EXTENSIONS OF TOTALLY
BOUNDED PSEUDOMETRICS

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1. <u>Introduction</u>. The concept of extending to the topological space X a continuous real-valued function defined on a subspace S of X has been shown to be very useful. When every continuous real-valued function on S extends to X, S is said to be C-<u>embedded</u> in X. If bounded continuous real-valued functions are considered, then S is said to be C*-<u>embedded</u> in X. Thus every completely regular Hausdorff (Tychonoff) space is C-embedded in the real compactification uX and is C*-embedded in the Stone-Cech compactification f3X. Results in this line have been extensively studied in [5].

A similar problem is concerned with extending a pseudometric defined on a subspace S to the space X. This problem was first studied by .P. Hausdorff for the metric case in 1930 [6]. Hausdorff showed that a continuous metric defined on a closed subset S of a metric space X can be extended to a continuous metric on X. R* H, Bing [2] and R. Arens [1] rediscovered this result independently. Recently H. L. Shapiro [10] studied the problem by considering subspaces S for which every continuous pseudometric defined *on* S has a continuous pseudometric extension to X. Such subspaces S are said to be P-<u>embedded</u> in X. Shapiro showed that if S is P-embedded in X, then S is C-embedded in X, but not conversely. However^ S is P-embedded in X if X is completely regular and S is a pseudocompact C*-embedded subset of X.

In this paper we introduce the concept of T-embedding , a particular case of P-embedding. A subset S is T-<u>embedded</u> in X in case

every totally bounded continuous pseudometric on S has a totally

bounded continuous pseudometric extension to X* In §2 we will

characterize T-embedding in terms of various types of open covers

of the space X.   In addition we shall show that in order for a

subspace S to be T-embedded in X it is sufficient to extend every

totally bounded continuous pseudometric on S to a continuous

pseudometric on X.   In §3* we show that a subset is T-embedded

if and only if it is C -embedded.   Finally, using the concept of

T-embedding<, we give some new characterizations for a topological

space to be normal.

The notation and terminology in this paper will follow that 1

of Gillman and Jerison in [5]» Other terms used here are defined

below.

DEFINITIONS.   If $(\bar{u}) = (U)$ $_T$ is a family of subsets of a

topological space X, arid if S is a subset of X, then by U JS we mean

the family (U n S) $_T$.   We say that $(\bar{u})$ is discrete at a point

x e X if there is a neighborhood G of X that meets at most one

member of $(\bar{u})$.   We say that X is a collectionwise normal space if

for every discrete family $(\bar{u})$ = $(U_.)$ $_T$ of closed subsets of X •

there is a family $(G_,)_,$ $_c$ $_T$ of mutually disjoint open subsets of X

such that y_a c G_Q for each a e I.   (This definition is due to R. H.

Bing [2].)

If ® is a cover of X and if (v) = $(V_{ft})_A$ $_r$ $_T$ is a cover of S,

then @ is an ~~extension~~ of © if I = J and if U n S = V for all

a e I.   A sequence $((\bar{u})_n)_{n \in N}$ of covers of a set X is said to be a

~~normal sequence~~ in case $(v)_{n+1}$ is a star refinement of $(u)_n$.   A

cover (u) of X is said to be a ~~normal cover~~ in case there exists a

normal sequence $(\widehat{CD})$ $\ast_T$ of open covers of X such that $@_n$ is a

refinement of $(\hat{u})$. (This definition is due to Tukey [11].) If

$(\widehat{\mathcal{U}}_n)_{n \ \epsilon \ N}$ is a normal sequence of open covers of a space X and

if d is a pseudometrie on X, then d is associated with $(\mathcal{E})_{,,}$,, M

if the following three conditions are satisfied:

(1) d is bounded by the identity function $\mathbb{1}$.

(2) If k e N and if $d(x,y) < 2 \sim (^{k+1})$, then x e st(y, $@_v$) (the

star of y with respect to $(\tilde{U})_{,}^{K}$.

(3) If k e N and if x e $_s$t(y, $@_v$) , then $d(x,y) < 2"^{k"3^}$.

For any real-valued continuous function defined on $X^$ the set,

Z(f), of all points x in X for which f(x) $^$ 0 is the zero set of f.

The complement of Z(f) is called the cozero-set of f. The family

(jj) = $(U^)^$ $_T$ is a cozero-set cover of X if (u\ is a cover of X

and if each U_Q is a cozero-set• A zero-set cover is defined in an

analogous manner.

If $(@ - IJ \ \bullet\bullet\bullet\ast \ @^)$ is a finite sequence of covers of a set X

and if $(\&_\pm = (A_\pm(a))_{a \ g} j$ for each i .= 1, ..., n, then by $AJ_{=1}(K\mathcal{T}_\pm$

we mean the family

$$(A^{^^})_n \ \bullet\bullet\bullet \ \mathbf{n} \ A_n(^{\alpha}_n))_{(\alpha_1, \ ..., \ \alpha_n)} \ \epsilon \ J_1 \ x \ ... \ x \ J_n \cdot$$

2. Equivalent formulations of T-embedding> We will now

characterize T-embedding in terms of finite cozero-set covers, finite

normal cozero-set covers, and finite normal open covers. To do this

we will need some preliminary results that are interesting in

•themselves. These first results are worthy of note since they state

the relationship between the topological structure induced by a

pseudometric and the given topology on the space.

PROPOSITION 2.1 If. (X, ©) is a topological space and if d is;

a, pseudometric on X then d ^is continuous if and only ±f @ $_d$C (?).,,
where (FY ^is the topology induced by d.

PROPOSITION 2o2 Let (X, ($\tilde{T}$}) be a, topological space and let d
be a, continuous pseudometric on X. . If G i^s an £P£H subset of X
relative to ©$_H$* then G is a, cozero-set relative to ©.

THEOREM 2.3 [11, Theorem 7.1] If. (©)   ™ lJL §L normal
                                      ^•^ n n G IN
_____ __ ____ _____ __ _, _____ _____ ___ ____ __ _____ _____
sequence of open covers of a, topological space X, then there exists
a continuous, pseudometric on X that is associated with ((u)^) $_{M.}$
                                                                    $_{rv}$

Since we will be considering totally bounded continuous pseudo-
metrics we will now make the necessary modifications of previously
known results. These will then be applied to obtain our desired result.

PROPOSITION 2.4 Let X be a topological space and let ((g)) $_M$
be a, normal sequence of finite open covers of X. If d j^s a continuous
pseudometric on X that i£ associated with (© $_n$)$_{n}$ $_ε$N, then d j^is
totally bounded.
We omit the proof of 2A.

PROPOSITION 2.5 Let X b£ a topological space, let S ε x , and
let (tp = (U$_a$)$_a$ $_ε$ j be an open cover of S. If; © i^ a normal open
cover of X such that ^ | S refines (§), then there j^ a. normal locally
finite cozero-set cover (^) = (W$_~$)$_~$ $_τ$ of X such that
W$_a$ n S c u$_a$ for each a $_ε$ i.
The reader is referred to [10, Theorem 2.5] for the proof.

LEMMA 2.6 <u>rf</u> (u^) = (U_a)$_{a g}$ j <u>is a finite normal open cover of the</u> <u>topological space</u> X, <u>then there i^ a normal sequence</u> ($\{v_n\}$)$_{n \epsilon N}$ <u>jof open covers ff</u> X <u>such that</u> Q^.$_1$ <u>refines</u> (f^ <u>and such that</u> $(v)_n$ <u>is finite for each</u> n e N.

Proof. By hypothesis, there exists a normal sequence (/Ul^)^ _ M$_{(V}$ of open covers of X such that (fj)$_1$ refines (\J) . Let d be a continuous pseudometric that is associated with ((u)^)^ - M (2.3).

For each a $_f$ I, let W_a = U$\{S_d(x, 2\sim^3)$: $S_d(x, 2\sim^3)$ c U_Q]. Then . (\P) = (W_a)$_a^\epsilon$ -j is a finite normal open cover of X relative to (?)$_d$ (the topology on X determined by d) such that A_a c U_Q for each a e I. A repeated application of [8, Theorem 1.2], and the observation that the covers constructed therein are finite, give us a normal sequence (VY)$^{n \epsilon}$ S$_N$ of open covers of X, relative to $\times$/d such that (V)$_1$ refines ^^ and such that, for each n e N, f$\{f\}_n$ is finite. Since (T^$_d$ c C^ and since nf) refines (l^), the result now follows.

THEOREM 2.7 <u>If</u> S <u>is a subspace of a topological space</u> X, <u>then</u> <u>the following statements are equivalent</u>:

(1) S <u>if</u> T-<u>embedded in</u> X.

(2) <u>Every totally bounded continuous pseudometric on</u> S <u>can be</u> <u>extended to</u> ^a <u>continuous pseudometric on</u> X.

(3) <u>Every finite normal cozero-set cover of</u> S <u>has a refinement</u> <u>that can be extended to a normal open cover of</u> X.

(4) <u>Every finite normal open cover of</u> S <u>has a refinement that</u> <u>can be extended to a finite normal cozero-set cover of</u> X.

Proof. (1) <u>implies</u> (2). This implication is immediate.

(2) <u>implies</u> (3)* Assume (2) and suppose that (u) is a finite normal cozero-set cover of S. By 2.6, there exists a normal sequence (V_i)$_{i e N}$ of open covers of S such that (v)-$_i$ refines ^j) and such

that, for each i t $\underset{\sim}{N}$, $(\tilde{v})_i$ is finite. Then,, by 2.3* there exists a

continuous pseudometric d on S that is associated with ($(\overset{\sim}{V^*}).)$. $\overset{\wedge}{\underset{\sim}{N}}$

and,, by 2.4, d is totally bounded. Therefore, by (2)$\diamond$ there is a

continuous pseudometric $\overline{d}$ on X such that $\overline{d}|S \times S = d$. Let

$\overset{\wedge}{\underset{a}{R}}$ $\overset{-h}{}$ $\underset{x}{\in}$ $-$

$\gg = (S-r(x, 2^{}))_x \ll$ Since $(X_5 d)$ is a pseudometric space,,

it is paracompact, so there is a locally finite open cover ($\wedge$) of

X such that $(vh$ refines $(w)^f$ • By 2.1, 2.2, and the fact that a

locally finite cozero-set cover is normal, it follows that $\wedge y$ is a

normal open cover of X relative to the given topology on X and one

easily verifies that (5f)lS refines H$\wedge \ll$

(3) implies (4)$_\circ$ This implication follows from [9, Theorem 1.2]

and 2.5. _____

(4) implies (1). Assume (4). Let ($\wedge$) be the given topology

on X and suppose that d is a totally bounded continuous pseudometric

on S. For each m e $\underset{\sim}{N}$ there exists a finite subset F_m of S such that

$$\overset{\frown}{W}{}^m = (S_d(x, 2^{-(m+3)}))_{x \in F_m}$$

covers So

Now consider any m c $\underset{\sim}{N^*}$ Note that $\overset{\frown}{Q?})^m$ is normal relative to

$\overset{\frown}{T}_d$ and therefore, by (4), there exists a refinement of $\overset{\sim}{fw})^m$

that extends to a finite normal cozero-set cover $\overset{\sim}{£v})^m$ of X, Then

there exists a normal sequence $(\overset{\frown}{Cy}\backslash).^m)._{\_N}$ of open covers of X

such that $(\overset{\frown}{v^{\wedge}}$-$L^{m1}$ refines @ ^{m} and such that @ $._i^m$ is finite for

each i e N.

Now for all i, m e $\underset{\sim}{N}$, let

$$\overset{\frown}{U}{}^m = \wedge^m_{j=1} \overset{\frown}{V}{}^j$$

and

$$\underset{1}{\bigcup} \quad \underset{J=1}{} \quad \underset{1}{\bigcap} .$$

Then for all i, m e $\mathbb{N}^*$ one easily verifies that

(i) ©^{m} and ©.^m_i are finite open covers of X,

(ii) $\{u\}$-$_{+1}^m$ <* ®i_^{m} and and ©i ™ refines $(w)^{"1}$,

(iii) $(O)^{\wedge 1}$ refines $ST)_{\pm}^{...}$ and $(\backslash p^{\wedge\wedge}$ refines $(j/^1$, and

(iv) fiT^m IS refines $\overline{QQ}^n$.

Now again consider any m e $\mathbb{N}$. It follows from (i) and (ii)
that $(\wedge\wedge_1^m)_{1\ e\ N}$ is a normal sequence of finite open covers of X.
Then, by 2.3, there exists a continuous pseudometric r$^\wedge$ on X that
is associated with $(uD.^m)o\ _M$ and, by 2.4, r is totally bounded.

By (ii) and (iv), we also have

(*) <u>If</u> x, y e S <u>and if</u> r$_{mr}$ (x,y) < 2~^{3}, <u>then</u> d(x,y) < 2$^{-(m+2)}$.

Define r: X x X - $\mathbb{R}$ by r(x,y) = $^s_{m\ \in\ N}$ 2 $^{\wedge-m}$.$^\wedge$ (x$^\wedge$y). Then r is a
continuous pseudometric on X. Moreover, r is totally bounded, for
if c > 0, let k e N such that $2'^{\wedge'\wedge}$ < e. Since r$_{,r}$ is totally
bounded there exists a finite subset F of X such that

$U_{x\ \epsilon\ F}S^\wedge_{\dot{}k}$ (x, 2~$^{(k+4)}$) = X. If z e X then z e S$^\wedge$ (x, $2'^{\wedge\wedge\wedge}$) for
some x c F and it follows that z c st(x, $^{\wedge}Jwo^k$). Furthermore,
if 1 $\leq^m <^{k*}$ then by $(i:Li)$ $^{we\ havs}$ $(^\wedge)i_{+}Q^k$ refines $^\wedge)_{k+}3^{m\ and}$ hence
. z e st(x, $(uX,_Q^m)o$ Hence$-\frac{}{in}$ (x,y) < 2$^{"k}$ < e/2, whenever 1 < m < k.
Then r(x,z) = $E^k_{m=1}$ 2$^{\wedge}$-r$_m$(x,z) + $_{m=k}^{s}V^\wedge$ 2$^{"""m}$.r$_m$(x,z) < $\Sigma^k_{m=1}$ 2^{-m}.(e/2) $\div$
$\Sigma^\infty_{m=1}$ 2^{-m} < ϵ.

From (^) it follows that

(**) <u>If</u> x, y e S, <u>if</u> i > 3, <u>and if</u> r(x,y) < 2$^{"1}$, <u>then</u> d(x,y) < 2$^{"""\wedge1"1)}$.

Define a relation R on X as follows:

$$x\ R\ y\ \text{in case}\ r(x,y) = 0 \qquad (x,\ y\ c\ X).$$

Observe that R is an equivalence relation on X. Let X* = X/R be
the quotient space of X modulo R and let A: X - X* be the canonical
map. Then the formula r*(A(x), $?\backslash$(y)) = r(x,y) (x, y e X)

determines a well defined map r*: X* X X* - $\underset{\sim}{R}$.   One easily verifies

that (X*, r*) is a metric space., that $(\mathfrak{f}\}_r$# is the quotient

topology on X*, and that the canonical map V. X -* X* is an isometry.

It follows that r* is totally bounded.

Let S* = A(S).   By (**) it follows that we can define a map

d*: S* X S* - $\underset{\sim}{R}$ as follows

$$d*(A(a), *(b)) = d(a, b) \qquad (a, b \ e \ S).$$

Then one easily verifies that d* is a totally bounded pseudometric

on S*.

Let $(\bar{u})$* be the uniformity on X* generated by the metric

r*|S* X S*.   Let @ * = (u)* | S* x S* and note that (S*, ® *) is a

uniform subspace of (X*, (tf)*) .   Using (**) and the fact that

A: X -* *X* is an isometry one easily shows that d* is a uniformly

continuous pseudometric on S*.   Therefore, by [7* page 42* Corollary

16], there exists a uniformly continuous bounded pseudometric e on X*

such that e|S* X S* = d.   To show that e is totally bounded, let

e > 0 and let B* = [(x*, y*) e X* x X^ : $e(x*, y*)$ < e}.   Since e

is uniformly continuous, B^ e $^{\wedge}\bar{u}$)* and therefore there exists 5 > 0

such that £(x*, y*) e X^ x X^ : r*(x*, y^) < 5} c B*.   NOW r* is

totally bounded, so there is a finite subset F* of X* such that

^{U}x* e F* ^{S}r*(^{X*} ' $^{6\wedge}$ $^{\wedge X\wedge*}$ ' $^{One \ easil}$y shows that

U o   TH^. $\underset{\sim}{S}$ (x^-, e) = X* and it follows that e is totally bounded.

Define "d on X x X by "d = eo.(A x ~h) .   Since A is continuous

relative to $C\bar{I}$), $\bar{d}$ is a continuous pseudometric on X (2.1).   Moreover,

if x, y e S, then $\bar{d}$(x, y) = e(?v(x), A(Y)) - d*(A(x), A(y)) - d(x, y) .

Therefore $\bar{cf}$ | S x s = d.   Since A is an isometry and since e is

totally bounded, it follows that $\bar{d}$" is totally bounded.   Therefore

(1) holds.

The proof is now complete.

3. _The equivalence of_ T-_embedding and_ C _-embedding for completely-regular Hausdorff spaces_. We shall prove this equivalence in several stages. In our final result we give several new necessary and sufficient conditions for a topological space to be normal. (Note that a normal space need not be Tj.)

PROPOSITION 3.1 ([5. 15E.1]) _Suppose that_ X _is a topological space and that_ d _jls a, continuous pseudometric on_ X. _Then_ d _jus totally bounded if and only if for each_ e > 0, X _jls^ a, finite union °^ zero-sets of diameter at most_ e.

LEMMA 3.2 _Suppose that_ S _3js_ T-_embedded in_ X. _If_ f € C*(S), _if_ $Z_q(f) \; / fl_3$ _and if_ f > 0, _then there exists_ g e C (X) _such that_ g|S = f.

Proof. Let f e C*(S) and suppose that f > 0 and that $Z = Z_g(f) \; ^ \; 0$. Define Y_f: S >< S - $\underset{\sim}{R}$ by

$$*_f(x, y) = lf(x) - f(y)| \qquad (x, y \in S).$$

Then Y_f is a continuous pseudometric on S. To show that $\underset{\mathbf{f}}{¥}$ is totally bounded, let e > 0 and choose k e $\underset{\sim}{N}$ such that f(x) $\leq$ (k+l)«e for all x e S. For n = 1, ..., k, let

$$Z_{\sim} = \text{fx e S : n-e < f(x) < (n+l).e}\}.$$

Then $(Z^, ..., Z_{fc})$ is. a finite number of zero-sets of diameter at most € and one easily verifies that S = $U^k -, Z$. Since S is
$$n=i \quad n$$
T-embedded in X, there exists a continuous pseudometric d on X such that d | S >< S = Y_f. Let g: X - $\tilde{R}$ be defined by g(x) = $\inf^{y \; \in}_{z} d(x',y)$ (x e X). Then g e C(X) and gfs = f.

.THEOREM 3.3 _Suppose that_ X _JLS a topological space. If_ S _is_ T-_embedded in_ X, _then_ S _is_ C*-_embedded in_ X.

Proof. Assume S / $\not$ and let f e C*(S). Fix an arbitrary a e. B_3
let f(a) = a, let g = (f v $\underset{\sim}{a}$) - $\underset{\sim}{a}$ and h - -((f A a) - a). By
3.2/ there exist $\overline{g}$, E e C(X) such that $\overline{g}$|S = g and E|S = h. Let
k = ($\overline{g}$ - E) + $\underset{\sim}{a}$. Then one easily verifies that k e C(X) and that
klS = f.

THEOREM 3.4 <u>Suppose</u> <u>that</u> X <u>is</u> <u>a</u> <u>topological</u> <u>space</u>, If S is
$\aleph_o$
P -<u>embedded</u> <u>In</u> X (<u>every</u> <u>separable</u> <u>continuous</u> <u>pseudometric</u> <u>on</u> S <u>can</u>
<u>be</u> <u>extended</u> <u>to</u> <u>a</u> <u>separable</u> <u>continuous</u> <u>pseudometric</u> <u>on</u> $X)_s$ <u>then</u> S <u>JLf</u>
<u>T-embedded</u> <u>in</u> X.

Proof. This follows Immediately from Theorem 2*7 and the fact
that a totally bounded pseudometric space is separable.

COROLLARY 3*5 <u>Suppose</u> <u>that</u> X <u>is</u> <u>&</u> <u>topological</u> <u>space</u>. <u>If</u> S <u>Isi</u>
C-<u>embedded</u> <u>In</u> X_s <u>then</u> S <u>is</u> <u>T-embedded</u> <u>in</u> X.
Proof. Since a C-embedded subset Is P °-embedded ([10^ Theorem
4.7])* the result follows from 3.^{1}!-*.

REMARK. Actually, T. E. Gantner has shown that a subset S of a
topological space is P °«embedded if and only If it Is•C-embedded [4].

THEOREM 3«6 <u>If</u> S <u>is</u> <u>a</u> <u>dense</u> <u>C*-embedded</u> <u>subset</u> <u>of</u> <u>a</u> <u>topological</u>
<u>space</u> X^ <u>then</u> S <u>i^</u> <u>T-embedded</u>.
Proof. Let (G_1, .„o, G_n) be a finite normal open cover of 3.
Then there Is a cdzero-set cover (U-^ ..., U) of S such that cl_a U.
is completely separated (in S) from 3~G_i for 1 = 1 , ..., n (['9,
Theorem 1.2]). Since S Is C*-embedded In X^ cl_o U. Is completely
separated (In X) from S-$G_\pm$. Hence there is an f_1 e C(X) such that
f_1(cl_g Uj) - fO] and $f_\pm$(S-G_1) - {!]. For each 1 = 1 , ..., n,

let $Z_\pm = Z(f_\pm)$ and let $V_\pm = [x \in X : f_\pm(x) < 1/2)$. We note that
$V_\pm \cap S \subset G_\pm$ and $cl_x U_\pm \subset cl_x Z_\pm = Z_\pm \subset V_i$o Therefore $X = cl_x S -$
$c_M u_{i=1}^n V = u_{i=1}^n cl_x u_i \subset u_{i=1}^n v_i$ and hence $(\sim V -\bullet^* V_n)$ is a
cozero-set cover of X that* on S_5 refines $(G,,_\perp . <> ,^* G_n)$. It
follows that S is T-embedded in X.

THEOREM 3.7 <u>Suppose that X ^s a completely regular Hausdorff</u>
<u>space and that S</u> 3J3 <u>a, subspace</u> of X. <u>If</u> S ^s C^-<u>embedded in</u> X, <u>then</u>
S i£ T-<u>embedded in</u> X.

Proof. Since S is C*-embedded in X^ $cl_{ftv} S = PS$. Moreover^ S
pA
is C*-embedded in PS so^ by Theorem 3*&> S is T-embedded in PS.
But PS is a closed subset of the normal space px<, so pS is C-embedded
in PX and therefore by 3-5^ PS is T-embedded in PXO Since T-embedding
is transitive^ S is T-embedded in PX, It follows that S is T-embedded
in X.

GORCLLA.RY 3.8 ~~If~~ X ~~is a, completely regular Hausdorff space and,~~
~~if~~ S ~~is a, subspace~~ of X^ ~~then the following statements are equivalent;~~

(1) S ~~is~~ C*-~~embedded in~~ X_e

(2) S ~~is~~ T-~~embedded iri~~ X.

As an immediate result of Theorems 2.7 and 3.8 and known
characterizations of a normal space we have

THEOREM 3*9 <u>JIf X is a, topological space,, then the following</u>
<u>statements are equivalent</u>%

(1) X <u>is</u> normal.

(2) <u>Every closed subset of X is</u> -T-<u>embedded in</u> X,

(3) <u>For every closed subset F of X, every totally bounded •contin-</u>
<u>uous pseudometric on F can be extended to a, continuous pseudometric on X.</u>

(4) <u>For every closed subset F of X^ every finite normal cozero-set</u>
<u>cover of S has a, refinement that extends t£ a, normal open cover of X.</u>