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A CONVOLUTION PRODUCT FOR THE SOLUTIONS
OF PARTIAL DIFFERENCE EQUATIONS

by

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and

Joan Rohrer

Report 67-11

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Abstract

This paper is concerned with linear partial difference operators L having constant coefficients. The functions considered are defined only on the lattice points of the complex plane. It is shown that with any two solutions of $Lu(z) = 0$ there is associated a new solution which is represented as a convolution product. This product may be considered as a type of line integral and is based upon a discrete analogue of Green's formula. This development may be regarded as an analogy to the pioneering work of H. Lewy concerning the composition of solutions of partial differential equations. It may also be considered a continuation of the investigation pursued by Duffin and Duris in the introduction of a convolution product for discrete analytic functions.

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1. Introduction

The purpose of this paper is to establish a formula which associates with any two solutions of a partial difference equation with constant coefficients a new solution which is represented as a convolution product. This product is based upon a discrete analogue of Green's formula in the plane.

The lattice points of the complex plane are the points $z = m + ni$, where m and n may assume the values $0, \pm 1, \pm 2, \dots$. Let $u(z)$ be a complex-valued function defined on the lattice points of the plane. The translation operators are defined as follows:

$$(1) \quad X^m u(z) = u(z + m); \quad Y^n u(z) = u(z + in), \quad m, n = 0, \pm 1, \pm 2, \dots$$

We are concerned with solutions of the partial difference equation

$$(2) \quad Lu(z) = 0,$$

where $L = \sum_{i=1}^k c_i T_i$, $T_i = X^{m_i} Y^{n_i}$, m_i, n_i are real integers and c_i are complex constants. Note that the operator $T_i^{-1} = X^{-m_i} Y^{-n_i}$

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is the inverse of T_i .

Particular examples of the partial difference equations of concern are:

$$(3) \quad (X - 2I + X^{-1} + Y - 2I + Y^{-1})u(z) = 0 \quad (\text{Laplace's equation})$$

$$(4) \quad [(X - 2I + X^{-1}) - c^{-2}(Y - 2I + Y^{-1})]u(z) = 0 \quad (\text{Wave equation})$$

$$(5) \quad (X - 2I + X^{-1} + Y - 2I + Y^{-1})^2 u(z) = 0 \quad (\text{Biharmonic equation})$$

$$(6) \quad [(X - 2I + X^{-1}) - c(Y - I)]u(z) = 0 \quad (\text{Heat equation})$$

$$(7) \quad (I + iX - XY - iY)u(z) = 0 \quad (\text{Cauchy-Riemann equation, complex form})$$

A problem of interest concerning such difference equations is the generation of new solutions from a given solution. One approach to this problem was given by Duffin and Shelly [2] by the definition of operators under which the solution set is invariant. A new class of such operators is studied here.

The first four of the above equations are self-explanatory; the last refers to the theory of discrete analytic functions, and solutions of this equation are termed discrete analytic. It was shown by Duffin and Duris [1] that, given two solutions $w(z)$ and $u(z)$ of (7) there is a new solution $\Phi(z) = w*u$, where "*" was termed a convolution product. This product is both commutative and associative. In this paper we also introduce a convolution product for solutions of an arbitrary difference equation of the class described. We shall designate this product as $\Psi(z) = w*u$; in general, however, this product is neither commutative nor associative.

We pattern our product after a formula given by Hans Lewy [5], who was concerned with the corresponding problem for partial differential equations. The formula of Lewy has several applications

in the theory of partial differential equations; in particular we refer to the paper by Lehman and Lewy [6].

To develop the formula in the context of discrete function theory, we first of all need a close analogue of Green's formula in the plane. We express this as:

$$\sum_E [v(z) Lu(z) - u(z) Mv(z)] = \sum_{i=1}^k c_i \int_{MB} v(\alpha) u(T_i \alpha) d\mu_i ;$$

here M is the adjoint difference operator, and E is a set of lattice points in the complex domain. On the right of this formula is a Stieltjes-type integral evaluated on a closed contour MB termed the median boundary of E , and α denotes certain points in the vicinity of this boundary. We then proceed to define an open contour integral given by the formula

$$(w^*u)(z) = \sum_{i=1}^k c_i \int_0^z w(z - \alpha) u(T_i \alpha) d\mu_i .$$

It is clear that this is a product of convolution type. By virtue of Green's formula it results that this product is independent of the path if w and u are both solutions of the difference equation. By making use of this property it is then shown that this convolution product is itself a solution of the difference equation.

2. The adjoint operator and Green's formula

Let $u'(z)$ and $v'(z)$ be lattice functions which both vanish for $|z| \geq R$ where R is a constant. If Σ indicates the sum taken over all lattice points, it is clear that

$$(1) \quad \Sigma v'(z) X^m Y^n u'(z) = \Sigma u'(z) X^{-m} Y^{-n} v'(z) .$$

Corresponding to the operator $L = \sum_{i=1}^k c_i T_i$, the adjoint operator M is defined in this treatment by $M = \sum_{i=1}^k c_i T_i^{-1}$. Then by (2-1) it is seen that

$$(2) \quad \sum v'(z) Lu'(z) - u'(z) Mv'(z) = 0 .$$

In particular for functions u' and v' which vanish outside a finite set E of lattice points, (2-2) becomes

$$(3) \quad \sum_E v'(z) Lu'(z) - u'(z) Mv'(z) = 0 .$$

Now let $u(z)$ and $v(z)$ be lattice functions; let $u' = u$ and $v' = v$ in E , and let $u' = v' = 0$ in the complement E' . Define the sets of neighboring points for a lattice point z as determined by L and M :

$$\begin{aligned} \mathcal{N}_L(z) &= \{ \zeta \mid \zeta = T_i z \text{ for some } i, 1 \leq i \leq k \} ; \\ \mathcal{N}_M(z) &= \{ \zeta \mid \zeta = T_i^{-1} z \text{ for some } i, 1 \leq i \leq k \} . \end{aligned}$$

Note that the sets are not necessarily disjoint and also that $\zeta \in \mathcal{N}_L(z) \iff z \in \mathcal{N}_M(\zeta)$. Now for $z \in E$, define

$$T_i \partial u(z) = \begin{cases} 0 & \text{if } T_i(z) \in E \\ u(T_i z) & \text{if } T_i(z) \in E' ; \end{cases}$$

$(T_i^{-1}) \partial$ is defined similarly.

Then let $L \partial u(z) = \sum_{i=1}^k c_i T_i \partial u(z)$; likewise $M \partial u(z) = \sum_{i=1}^k c_i (T_i^{-1}) \partial u(z)$.

At a point $z \in E$ such that $\mathcal{N}_L(z) \cap E' = \emptyset$, $Lu' = Lu$; when $\mathcal{N}_L(z) \cap E' \neq \emptyset$, $Lu' = Lu - L \partial u$. Thus at each point $z \in E$,

$$\begin{aligned} Lu' &= Lu - L \partial u \\ Mu' &= Mu - M \partial u . \end{aligned}$$

Then (2-3) may be written

$$\sum_E \{v(z) [Lu(z) - L^\partial u(z)] - u(z) [Mv(z) - M^\partial v(z)]\} = 0 ,$$

or

$$(4) \quad \sum_E [v(z) Lu(z) - u(z) Mv(z)] = \sum_E [v(z) L^\partial u(z) - u(z) M^\partial v(z)] .$$

This may be termed a first analogue of Green's formula for partial difference equations.

Now if $Lu(z) = Mv(z) = 0$ in E , (2-4) gives

$$(5) \quad \sum_E [v(z) L^\partial u(z) - u(z) M^\partial v(z)] = 0 .$$

3. Green's formula as a line integral

It is proposed to formulate the right side of Green's formula (2-4) as a line integral. To facilitate introduction of this line integral the device of the "median boundary" is used.

The edges of the primal lattice are those lines of length one connecting lattice points z_i and z_j , where $|z_i - z_j| = 1$. Now consider a dual lattice whose edges, again having length one, are the perpendicular bisectors of edges of the primal lattice. About each point z of E , construct a unit square having z as center. The sides of such a square are edges of the dual lattice; the union of all such squares will be designated the associated region of E . See Figure 1.

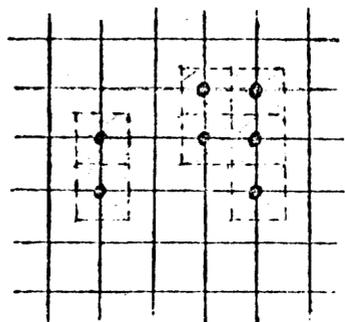


Figure 1. A set E together with its associated region

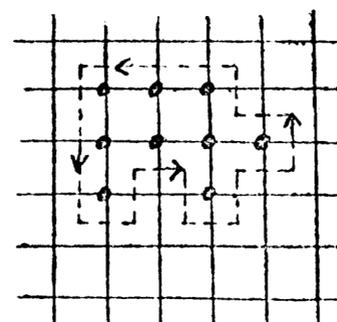


Figure 2. A set E together with its median boundary

The boundary of the associated region of E consists of a set of closed "curves" which are unions of edges of the dual lattice. Hereafter we are concerned only with a set E whose associated region has a boundary consisting of a single simple closed curve (or Jordan curve, in the usual sense). This boundary will be designated as the median boundary MB of E ; see Figure 2.

Let MB have direction of circulation so that the enclosed set E remains on the left. For an arbitrary translation T , construct a translation vector from every lattice point α to the lattice point $T\alpha = \beta$, observing the following convention: The vector $\overline{T\alpha}$ shall be composed only of edges of the primal lattice and shall exhibit at most a single right turn. For example, if $T = XY^2$, the translation vector $\overline{T\alpha}$ would be constructed by moving along consecutive edges beginning at α , first two units upward and then one unit to the right. See Figure 3.

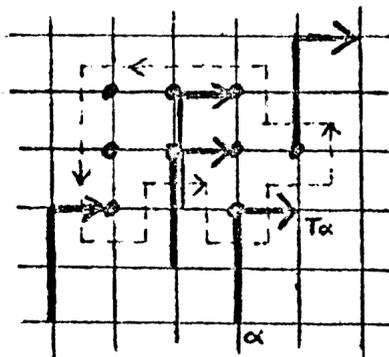


Figure 3. Construction of the translation vector for $T = XY^2$.

Let p_1, p_2, \dots, p_N be the points at which MB crosses an edge of the primal lattice. For a specified translation T , define the sets of lattice points:

$$A(p_s) \equiv \{\alpha \mid \overline{T\alpha} \text{ crosses } MB \text{ at } p_s\}, \text{ for } s = 1, 2, \dots, N.$$

Noting that if $A(p_s)$ contains more than one element, $\bar{T}\alpha$ crosses MB in the same direction, $\forall \alpha \in A(p_s)$, we now define a step function on the set p_1, p_2, \dots, p_N as follows:

$$(1) \quad d\mu(p_s) = \begin{cases} 0 & \text{if } A(p_s) = \emptyset ; \\ +1 & \text{if } \bar{T}\alpha \text{ crosses MB at } p_s \text{ left to right;} \\ -1 & \text{if } \bar{T}\alpha \text{ crosses MB at } p_s \text{ right to left.} \end{cases}$$

Now suppose $f(\alpha, \beta)$ is a function defined on pairs of lattice points α and β , where $\beta = T\alpha$. Then the line integral of f around the contour MB is symbolized by

$$\int_{MB} \{f\} d\mu$$

and in this context this line integral is defined by

$$(2) \quad \int_{MB} \{f\} d\mu \equiv \sum_{s=1}^N \sum_{\alpha \in A(p_s)} f(\alpha, T\alpha) d\mu(p_s) .$$

The symbol $\{ \}$ is used to indicate that it is in general a set of functional values of f that is to be multiplied by the measure function $d\mu$ at a particular point. With this understanding, however, there will be no danger of confusion if we omit the braces and write simply

$$\int_{MB} f d\mu .$$

Of course it may happen that a single translation vector $\bar{T}\alpha$ crosses MB N_α times, where $N_\alpha > 1$; however it is impossible that $\bar{T}\alpha$ should cross MB in the same direction twice in succession. To see this, let the crossing points be labeled $p_{s_1}, p_{s_2}, \dots, p_{s_{N_\alpha}}$ as they occur consecutively along $\bar{T}\alpha$ from its tail towards its head. Now suppose $\alpha \in A(p_{s_m}) \cap A(p_{s_{m+1}})$, and that the crossings

at p_{s_m} and $p_{s_{m+1}}$ both occur in the direction left to right. Since MB circulates with the associated region on its left and $\bar{T}\alpha$ always crosses MB perpendicularly, this would imply that $\bar{T}\alpha$ passes from the inside to the outside of the associated region twice in succession, which is clearly impossible. Likewise it is impossible that two consecutive crossings should occur from right to left. Thus we have $d\mu(p_{s_m}) = -d\mu(p_{s_{m+1}})$, and it may be concluded that if $\bar{T}\alpha$ crosses MB an even number of times, there is no net contribution to the sum in (3-2).

If $\bar{T}\alpha$ crosses MB an odd number of times, then either
 (i) $\alpha \in E$ and $T\alpha \in E'$, in which case the left-to-right crossings must exceed the right-to-left crossings by exactly one, and the net contribution to the sum is $+f(\alpha, T\alpha)$; or
 (ii) $\alpha \in E'$ and $T\alpha \in E$, in which case the right-to-left crossings must exceed the left-to-right crossings by exactly one, with a net contribution of $-f(\alpha, T\alpha)$ to the sum.

Now let a and b denote two points on MB which are intersection points of edges of the dual lattice. Then the definite line integral is defined by

$$(3) \quad \int_a^b f d\mu \equiv \sum_{\alpha \in A(p_s)} \sum f(\alpha, T\alpha) d\mu(p_s) ,$$

where the sum is taken over all points of the set p_1, \dots, p_N which occur between a and b in the positive sense of circulation. The integral thus defined satisfies

$$(4) \quad \int_a^b f d\mu + \int_b^c f d\mu = \int_a^c f d\mu ;$$

if the direction of path be reversed,

$$(5) \quad \int_b^a f d\mu = - \int_a^b f d\mu.$$

Clearly the integral is a linear operator. Thus

$$\int_a^b (f + g) d\mu = \int_a^b f d\mu + \int_a^b g d\mu.$$

Taking $f(\alpha, T\alpha) = v(\alpha)u(T\alpha)$ we have

Lemma 3.1.

$$\sum_E [v(z)T^\partial u(z) - u(z)(T^{-1})^\partial v(z)] = \int_{MB} v(\alpha)u(T\alpha) d\mu$$

Proof: In view of the preceding remarks, it is seen that a non-zero contribution occurs on the right side only for those lattice points α for which $\overline{T\alpha}$ crosses MB an odd number of times. For those cases either α or $T\alpha \in E$, and a corresponding term appears on the left side:

- (i) For $\alpha \in E$, $T\alpha \in E'$, the term $v(\alpha)u(T\alpha)$ appears on the right; taking $z = \alpha$, $T^\partial u(z) = u(Tz) = u(T\alpha)$ and the same term appears on the left.
- (ii) For $\alpha \in E'$, $T\alpha \in E$, the term $-v(\alpha)u(T\alpha)$ appears on the right; taking $z = T\alpha$, $T^{-1}z = \alpha \in E'$, so that $(T^{-1})^\partial v(z) = v(\alpha)$ and the term $-u(T\alpha)v(\alpha)$ appears on the left.

It is clear that such a correspondence accounts for all the terms on the left side, since the sum is taken only over points $z \in E$ and the presence of either T^∂ or $(T^{-1})^\partial$ in each term decrees that a non-zero contribution can occur only if either $Tz \in E'$ or $T^{-1}z \in E'$. This in turn necessitates that the translation vector $\overline{T\alpha}$ cross MB an odd number of times, with either $\alpha = z$ or $\alpha = T^{-1}z$, respectively.

Consider now the set of translations T_1, T_2, \dots, T_k appearing in L . Define the sets $A_i(p_s)$ and the step functions $d\mu_i(p_s)$ as above to correspond with the translations T_i , $i = 1, 2, \dots, k$. Then from the preceding lemma it follows that for $L = \sum_{i=1}^k c_i T_i$

$$(6) \quad \sum_E [v(z) L^\partial u(z) - u(z) M^\partial v(z)] = \sum_{i=1}^k c_i \int_{MB} v(\alpha) u(T_i \alpha) d\mu_i .$$

This together with the first form of Green's formula (2-4) gives

$$(7) \quad \sum_E [v(z) Lu(z) - uMv(z)] = \sum_{i=1}^k c_i \int_{MB} v(\alpha) u(T_i \alpha) d\mu_i .$$

This is the second analogue of Green's formula.

It follows that for functions u and v satisfying $Lu = Mv = 0$ in E ,

$$(8) \quad \sum_{i=1}^k c_i \int_{MB} v(\alpha) u(T_i \alpha) d\mu_i = 0 .$$

This together with (3-5) implies that the line integral

$$(9) \quad \Psi(u, v, a, b) \equiv \sum_{i=1}^k c_i \int_a^b v(\alpha) u(T_i \alpha) d\mu_i$$

is independent of path between points a and b on the dual lattice, provided any two paths taken together enclose a region E where $Lu = Mv = 0$. This line integral exhibits a useful translation property.

Lemma 3.2. $\Psi(u, v, T_j a, T_j b) = \Psi(T_j u, T_j v, a, b)$

Proof. Assuming that the conditions for independence of path continue to hold, choose a path P to evaluate the left side. Then the "parallel" path $T_j^{-1}P$ may be used to evaluate the right

side, and it is observed that the crossing points on the two paths are in 1-1 correspondence. The functions u and v as evaluated on path P are replaced by the translated functions $T_j u$ and $T_j v$ on the path $T_j^{-1} P$.

4. The convolution product

Let O represent the origin of the xy plane, Q the point $z = x + iy$, P the point $\hat{z} = \hat{x} + i\hat{y}$. A lattice function $w(z)$ may be considered a function of the vector \vec{OQ} ; also $w(z) = w(\vec{OQ}) = w(\vec{QP}) = w(\hat{z} - z)$ provided the vector \vec{QP} has the same components as the vector \vec{OQ} . Let the notation $X^m Y^n w(\vec{QP})$ indicate that the operation $X^m Y^n$ is to be applied treating the point P as independent variable while the point Q is held fixed; i.e., $X^m Y^n w(\hat{z} - z) = w(\hat{z} + m + in - z)$. Correspondingly, let $\hat{T}_i = X^m Y^n$, $\hat{L} = \sum_{i=1}^k c_i \hat{T}_i$

Lemma 4.1. Suppose the vectors \vec{QP} and \vec{OQ} have the same components.

Then if $Lw(z) = 0$, $Mw(\hat{z} - z) = 0$.

Proof. First it is observed that $Lw(\vec{OQ}) = \hat{L}w(\vec{QP})$, and $Lw(z) = 0$

$$\begin{aligned} \Rightarrow \hat{L}w(\hat{z} - z) &= 0. \text{ Now } Mw(\hat{z} - z) = \sum c_i T_i^{-1} w(\hat{z} - z) \\ &= \sum c_i w(\hat{z} - T_i^{-1} z) \\ &= \sum c_i w(\hat{T}_i \hat{z} - z) \\ &= \hat{L}w(\hat{z} - z) \\ &= 0. \end{aligned}$$

Thus if $Lu = Lw = 0$ in E , $O \in E$, the function $v(\alpha)$ in (3-9) may be replaced by the function $w(z - \alpha)$; the integral remains independent of path provided any two paths enclose a region $E^* \subset E$ such that for any point $\alpha \in E^*$, the vector \vec{QP} directed from α

to z has the same components as a vector $\vec{OQ}' \in E$. See Figure 4.

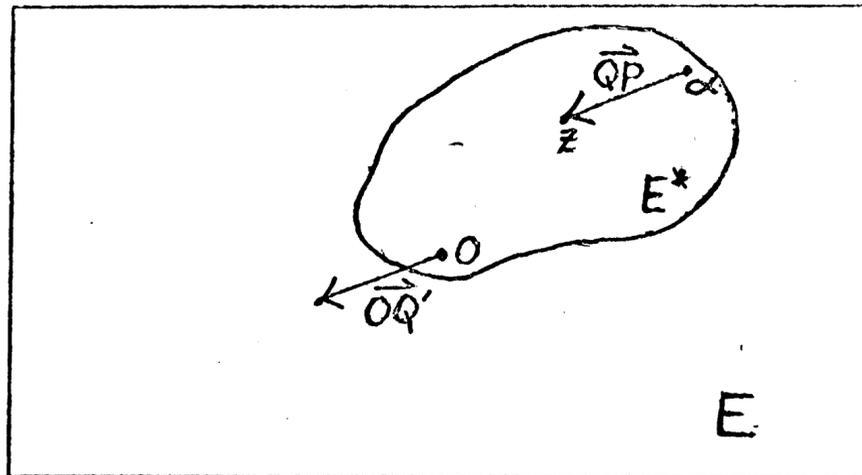


Figure 4. $\vec{QP} = \vec{OQ}'$

The concept of the median boundary was introduced merely for convenience in computing the line integral. Without changing the meaning of the integral, let the limits a and b on the dual lattice now be identified with the lattice points $z_1 = a - \epsilon$, $z_2 = b - \epsilon$, where $\epsilon = \frac{1}{2} + \frac{1}{2}i$. It then results that

$$(1) \quad \Psi(u, v, a, b) = \Psi(u, w(z - \alpha), a, b) = \sum_{i=1}^k c_i \int_{z_1}^{z_2} w(z - \alpha) u(T_i \alpha) d\mu_i .$$

In particular, the lattice points 0 and z may be chosen as limits, provided that the conditions for independence of path are fulfilled. This establishes Ψ as a function of z , and finally the convolution product of two functions u and w satisfying (1-2) is here defined as

$$(2) \quad (w^*u)(z) = \Psi(z) = \sum_{i=1}^k c_i \int_0^z w(z - \alpha) u(T_i \alpha) d\mu_i .$$

If $Lw = Lu = 0$ in the entire plane, then the convolution product is independent of path for any lattice point z . For $Lw = Lu = 0$ in a rectangular region E surrounding the origin, the conditions for independence of path are fulfilled whenever $z \in E$ and paths are restricted to the quadrant of E containing z . If z lies on an

axis, then paths restricted to the half-plane containing z will be suitable.

5. The convolution product as a solution of the difference equation

For clarity of presentation and emphasis of the main result of this development, the following theorem is stated and proved under the specialized hypothesis that u and w are solutions of (1-2) in the entire plane. Remarks concerning adaptation to more general situations follow the proof.

Theorem: If $Lu(z) = Lw(z) = 0$ for all z , then likewise
 $L(w*u)(z) = 0$ for all z .

Proof: $L(w*u)(z) = L\Psi(z) = \sum_{j=1}^k c_j \Psi(T_j z)$; a convenient path is chosen for the evaluation of $\Psi(T_j z)$. Let

$$(1) \quad \rho_j = \sum_{i=1}^k c_i \int_0^{T_j 0} w(T_j z - \alpha) u(T_i \alpha) d\mu_i ; \quad \varphi_j = \sum_{i=1}^k c_i \int_{T_j 0}^{T_j z} w(T_j z - \alpha) u(T_i \alpha) d\mu_i$$

Then by property (3-4) and independence of path, $\Psi(T_j z) = \rho_j + \varphi_j$, so that

$$L\Psi(z) = \sum_{j=1}^k c_j \rho_j + \sum_{j=1}^k c_j \varphi_j.$$

Now by the translation property,

$$\begin{aligned} \varphi_j &= \sum_{i=1}^k c_i \int_0^z w(T_j z - T_j \alpha) u(T_j T_i \alpha) d\mu_i \\ &= \sum_{i=1}^k c_i \int_0^z w(z - \alpha) u(T_j T_i \alpha) d\mu_i, \end{aligned}$$

so that $\sum_{j=1}^k c_j \varphi_j$

$$= \sum_{i=1}^k c_i \int_0^z w(z - \alpha) \sum_{j=1}^k c_j T_j u(T_i \alpha) d\mu_i$$

$$(2) \quad = \sum_{i=1}^k c_i \int_0^z w(z - \alpha) Lu(T_i \alpha) d\mu_i$$

$$= 0.$$

It remains to show that $\sum_{j=1}^k c_j \rho_j = 0$. Observe that $\rho_j = \sum_{i=1}^k c_i \rho_{ij}$,

where

$$(3) \quad \rho_{ij} = \int_0^{T_j^0} w(T_j z - \alpha) u(T_i \alpha) d\mu_i .$$

We propose to show that the matrix (ρ_{kj}) is skew symmetric.

Letting $T_i \alpha = \beta$, $\alpha = T_i^{-1} \beta$, we have

$$(4) \quad \begin{aligned} \rho_{ij} &= \int_0^{T_j^0} w(T_j z - T_i^{-1} \beta) u(\beta) d\mu_i \\ &= \int_0^{T_j^0} w(T_i T_j z - \beta) u(\beta) d\mu_i ; \end{aligned}$$

similarly,

$$(5) \quad \rho_{ji} = \int_0^{T_i^0} w(T_i T_j z - \beta) u(\beta) d\mu_j .$$

Now since ρ_j is independent of dual lattice path between 0 and $T_j^0 = m_j + n_j i$, it is possible to consistently choose a path P_j which makes at most one left turn. Also it is important to note here that for a fixed i and j , the integrand of ρ_{ij} and ρ_{ji} is a function of β alone; i.e., the usual $f(\alpha, \beta)$ of (3-2) may here be written $f(\beta)$.

The desired result is thus a special case of the more general result

$$(6) \quad \int_0^{T_j^0} f(\beta) d\mu_i = - \int_0^{T_i^0} f(\beta) d\mu_j ,$$

under the condition that the respective paths on the dual lattice, P_j and P_i , can exhibit at most a single left turn and $f(\beta)$ is an arbitrary function. However, to establish this it is sufficient to consider the case $f(\beta) = 0$ except if β has the special value

$\beta = \beta_0$. The general case will then follow by the linearity of the line integral.

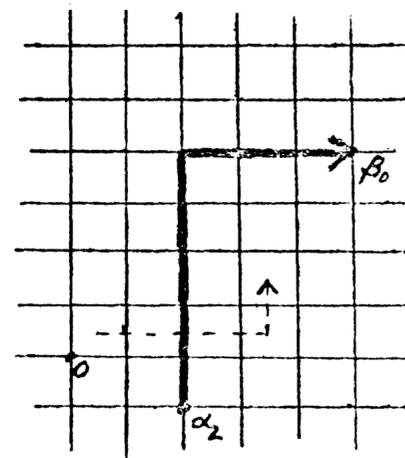
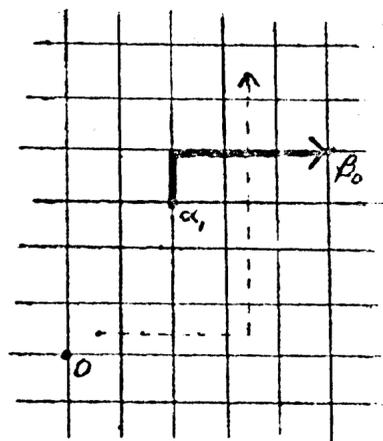
That the left and right sides of (5-6) must have the same value is a simple geometrical consequence of the consistent choice of left-turn paths and right-turn translation vectors. Consider the path P_j ; set $\alpha_1 = T_i^{-1}\beta_0$ and consider the translation vector $\bar{T}_i\alpha_1$. There are three possibilities:

- (i) $\bar{T}_i\alpha_1$ does not cross P_j ;
- (ii) $\bar{T}_i\alpha_1$ crosses P_j twice;
- (iii) $\bar{T}_i\alpha_1$ crosses P_j once.

In the first two cases, the left side of (5-6) vanishes. In the third case, the left side of (5-6) becomes $\pm f(\beta_0)$, the sign being determined by the direction of crossing.

Next consider the path P_i ; set $\alpha_2 = T_j^{-1}\beta_0$ and consider the translation vector $\bar{T}_j\alpha_2$. The same three possibilities ensue:

- (i) $\bar{T}_j\alpha_2$ does not cross P_i ;
- (ii) $\bar{T}_j\alpha_2$ crosses P_i twice;
- (iii) $\bar{T}_j\alpha_2$ crosses P_i once.



Path P_j , translation vector $\bar{T}_i\alpha_1$. Path P_i , translation vector $\bar{T}_j\alpha_2$.

Figure 5.

If the diagram of P_j and $\bar{T}_i\alpha_1$ be rotated 180° , it is found to be identical with the diagram of P_i and $\bar{T}_j\alpha_2$ (except for a uniform shift of $\epsilon = \frac{1}{2} + \frac{1}{2}i$); in fact, P_j and $\bar{T}_j\alpha_2$, and P_i and $\bar{T}_i\alpha_1$, respectively, are indistinguishable except for the sense of direction. A typical case is illustrated in Figure 5. Thus it may be concluded that

$$(i) \leftrightarrow (i)', \quad (ii) \leftrightarrow (ii)', \quad (iii) \leftrightarrow (iii)',$$

and that the direction of crossing in the cases (iii) and (iii)' must be opposite. It is also important to note that β_0 serves as the terminus of both $\bar{T}_i\alpha_1$ and $\bar{T}_j\alpha_2$, assuring that, in the case of (iii), both sides of (5-6) have a common value of $\dagger f(\beta_0)$. It is interesting to note that when $i = j$, case (iii) is impossible.

Thus (5-6) is demonstrated and from it follows the desired result $\rho_{ij} = -\rho_{ji}$, $\rho_{ii} = 0$, $i = 1, 2, \dots, k$. Then

$$\sum_{j=1}^k c_j \rho_j = \sum_{j=1}^k \sum_{i=1}^k c_j c_i \rho_{ij} = 0,$$

as was to be shown.

QED.

For the case where $Lu = Lw = 0$ in a bounded region E , it may be concluded that $L(w*u)(z) = 0$ provided the following additional conditions are met:

1) $\Psi(z)$ must be independent of path throughout a sufficiently large region $E^* \subset E$ containing both the origin and the point z , so that the evaluation of $\Psi(T_j z)$ may include the term ρ_j as defined in (5-1);

2) There must exist a dual lattice path P' joining the points $a = 0 + \epsilon$ and $b = z + \epsilon$ having the property that $T_i A_i(p) = \{\beta \mid \beta = T_i \alpha \text{ for } \alpha \in A_i(p)\} \subset E$, for $i = 1, 2, \dots, k$ and

for each point p lying on P' between a and b . This condition makes it possible to conclude $Lu(T_i\alpha) = 0$ in (5-2).

6. Applications

(i) Suppose $u(z)$ is a real discrete harmonic function; i.e., $u(z)$ satisfies (1-3). Then the function $\Psi(u, l, a, b) = l^*u$ is a conjugate harmonic function of $u(z)$. In this context, the conjugate harmonic function is defined on the nodes of the dual lattice, so that we revert to the original formula (3-9). The "Cauchy-Riemann" equations to be satisfied are $\Delta_y \Psi = \Delta_x u$, $\Delta_x \Psi = -\Delta_y u$, where the symbol " Δ " denotes a forward difference. Consider an edge S of the primal lattice with left end-point z . Let a denote the lower end-point of the edge of the dual lattice which bisects S . (See Figure 6.) Then the "Cauchy-Riemann" equations become

$$\Psi(u, l, a, a + i) = u(z + 1) - u(z),$$

$$\Psi(u, l, a + i, a + 1 + i) = -[u(z + 1 + i) - u(z + 1)].$$

That these equations are satisfied is a direct consequence of the definition of $\Psi(u, v, a, b)$.

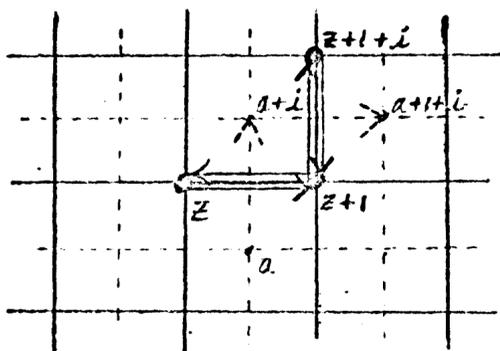


Figure 6.

(ii) Consider the discrete form of the heat equation (1-5). Here $L = \sum_{i=1}^k c_i T_i = X + X^{-1} - cY + (c - 2)I$. We generate the solution $(w^*u)(z)$ by choosing a path exhibiting a single left

turn. For $z = m + ni$, $m, n > 0$,

$$\begin{aligned} \Psi(z) &= \sum_{i=1}^k c_i \int_0^z w(z - \alpha) u(T_i \alpha) d\mu_i \\ &= c \sum_{s=1}^m w(z-s) u(s+i) + \sum_{s=1}^n [w(z-m-si) u(m+1+si) - w(z-m-1-si) u(m+si)] \\ &= c \sum_{s=1}^m w(z-s) u(s+i) + \sum_{s=1}^n [w((n-s)i) u(m+1+si) - w(-1+(n-s)i) u(m+si)] \end{aligned}$$

Figure 7 illustrates the translation vectors $\bar{X}\alpha$, $\bar{X}^{-1}\alpha$, $\bar{Y}\alpha$ as they cross the dual lattice path and the points (circled) at which w is to be evaluated.

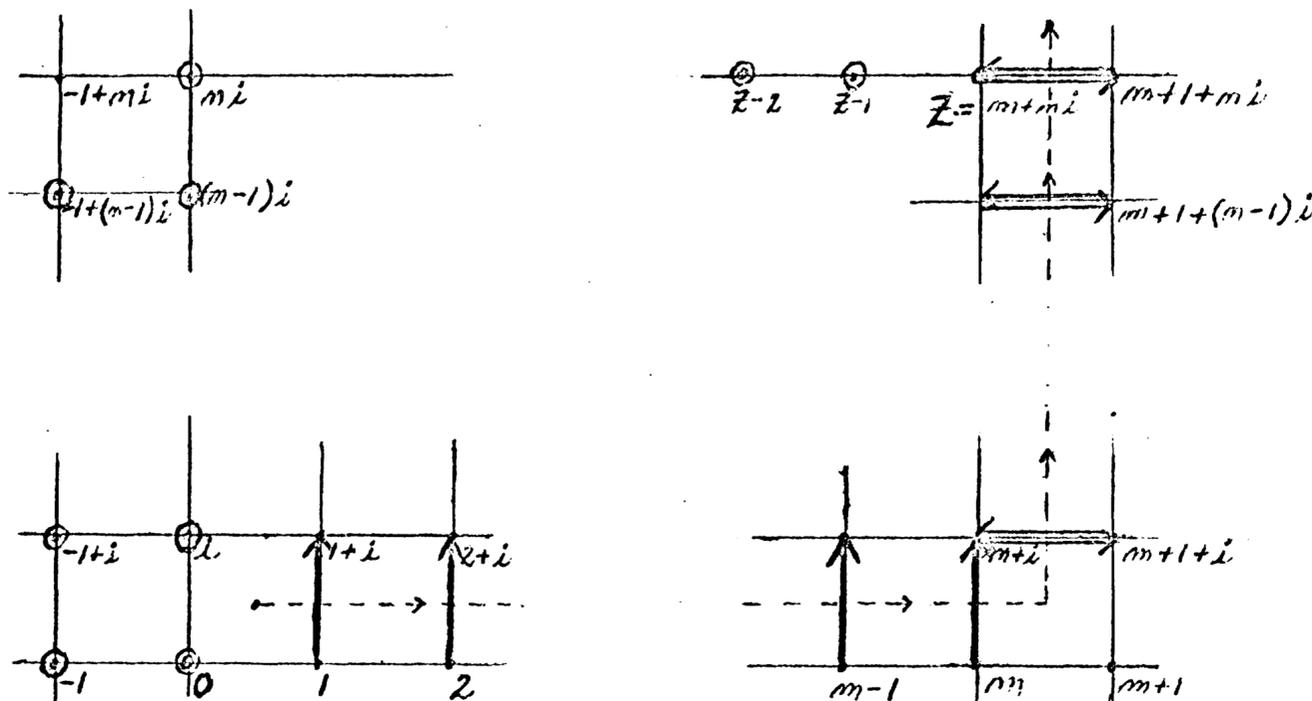


Figure 7.

(iii) It is known that the polynomials $1, z, z^2$ are discrete analytic in the entire plane; i.e., they satisfy $Lu(z) = 0$, where $L = I + iX - XY - iY$. Then the functions $\Psi(z) = (1*u)(z)$ for $u(z) = 1, z$, or z^2 will also be discrete analytic in the entire plane. For the sake of brevity, the lattice points of the plane may be listed as a single sequence z_0, z_1, z_2, \dots ; this sequence is termed the spiral coordinate system. See Figure 8. In Table 1 the first sixteen of these points are listed together with the values of the functions $\Psi(z) = (1*u)(z)$ at these points,

for $u(z) = 1, z, z^2$. The formulas (adapted from (4-2)) used to obtain the functional values are also tabulated.

Notation: $u_j = u(z_j)$.

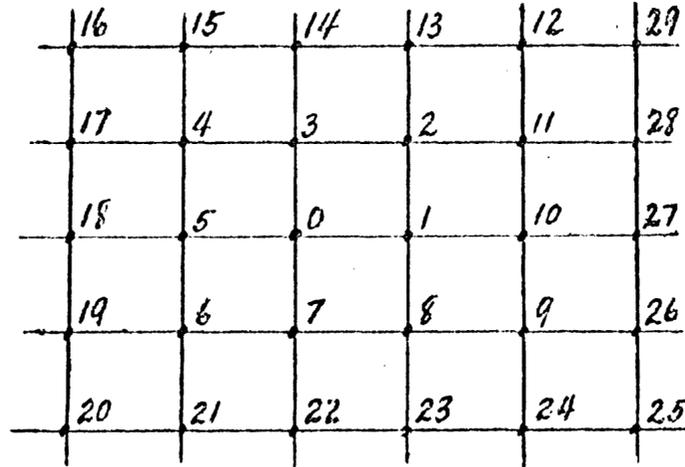


Figure 8. The spiral coordinate system

		$\Psi(z_j)$				
j	z_j	$1 \cdot 1$	$1 \cdot z$	$1 \cdot z^2$	$1 \cdot u(z)$	
0	0	0	0	0	0	
1	1	1+i	1+2i	1+4i	$iu_2 + u_{11}$	
2	1+i	2i	-2+3i	-6+3i	$iu_2 + iu_{11}$	
3	i	-1+i	-2	-2-2i	$iu_2 - u_2$	
4	-1+i	-2	-1-2i	+1-2i	$-u_2 - u_3$	
5	-1	-1-i	-i	-i	$-iu_3 - u_2$	
6	-1-i	-2i	-i	-i	$-iu_3 - u_2 - iu_0 + u_0$	
7	-i	1-i	1-i	+1-i	$-iu_1 + u_1$	
8	1-i	2	3	5	$u_1 + u_{10}$	
9	2-i	3+i	6+2i	14+4i	$u_1 + u_{10} + iu_{10} + u_{27}$	
10	2	2+2i	3+5i	5+13i	$iu_2 + u_{11} + iu_{11} + u_{28}$	
11	2+i	1+3i	-1+7i	-9+15i	$iu_2 + iu_{11} + iu_{12} + u_{29}$	
12	2+2i	4i	-6+8i	-26+8i	$iu_2 + iu_{11} + iu_{12} + iu_{29}$	
13	1+2i	-1+3i	-6+3i	-14-5i	$iu_2 + iu_{11} + iu_{12} - u_{12}$	
14	2i	-2+2i	-5-i	-3-9i	$iu_2 - u_2 + iu_{13} - u_{13}$	
15	-1+2i	-3+i	-3-4i	5-6i	$-u_2 - u_3 + iu_{14} - u_{14}$	
.						
.						
.						
27	3					
28	3+i					
29	3+2i					

Table 1.

It is found that the resulting functional values $\Psi(z)$ are those which would be obtained by the following special cases of the discrete line integral as defined in [1] by Duffin and Duris:

$$1*1 = \int_0^z (1 + i) 1 \delta z = (1 + i) z ;$$

$$1*z = \int_0^z [(1 + i) z + \frac{(1 + 3i)}{2}] \delta z = \frac{(1 + i) z^2}{2} + \left(\frac{1 + 3i}{2}\right) z ;$$

$$1*z^2 = \int_0^z [(1 + i) z^2 + (1 + 3i) z + 2i] \delta z = \frac{(1+i)(z^3 + \bar{z}/2)}{3} + \left(\frac{1+3i}{2}\right) z^2 + 2iz .$$

This example raises the question of the general relationship between the convolution product presented here and that introduced previously by Duffin and Duris.

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