then let $\leq_n$ be the partial order on S_n generated by $\leq_{n-1} \cup \{(y,x) \mid y \in S_x\}$. These orders will be used in section 5.

Also for later use, note that this second construction yields a natural embedding $\varphi_n : S_n \longrightarrow S_{n+1}$.

The properties claimed for the S_n in the opening paragraph of this section follow immediately from Propositions 2.1 and 2.2, and from

3.1 PROPOSITION. <u>If a Hausdorff sequential space</u> X <u>contains a copy of</u> S_n, <u>then</u> $\sigma(X) \geq n$. <u>Conversely, if</u> $\sigma(X) \geq n$, X <u>contains a subspace whose sequential closure is homeomorphic to</u> S_n.

Proof. Let L_n be the level n points of $S_n \subseteq X$. If $\sigma(X) = k$ for $k < n$, there are countably many points $x_j \in L_n^{\wedge} \setminus (L_n \cup L_{n-1})$ with

0 (the zero level point of S_n) in $\{x_j | j \in N\}^{k-1}$. Let A_j be the range of a sequence in L_n converging to x_j, and for each $y_i \in L_{n-1}$, let B_i be the L_n points under y_i. Then by Hausdorffness $A_j \cap B_i$ is finite for all $i,j \in N$. Hence there are disjoint sets A and B such that $A \setminus A_j$ and $B \setminus B_i$ are finite for all $i,j \in N$. Then $B \cup S_n \setminus L_n$ is an open set in S_n containing 0. Hence there is an open set U in X such that $U \cap S_n = B \cup S_n \setminus L_n$. Hence $A \cap U = \emptyset$. Thus for each j, $x_j \notin U$. This contradicts $0 \in d_X\{x_j | j \in N\}$.

The second assertion is obvious for $n = 0$ and $n = 1$ and for $n = 2$ it follows from Proposition 7.3 of [9]. As is frequently the case, in order to complete the induction, it is easier to prove something a little stronger: <u>if A is a subset of a Hausdorff sequential space X and if $x \in A^n \setminus A^{n-1}$, there is a subset S_n' of A^n whose sequential closure is homeomorphic to S_n, and whose level k points lie in $A^{n-k} \setminus A^{n-k-1}$</u>. Stated in this form, the inductive proof is trivial when it is noted that a sequentially bi-continuous bijection is a homeomorphism from the sequential closure of its domain to that of its range. The second assertion of the proposition is now immediate.

4. CONSTRUCTION OF THE K_α. In this section we shall construct for each $\alpha < \omega_1$ a countable space K_α (again with 'nice' properties) such that $\kappa(K_\alpha) = \sigma(K_\alpha) = \alpha$.

Let $K_0 = S_0 = \{0\}$ and suppose K_β is defined for each $\beta < \alpha$. If α is a limit ordinal, let K_α be the disjoint topological sum of the K_β with $\beta < \alpha$. By Proposition 1.6, $\kappa(K_\alpha) = \sigma(K_\alpha) = \alpha$. If $\alpha = \beta + 1$, choose a sequence of non-limit ordinals $\{\beta_i\}$ with supremum β. By Proposition 2.1, we may choose $0_i \in K_{\beta i}$ so that $\kappa(K_\alpha) = (\sup \beta_i) + 1 = \alpha$, where K_α is the sequential sum of $K_{\beta i}$.

By Proposition 1.5, $\sigma(K_\alpha) = \alpha$ also.

Recapitulating we have

4.1 THEOREM. For each ordinal $\alpha < \omega_1$, there is a countable, zero-dimensional Hausdorff space K_α such that $\kappa(K_\alpha) = \sigma(K_\alpha) = \alpha$.

Note that we may also define the space K_{ω_1} as the disjoint topological sum of the K_α for $\alpha < \omega_1$. Then K_{ω_1} is a zero-dimensional Hausdorff space of cardinality and local weight $\aleph_1$, with $\kappa(K_{\omega_1}) = \sigma(K_{\omega_1}) = \omega_1$. In the next section we shall construct another such space which is countable and homegeneous.

5. CONSTRUCTION OF S_ω. Using the maps $\varphi_n : S_n \to S_{n+1}$ defined in section 3, we define for each pair $m < n < \omega_o$, a map $\varphi_m{}^n : S_m \to S_n$ by $\varphi_m{}^n = \varphi_{n-1}{}^n \circ \ldots \circ \varphi_m{}^{m+1}$, creating an inductive system $\langle S_n, \varphi_m{}^n \rangle$ of spaces and maps. Denote by S_ω the inductive limit of this system.

5.1 THEOREM. S_ω is a countable, sequential, zero-dimensional, homogeneous, Hausdorff space with $\kappa(S_\omega) = \sigma(S_\omega) = \omega_1$, which contains a copy of K_α for each $\alpha < \omega_1$.

Proof. S_ω is clearly countable and is sequential by [8] Corollary 1.7. Hence by Propositions 1.3 and 1.5 $\kappa(S_\omega) = \sigma(S_\omega) \leq \omega_1$ (S_ω is clearly T_1 and is therefore Hausdorff since it will be shown to be zero-dimensional.). The opposite inequality will result from $K_\alpha \subseteq S_\omega$ for each $\alpha < \omega_1$.

Denoting by $\Psi_n : S_n \to S_\omega$ the canonical map we define a partial order on S_ω by $x \leq y$ if only if for some n, a,b we have $a \in \Psi_n^{-1}(x)$, $b \in \Psi_n^{-1}(y)$ and $a \leq_n b$ (see section 3.)

Noting that $l_n(x) = k$ implies that $l_{n+1}(\varphi_n(x)) = k$, one may unambiguously define the level $l(x)$ of a point S_ω by choosing some n and a with $a \in \Psi_n^{-1}(x)$ and setting $l(x) = l_n(a)$. It is easy to verify that $x \leq y$ implies $l(x) \geq l(y)$.

For each $x \in S_\omega$, let $I(x) = \{y \in S_\omega | y \leq x\}$, ie., $I(x)$ is the principal ideal generated by x. We shall show by an induction on the level of x that <u>each</u> $I(x)$ <u>is homeomorphic to</u> S_ω. For $l(x) = 0$ the assertion is trivial. Suppose $l(x) = 1$ and let $T_n = \Psi_n^{-1}(I(x))$ for each $n < \omega_0$. $T_0 = \emptyset$, and for $n > 0$, T_n is homeomorphic to S_{n-1}. But clearly $I(x)$ is the inductive limit of the system $\langle T_n, \varphi_n^m | T_n \rangle$ and hence is homeomorphic to S_ω. Now suppose our assertion is true for points at level $n-1$ and that $l(x) = n$. Then there is exactly one $y \in S_\omega$ with $l(y) = n-1$ and $x < y$. Then x is a level one point with respect to $I(y)$ which is homeomorphic to S_ω by the inductive assumption, and hence $I(x) \cong S_\omega$ by the level one argument.

Denote the level one points of S_ω by 0_i. Then S_ω is the sequential sum of the family $\langle I(0_i), 0_i \rangle$ and so S_ω <u>is the sequential sum of countably many copies of itself</u> with the level zero point of each as base point.

It is easily verified that a sequence $\{x_n\} \subseteq S_\omega$ of distinct points converges to $x_0 \in S_\omega$ if and only if eventually $l(x_n) = l(x_0) + 1$ and eventually $x_n \leq x_0$. We will write $x \sim y$ if $l(x) = l(y)$ and for some z, $l(z) = l(x) - 1$, $x \leq z$, and $y \leq z$. Hence $x_n \longrightarrow x_0$ implies that eventually $x_n \sim x_m$ or $x_n = x_0$. In fact, in order that a sequence of distinct points in S_ω converge, it is necessary and sufficient that it be eventually composed of points pairwise related by $\sim$. Using this characterization of sequential convergence and the fact that S_ω is sequential, one sees that not only is each $I(x)$ clopen but given any family $\{x_i\}$, no infinite subfamily of which is related by $\sim$, $\cup I(x_i)$ is clopen. It then follows immediately that S_ω is zero-dimensional.

Let $x,y \in S_\omega$ be distinct points. If x and y are not comparable, then $I(x)$ and $I(y)$ are homeomorphic disjoint clopen neighborhoods of x and y respectively. If $x \leq y$, then $I(x)$ and $I(y) \setminus I(x)$ are such neighborhoods and so S_ω is homeogeneous.

We will now recursively imbed each K_α in S_ω. Suppose this has been accomplished for each $\beta < \alpha$, so that the base point O_β of K_β is the level zero point of S_ω whenever β is not a limit ordinal. For each such β, let L_β be a copy of S_ω with K_β so embedded. If α is a limit ordinal K_α is the disjoint topological sum of the K_β and is homeomorphic to a subset of any sequential sum of the L_β. If $\alpha = \beta + 1$, K_α is the sequential sum of some sequence $K_{\beta i}$. Then K_α is embedded in the sequential sum of the corresponding $\langle L_{\beta i}, O_{\beta i} \rangle$ which is again S_ω.

Since for each non-limit ordinal α, K_α is homeomorphic to a closed subspace of S_ω, $\sigma(S_\omega) = \omega_1$ and the proof is complete.

Dudley has shown ([6] Theorem 7.8) that the sequential closure (i.e., the smallest sequential topology containing the given one) of the weak topology of a separable, infinite dimensional Banach space is the 'bounded topology' (see [7] 425-30). We shall apply 5.1 to show that $\sigma(\ell_2) = \omega_1$ if ℓ_2 is provided with its bounded topology. The authors are indebted to C. V. Coffman for a key idea in the proof.

5.2 THEOREM. S_ω can be embedded as a sequentially closed subset of ℓ_2 taken with its bounded topology. Hence, $\sigma(\ell_2) = \omega_1$.

Proof. Using the second description of S_n in section 3, we will embed recursively each S_n into ℓ_2, via $\Theta_n : S_n \to \ell_2$, in such a way that for each $m < n$, $\Theta_m = \Theta_n \varphi_m^n$. Since S_ω is the inductive limit of the S_n, this will map S_ω into ℓ_2.

We first represent each S_n as a collection of finite sequences of natural numbers as follows. Represent the single point of S_o by the empty sequence. Let $S_1 = S_o \cup \{(i) \mid i \in \mathbf{N}\}$. Supposing that S_n has been defined, and for $x = (i_1, i_2, \ldots, i_n) \in S_n$ and of level n, let $S_x = \{x\} \cup \{(i_1, \ldots, i_n, j) \mid j > i_n\}$. Now consturct S_{n+1} as in section 3. S_ω can be thought of as the union of the S_n in this representation, i.e., the collection of all finite strictly increasing sequences of natural numbers.

Convergence of sequences in S_ω (or in any S_n) can be easily described in terms of this representation: essentially a sequence converges if and only if it is eventually of the same level (i.e., length), say n, and eventually constant in each of the first $n-1$ co-ordinates, and further is either unbounded in it's eventual last co-ordinate or eventually constant there. In the first case the limit point is represented by the sequence of the first $n-1$ eventual values, and in the second case the sequence is eventually constant. We shall now embed each S_n as a sequentially closed subset of ℓ_2 via Θ_n in such a way that sequential convergnece in $\Theta_n(S_n)$ has this same description. Hence each Θ_n will be a homeomorphism (see for example Moore [11] Theorem 6.13), and so will their limit Θ.

Let $\{b^i\}$ be the standard orthonormal basis for ℓ_2 defined by $b_k^i = 0$ if $i \neq k$ and $b_i^i = 1$. Define Θ_o by $\Theta_o(0) = 0$, the origin in ℓ_2. Define Θ_1 by $\Theta_1(i) = b^i$ and $\Theta_1(\emptyset) = \Theta_o(\emptyset) = 0$. Having defined Θ_n as an extention of Θ_{n-1}, let $\Theta_{n+1} = \Theta_n$ on S_n and for $(i_1, \ldots, i_n, i_{n+1}) \in S_{n+1} \setminus S_n$, let $\Theta_{n+1}(i_1, \ldots, i_n, i_{n+1}) = \Theta_n(i_1, \ldots, i_n) + i_n b^{i_{n+1}}$. We must show that each Θ_n is a sequential homeomorphism onto $\Theta_n(S_n)$, and that $\Theta_n(S_n)$ is sequentially closed in ℓ_2.

It is clear from the fact that $\Theta_n(i_1.,,,.i_n)$ is non-zero only in the $i_1,i_2,\ldots,i_n$th places, that each Θ_n is one-to-one.

That Θ_1 is a sequential homeomorphism follows from the well known fact that the sequence $\{b^i\}$ converges weakly to zero. Clearly $\Theta_1(S_1)$ is sequentially closed in ℓ^2. If we suppose that Θ_n has been shown to be a sequential homeomorphism and that $\{x_k\}$ is a convergent sequence in S_{n+1}, then we may assume all x_k to have the same level. If $l_{n+1}(x_k) \leq n$, $\Theta_{n+1}(x_k) = \Theta_n(x_k)$ and the convergence is preserved. If $l_{n+1}(x_k) = n+1$, we may assume that $x_k = (i_1,\ldots,i_n,j_k)$ with the j_k unbounded. (Otherwise $\{x_k\}$ is an eventually constant sequence and convergence is preserved.) Then $\lim x_k = (i_1.,,,.i_n)$ and $\Theta_{n+1}(x_k) = \Theta_n(i_1,\ldots,i_n) + i_n b^{j_k}$ which converges weakly to $\Theta_n(i_1,\ldots,i_n) = \Theta_{n+1}(i_1,\ldots,i_n)$. Thus Θ_{n+1} is sequentially continuous.

Conversely, suppose $\{x^k\}$ is any sequence in $\Theta(S_\omega)$ which converges weakly to x^o in ℓ_2. Since $\{x^k\}$ must be bounded in norm there is a uniform bound, say q, on the number of non-zero co-ordinates of the x^k. Thus $\{x^k\} \subseteq \Theta_q(S_q)$. Hence <u>if each</u> $\Theta_n(S_n)$ <u>is sequentially closed, so is</u> $\Theta(S_\omega)$. Since weak convergence implies pointwise convergence, and since each co-ordinate of each x^k is an integer, the sequence $\{x^k\}$ must be eventually constant in each coordinate. Let r be the number of eventually non-zero coordinates and suppose the theorem proved for $r < n$. If $r = n$ we may assume that $x^k = \Theta_n(i_1^k,\ldots,i_n^k)$ where eventually $i_1^k = i_k,\ldots,i_{n-1}^k = i_{n-1}$. Then $x^o = \Theta_n(i_1.,,,i_{n-1}) \in \Theta_n(S_n)$ which is therefore sequentially closed. Clearly $\{i_n^k\}$ is unbounded and so $(i_1^k,\ldots,i_n^k)$ converges in S_n to $(i_1,\ldots,i_{n-1})$. This completes the proof.

Note that 0 is in the weak closure of $\Theta_2(S_2)\setminus\Theta_1(S_1)$ but is not the weak limit of any sequence therein. This is the well known example of von Neumann.

6. SOME REMARKS AND QUESTIONS. As suggested by Theorem 5.2, the functions σ and κ can be extended to the category of all topological spaces and continuous maps by means of the co-reflective functors s and k which assign to each space X, the spaces sX and kX where the underlying set is the same and the new topologies are the smallest sequential and k-space topologies containing the original. We may then define $\sigma(X) = \sigma(sX)$ and $\kappa(X) = \kappa(kX)$. Propositions 1.5 and 1.6 extend immediately. What else can be said?

Proposition 3.1 establishes the S_n as 'test spaces' for spaces X with $\sigma(X) = n$. It would seem that permitting all possible choices of the β_i (see the second paragraph of Section 4) the K_α could be used as 'test spaces' for $\sigma(X) = \alpha$. Are there 'test spaces' for κ?

The disjoint topological sum K of the K_α for $\alpha < \omega_1$ satisfies $\sigma(K) = \omega_1$ but contains no copy of S_ω. Can this happen with a countable space, or with a homogeneous space?

Is there for each $\alpha > \omega_1$ a space K_α with $\kappa(K_\alpha) = \alpha$. More particularly, if α is an ordinal corresponding to a cardinal $\tau(\alpha) > \aleph_1$, is there a k-space K_α with $\kappa(K_\alpha) = \alpha$ and $\bar{\bar{K}}_\alpha \leq \tau(\alpha)$ $(K_\alpha \leq 2^{\tau(\alpha)})$? Is there for each $\alpha < \beta \leq \omega_1$ a space $X_{\alpha,\beta}$ with $\kappa(X_{\alpha,\beta}) = \alpha$, $\sigma(X_{\alpha,\beta}) = \beta$?

What are the permanent properties of space with $\sigma(X) = \alpha$ or $\kappa(X) = \alpha$?

S_ω is something of a topological curiosity in itself. Are there other countable Hausdorff k-spaces with no point of first countability? If so are there other such which are homogeneous and sequential?

It is easily seen that the proof of Theorem 5.2 depends only on the existence of a sequence bounded away from 0 which converges weakly to 0. For what linear topological spaces do such exist?

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