Optimum Shape of a Cooling Fin on a Convex Cylinder

by

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ABSTRACT

The problem considered is that of maximizing the heat dissipation of a cooling fin of fixed weight attached to a cylinder with a convex cross-section by properly tapering the fin. It is assumed that Newton's law of cooling holds and that the boundary of the cylinder has a constant temperature. In a previous paper R. J. Duffin (A Variational Problem Relating to Cooling Fins, J. Math. Mech. 8 (1959), 47-56) considered the special case of cylinders of circular cross-section and proved that for the optimum taper the temperature gradient is a constant. Our method is to convert the differential equation for the heat flow into a saddle point variational problem. The solution of this variational problem shows that for the optimum taper the temperature gradient vector again has constant magnitude. This criterion leads to explicit formulae for the thickness of the fin at each point.

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Optimum Shape of a Cooling Fin on a Convex Cylinder

1. Introduction. Cooling fins are used to conduct heat away from machines to the ambient medium. A common example are the fins on the cylinders of air cooled internal combustion engines. It is not difficult to see that in order to economize on weight the fin should taper, narrowing in the direction of heat flow. This gives rise to a definite mathematical problem which may be phrased in this way--how should the thickness of the fin be tapered so as to minimize the weight of the fin for a given rate of dissipation of heat. This will be called Problem O.

In a previous paper [1] one of us gave an exact solution of this problem for the case of a disk fin on a circular cylinder assuming Newton's law of cooling. The solution proved to be surprisingly simple; the thickness of the fin is tapered so as to satisfy the following criterion--the temperature should be a linear function of the distance along a radius. This criterion had previously been proposed by E. Schmidt [2] but without convincing proof.

The present paper concerns a cooling fin on a convex cylinder. The two dimensionality complicates the problem because the equation of heat conduction is now a partial differential equation instead of an ordinary differential equation. Nevertheless, the criterion for the optimum tapering is still simple--the temperature gradient should be of constant magnitude. This results in the isothermal lines being a family of equidistant convex curves. The flow lines are the straight lines normal to the cylinder. Explicit formulae are found for the shape and tapering of the optimum fin.

The method of proof is to first consider an equivalent problem in which it is desired to maximize the rate of heat dissipation for a fin of given weight. This question can be transformed to a minimax problem of the calculus of variations. This is not a problem of linear type but the proofs are simplified because an explicit solution is found.

Attention is focused on convex cylinders more or less as a matter of convenience. It will be seen that many of the results also hold for a non-convex cylinder. However, such questions will not be treated here.

2. Formulation of a Minimax Problem

Consider a plane cooling fin having an inner boundary Γ_1 and an outer boundary Γ_2 defining a plane region S such as is shown in Figure 1. The inner curve Γ_1 is also to be the boundary of a convex cylinder, the surface of which has a constant temperature T. If we assume that the ambient temperature is zero, that Newton's law of cooling holds, and that the thermal conductivity is unity, then the steady state heat flow in a thin fin is governed by the equation

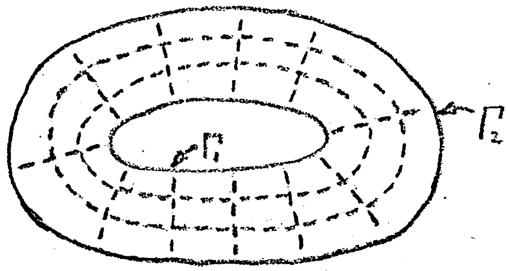

Figure 1

(1) $$\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(p\frac{\partial u}{\partial y}\right) - qu = 0 \quad \text{in} \quad S,$$

or in vector notation

(1') $$\nabla\cdot(p\nabla u) - qu = 0 \quad \text{in} \quad S.$$

Here (x,y) are the coordinates in the plane of the fin, $u = u(x,y)$ is the temperature of the fin, $p = p(x,y)$ is the thickness of the fin, and $q = q(x,y)$ is the cooling coefficient. Of course, $p \geq 0$ and $q > 0$. The boundary conditions to be satisfied are

(2) $$u = T \text{ on } \Gamma_1$$

(3) $$p\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_2.$$

Assuming Newton's law of cooling the heat dissipated per unit time by the fin is

(4) $$\mathcal{H}(u) = \iint_S qu \, dA.$$

The total weight of the fin (assuming a density of unity) is

(5) $$\mathcal{K}(p) = \iint_S p \, dA.$$

With the temperature determined by (1), (2), and (3) we can formulate the following optimization problem:

Problem 1: Find the maximum heat H that can be dissipated per unit time by a fin subject to the constraint that the weight is a constant K.

In order to solve this problem we find it desirable to change its form. This change is motivated by the following heuristic analysis.

First let us suppose that Γ_1 and Γ_2 are made up of 'regular arcs' and that $p, q,$ and u are sufficiently smooth functions so that the application of Green's theorem below is valid. We can permit Γ_2 to be composed of a finite number of simple closed curves. The following functional is basic in our analysis

(6) $$E(p,u) = \iint_S [p|\nabla u|^2 + qu^2]\, dA.$$

For reasons which will soon be evident E is called the saddle functional. Let v be an arbitrary smooth function. Then

(7) $$E(p,u+v) = E(p,u) + E(p,v) + 2\iint_S [p\nabla u \cdot \nabla v + quv]\, dA.$$

Applying Green's theorem to the third term gives

(8) $$\iint_S (p\nabla u \cdot \nabla v + quv]\, dA = \iint_S [qu - \nabla\cdot(p\nabla u)]\, v\, dA + \int_{\Gamma_1} vp\frac{\partial u}{\partial n}\, ds + \int_{\Gamma_2} vp\frac{\partial u}{\partial n}\, ds.$$

The first integral on the right vanishes because u satisfies the differential equation (1). The third integral vanishes because the boundary condition on Γ_2, $p\frac{\partial u}{\partial n} = 0$, is satisfied. If we impose the condition $v = 0$ on Γ_1, then the second integral also vanishes giving

(9) $$E(p,u + v) = E(p,u) + E(p,v) \geq E(p,u).$$

It follows that $E(p,u')$ is minimized, for the class of functions satisfying the boundary condition (2), by u satisfying (1) and (3). This is a standard result of the calculus of variations. In the calculus of variations (3) is termed a natural boundary condition because it is satisfied automatically by the minimizing function.

Now in the application of Green's theorem, (8), let $v = u$. This gives

(10) $$E(p,u) = \int_{\Gamma_1} p\frac{\partial u}{\partial n}\, u\, ds = \int_{\Gamma_1} p\frac{\partial u}{\partial n}\, T\, ds.$$

Next let $v = T$ in (8). This gives

(11) $$\iint_S qu\, T\, dA = \int_{\Gamma_1} p\frac{\partial u}{\partial n}\, T\, ds.$$

<u>Lemma 0</u>: <u>If the function</u> u <u>satisfies the Euler differential equation</u> (1) <u>corresponding to the saddle function</u> $E(p,u)$ <u>and</u> p <u>and</u> u <u>satisfy the boundary conditions</u> (2) <u>and</u> (3), <u>then</u>

(12) $$\mathcal{H}(u) = T^{-1}E(p,u)$$

<u>where</u> $\mathcal{H}(u)$ <u>is the heat dissipated per unit time</u>.

Proof: This follows directly from (10) and (11).

Note that Lemma 0 is proved only for Newton's law of cooling. For any other mode of cooling this proof will not carry through.

In particular, it does not apply for the Stefan-Boltzmann T^4 cooling law.

In view of relations (9) and (12) we pose an equivalent problem:

Problem 2: Find

(13) $$H = T^{-1} \max_{P} \min_{U} E(P,U)$$

subject to the constraints that $U = T$ on Γ_1 and that the weight $\mathcal{K}(P)$ is a constant K.

Thus the original maximizing problem has been replaced by a minimax problem. We continue the heuristic analysis and investigate this minimax problem.

Again consider the smooth functions p and u which satisfy the boundary value problem (1), (2), and (3) in the region bounded by the smooth curves Γ_1 and Γ_2. Given $\epsilon > 0$ let $\rho(x,y)$ be a smooth function satisfying

(14) $$\iint_S \rho \, dA = 0$$

(15) $$\rho(x,y) = 0 \quad \text{if} \quad p(x,y) < \epsilon.$$

Define

(16) $$P(x,y,t) = p(x,y) + t\rho(x,y)$$

Clearly $\iint_S P(x,y,t)\,dA = K$ and $P(x,y,t) \geq 0$ for $|t|$ sufficiently small. Let $u(x,y,t)$ be the solution of the boundary value problem for the fin of thickness P. Differentiating $E(P,u)$ with respect to t gives

$$\frac{dE}{dt} = \iint_S \rho|\nabla u|^2 dA + 2 \iint_S [P\nabla u \cdot \nabla u' + quu']\,dA$$

where $u' = \frac{du}{dt}$. For $t = 0$ let $u' = v$ and we have

$$\left.\frac{dE}{dt}\right|_{t=0} = \iint_S \rho |\nabla u|^2 dA + 2 \iint_S [p \nabla u \cdot \nabla v + quv] dA$$

On Γ_1 $u = T$ for all values of t and therefore $v = u' = 0$ on Γ_1. It follows, by the same reasoning as was applied to (8), that the second integral on the right vanishes, and we have

$$\left.\frac{dE}{dt}\right|_{t=0} = \iint_S \rho |\nabla u|^2 dA \tag{17}$$

If p and u solve the minimax problem then E is a maximum for $t = 0$ and $\left.\frac{dE}{dt}\right|_{t=0} = 0$. Thus

$$\iint_S \rho |\nabla u|^2 dA = 0 \tag{18}$$

It is then an easy deduction from (14), (15), and (18) that there is a constant C such that

$$|\nabla u|^2 = C^2 \tag{19}$$

wherever $p(x,y) \geq \epsilon$. Since ϵ is an arbitrary positive number it follows that (19) holds in all of S.

The above heuristic analysis suggests that <u>the optimum cooling fin has the thickness so tapered that the magnitude of the temperature gradient is constant</u>. A fin so tapered will be called a <u>constant gradient fin</u>.

If the solution of the equation $|\nabla u|^2 = C^2$ which satisfies the boundary condition $u = T$ on Γ_1 can be found, then it can be substituted in the differential equation (1), and the resulting differential equation can be solved for the thickness function p. This is accomplished using standard methods of solving first order partial differential equations in Appendix A.

There is a common geometric construction which leads to a constant gradient fin. We assume that the curve Γ_1 is convex and has a continuous curvature function. These hypotheses ensure that an outward normal exists at each point of Γ_1 and that no two of these normals intersect. Let ρ denote the distance along a normal. Let ν denote the counter-clockwise distance of a point on Γ_1 from some fixed point on Γ_1. Then (ρ, ν) constitute a coordinate system for the region exterior to Γ_1. We call the straight lines ν=constant streamlines and the curves ρ = constant isothermals. Given a positive constant C we define the temperature as

(20) $$u = T - C\rho,$$

and u so defined satisfies $|\nabla u|^2 = C^2$ and the boundary condition $u = T$ on Γ_1. Let Γ_2 be the isothermal $\rho = TC^{-1}$ so that $u = 0$ on Γ_2. The geometry insures that Γ_2 is a convex curve with a continuous curvature function. The isothermals and streamlines are shown as dotted lines in Figure 1.

We now choose p so that the differential equation (1) is satisfied. A simple analysis leads to the ordinary differential equation

(21) $$\frac{dp}{d\rho} + kp = -q(TC^{-1} - \rho)$$

where k is the curvature of the isothermal at the point in question. From (20) we see that $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial \rho} = -C$ on Γ_2. In order to satisfy the boundary condition on Γ_2, $p\frac{\partial u}{\partial n} = 0$, we must have

(22) $$p = 0 \text{ on } \Gamma_2, \text{ i.e. } \rho = TC^{-1}.$$

The solution of (21) subject to the boundary condition (22) is of the form

$$p = L^{-1}(\rho) \int_{\rho}^{m} L(\tau)\, q(m - \tau)\, d\tau \tag{23}$$

where $m = TC^{-1}$ and $L(\rho) = \exp(\int k \, d\rho)$.

Thus we see that for $\rho \leq TC^{-1}$ $p \geq 0$, p is a smooth function, and the boundary condition on Γ_2 is satisfied. This is an outline of the proof of the existence of a constant gradient fin. The explicit formulae in terms of parametric equations for Γ_1 are given in Appendix A.

The constant C is a parameter in the equation for p, (23). It is chosen so that the mass of the fin, defined by (5), is K. It is shown in Appendix B that there exists a unique C for any $K > 0$.

We have shown that a constant gradient fin exists. The heuristic considerations indicate that it is optimal, and this will be proved in Section 3.

3. Comparison Relations and the Main Proof.

In this section we propose to use rigorous arguments to obtain the results obtained heuristically in the preceding section. It is shown in Appendix A that a constant gradient fin exists, and an expression for the thickness of the fin for which $u = 0$ on Γ_2, the outer boundary of the fin, is obtained. This particular constant gradient fin, with zero temperature on the outer boundary, will be denoted hereafter as the CGO-fin. It is also shown in Appendices A and B that the thickness function for the CGO-fin is unique.

In general, given $u = T - C\rho$, if Γ_2 is any simple closed curve, the interior of which contains Γ_1, and which is contained in the region bounded by Γ_1 and the curve $u = 0$, then a thickness function for a constant gradient fin may be determined just as was done for the CGO-fin. Any choice of Γ_2 satisfying the above conditions will give a constant gradient fin, but the following lemmas and theorems show that the CGO-fin is optimum.

It is desirable to relax some of the restrictions of Problem 2. These relaxed restrictions are stated in the following problem:

Problem 3: Given Γ_1, a simple, closed, convex curve with a continuous curvature function, find

$$H = T^{-1} \sup_{P} \inf_{U} E(P,U) \tag{24}$$

subject to the following constraints: $P \geq 0$; P has finite support; P is continuous with piecewise continuous first derivatives in its support; the region of integration is the support of P, and the integral of P over this region is a constant K;

the boundary of the region consists of a finite number of regular arcs; the functions U are continuous, have piecewise continuous first derivatives, and on Γ_1 have a constant value $T > 0$.

Figure 2 shows the superposition of two cooling fins. Region S, the region between curves Γ_1 and Γ_2, corresponds to the CGO-fin. Region R, the region between curves Γ_1 and Γ_3, corresponds to another fin which has R as support.

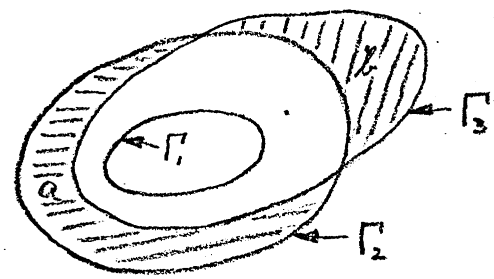

Figure 2

LEMMA 1: Let p_o and u_o be the thickness and temperature functions for the CGO-fin for which the magnitude of the gradient is C. Let P be an arbitrary thickness function. Then the saddle functional satisfies

$$(25) \qquad E(p_o,u_o) = E(P,u_o) + C^2 \iint_b P\,dA + \iint_a q u_o^2\,dA$$

where $a = S - R$, $b = R - S$, S is the support of p_o, R is the support of P, and u_o is defined to be zero outside of S.

Proof: Since $|\nabla u_o|^2 = C^2$ and $\iint_S p_o\,dA = K$ we have

$$E(p_o,u_o) = C^2 K + \iint_S q u_o^2\,dA.$$

Let $d = S \cap R$ so that $\iint_d P\,dA + \iint_b P\,dA = K.$

$$E(p_o,u_o) = \iint_d P|\nabla u_o|^2 dA + C^2 \iint_b P\,dA + \iint_d q u_o^2\,dA + \iint_a q u_o^2\,dA$$

$$= E(P,u_o) + C^2 \iint_b P\,dA + \iint_a q u_o^2\,dA.$$

THEOREM 1: The CGO-fin is optimum.

Proof: Suppose that Problem 3 has a minimax value H' and $H' > H$, where $H = T^{-1}E(p_o,u_o)$ for the CGO-fin. Then there is a sequence $\{P_n\}$ such that $\inf_u E(P_n,u) \rightarrow H'T$. Thus we can find $P = P_n$ for some n and a corresponding U such that

$$(26) \qquad TH < E(P,U) \leq E(P,u_o)$$

Then by Lemma 1 we have

$$(27) \quad E(p_o,u_o) = E(P,U) + [E(P,u_o) - E(P,U)] + c^2 \iint_b PdA + \iint_a qu_o^2 dA$$

But by (26) all terms on the right are non-negative which contradicts the assertion $E(p_o,u_o) = TH < E(P,U)$. This proves that the CGO-fin is optimum.

Theorem 1, which proves the optimum property of the CGO-fin, does not hold for an arbitrary constant gradient fin because for such a fin u will not be zero on Γ_2 and extending u as zero outside the support of the arbitrary constant gradient fin results in a discontinuous temperature function. Therefore, it is not a member of the class of functions over which the minimum is sought, and the second inequality in (26) no longer follows.

Having shown that the CGO-fin is optimum we turn our attention to demonstrating that it is the only optimum fin, i.e., the temperature function which gives the solution is unique.

THEOREM 2: The CGO-fin is the only optimum fin.

Proof: Let p_o and u_o be the thickness and temperature functions for the CGO-fin. Assume that P is another admissible thickness function, that U is the temperature function which minimizes

$E(P,u)$, and that $E(P,U) = E(p_o,u_o)$. Then $E(P,u_o) \geq E(P,U)$ and from (27)

$$[E(P,u_o) - E(P,U)] + c^2 \iint_b P\, dA + \iint_a qu_o^2 dA = 0,$$

where each of the three terms is non-negative. Hence

$$E(P,u_o) = E(P,U) \tag{28}$$

$$\iint_b P\, dA = 0 \tag{29}$$

$$\iint_a qu_o^2 dA = 0 \tag{30}$$

From (29) and (30) it follows that $a = b = 0$ so that P and p_o have the same support S. Now consider $E(P,u)$ as a quadratic function for u defined in S. By the parallelogram law for quadratic functionals

$$E(P,U - u_o) = 2E(P,U) + 2E(P,u_o) - 4E(P,w) \tag{31}$$

where $w = (U + u_o)/2$. Clearly w satisfies the boundary condition $w = T$ on Γ_1 so that $E(P,w) \geq TH$, where H is the optimum rate of heat dissipation. However, according to relation (28) we have $E(P,u_o) = E(P,U) = TH$ so that (31) gives

$$E(P,U - u_o) \leq 0.$$

Since E is a positive definite quadratic form this implies that $U = u_o$. It is shown in the Appendices that the thickness function for the fin with temperature function u_o is unique. Therefore, the CGO-fin is the only optimum fin.

4. Formulae for the Optimum Taper.

It is a direct consequence of what was shown in the preceding sections together with Appendices A and B that the optimum fin has thickness defined by (11 a) (a- and b-equations are in Appendix A and B respectively) and that the temperature in the fin is given by (6a), where the variables ρ and ν are defined by (3a) and (4a).

The infimum and supremum in the formulation of Problem 3 are actually assumed for the thickness and temperature functions of the CGO-fin. Hence the solution of Problem 3 is also a solution to Problem 2. The thickness and temperature functions and Γ_2 are sufficiently smooth that the use of Green's theorem in Section 1 is valid. It follows that the solution of the mini-max problem is the maximum heat which can be dissipated per unit time by a fin of mass K and is given by

(4) $$H = \iint_S q u_o \, dA$$

where S is the support of p_o. Substituting for u_o and changing variables gives

(4') $$H = \iint_S q(T - C\rho)(F\rho + 1)\, d\rho d\nu .$$

If q is constant the integrations in (11a), (5), and (4') can be carried out and explicit formulae obtained for p_o, C, and H. The results for constant and arbitrary q are summarized in the following theorem.

THEOREM 3: Let Γ_1 be a simple closed curve with a continuous curvature function which encloses a convex region and which is parameterized by $x = f(\nu)$, $y = g(\nu)$, where ν is the arc length on Γ_1. The thickness function p_o for the fin which dissipates the maximum amount of heat per unit time in the class of all fins which have Γ_1 as their inner boundary, temperature T on Γ_1, thickness functions with finite support which are continuous and piecewise continuously differentiable in their support and satisfy

(5) $$\iint_S p \, dA = K,$$

where S is the support of p, is given by

(11a) $$p_o(\rho,\nu) = \begin{cases} (F\rho + 1)^{-1} \displaystyle\int_\rho^{TC^{-1}} q(TC^{-1} - \tau)(F\tau + 1)\,d\tau, & 0 \le \rho \le TC^{-1} \\ 0, & TC^{-1} < \rho \end{cases}$$

Here ρ and ν are coordinates defined by

(3a) $$x(\rho,\nu) = g'(\nu)\rho + f(\nu)$$

(4a) $$y(\rho,\nu) = -f'(\nu)\rho + g(\nu)$$

and

$$F = f'(\nu)g''(\nu) - f''(\nu)g'(\nu)$$

The temperature function in this optimum fin is

(6a) $$u_o(\rho,\nu) = \begin{cases} T - C\rho \quad , & 0 \le \rho \le TC^{-1} \\ 0 \quad , & TC^{-1} < \rho \end{cases}$$

The value of C is determined by (5). The maximum rate of heat dissipation is

(4') $$H = \iint_S q(T - C\rho)(F\rho + 1)\,d\rho d\nu$$

where

$$S = \{(\rho,\nu) \mid 0 \le \rho \le TC^{-1}, \ 0 \le \nu \le L\}$$

and L is the arc length of Γ_1. If q is constant these reduce to

$$(11a') \quad \psi_0(\rho,\nu) = q(F\rho + 1)^{-1}\left[\frac{T^3F}{6C^3} + \frac{T^2}{2C^2} - \frac{T}{C}\rho - \left(\frac{FT + C}{2C}\right)\rho^2 + \frac{F}{3}\rho^3\right]$$

$$(4'') \quad H = q\left(\frac{TL}{2C} + \frac{T^2\pi}{3C^2}\right)$$

and C is the root of

$$(32) \quad \left(\frac{T}{C}\right)^4 + \frac{L}{\pi}\left(\frac{T}{C}\right)^3 - \frac{6K}{q\pi} = 0.$$

5. Solution of the Dual Problem.

The solution of the problem of tapering a fin of fixed mass so that it dissipates the maximum amount of heat per unit time from a convex cylinder with constant surface temperature is given in Theorem 3. The dual problem, which is Problem 0, tapering a fin so that it has the minimum mass of all those fins which dissipate a fixed amount of heat per unit time, is now easily solved.

LEMMA 2: H is a decreasing function of C.

Proof:
$$H = \iint_S q(T - C\rho)(F\rho + 1)\,d\rho d\nu .$$

If C increases the integrand decreases and the region of integration decreases. Hence H decreases.

LEMMA 3: H is an increasing function of K.

Proof: From Lemma 2B (in Appendix B) C is a decreasing function of K. By Lemma 2, H is a decreasing function of C. Therefore, H is an increasing function of K.

Define H as a function of K, $H = h(K)$. Then $h(K)$ is an increasing function.

THEOREM 4: For a fixed rate of heat dissipation H_o, the minimum weight of a fin which will have this rate of heat dissipation is the unique solution of $H_o = h(K)$.

Proof: The solution of $H_o = h(K)$ is unique because $h(K)$ is an increasing function. Let K_o be this solution. Assume that there exists a $K_1 < K_o$ and an admissible thickness function p such that the rate of dissipation is H_o. From the definition of

$h(K)$, the maximum rate of heat dissipation for a given weight K, it follows that $h(K_1) \geq H_o = h(K_o)$. But, since $h(K)$ is an increasing function of K, $h(K_1) < h(K_o)$. This is a contradition. Therefore K_o, the solution of $H_o = h(K)$, is the minimum weight of a fin which will give the rate of heat dissipation H_o.

The thickness function for the optimum fin of the dual problem is given by (11a), where C is determined by the condition

$$\iint_S p\,dA = K_o.$$

6. Related Papers.

Various papers have been written on the problems related to the optimum straight fin. Liu [4] considered the optimum fin problem for a mode of cooling somewhat more general than Newton's law. Wilkins [5,6] solved the problem for an arbitrary cooling mode. Minkler and Rouleau [7], Liu [8], and Wilkins [6] considered the straight fin with internal heat generation. A method of determining upper and lower bounds for problems of this type has been found by Appl and Hung [9].

All of the above papers consider a single fin. The interaction of a fin with other fins and with the base surface is considered by Heaslet and Lomax [10], Sparrow, Eckert, and Irvine [11], and Sparrow, Miller, and Jonsson [12].

It is sometimes thought that the constant gradient fin is the optimum fin for an arbitrary cooling mode. However, Wilkins [6] shows that for a straight fin Newton's law of cooling is the only mode of cooling for which the constant gradient fin is optimum. The results of our paper hold only for Newton's law of cooling because it is basic in our proof of Lemma 0.

Appendix A: Constant Gradient Fins.

We wish to solve $|\nabla u|^2 = c^2$ subject to the boundary condition $u = T$ on Γ_1. In scalar notation, this is

(1a) $$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = c^2.$$

Let Γ_1 be described parametrically by

(2a) $$x = f(\nu), \ y = g(\nu), \quad 0 \leq \nu \leq L$$

where ν is the arc length measured counterclockwise from some reference point, L is the length of Γ_1, and f and g periodic functions of period L.

The application of a standard technique for solving first order partial differential equations [3] leads to the consideration of the change of variables

(3a) $$x = g'(\nu)\rho + f(\nu)$$

(4a) $$y = -f'(\nu)\rho + g(\nu)$$

where $0 \leq \nu \leq L$, and $\rho \geq 0$. It can be seen geometrically in Figure 3 that this is a valid change of variables in the exterior of Γ_1.

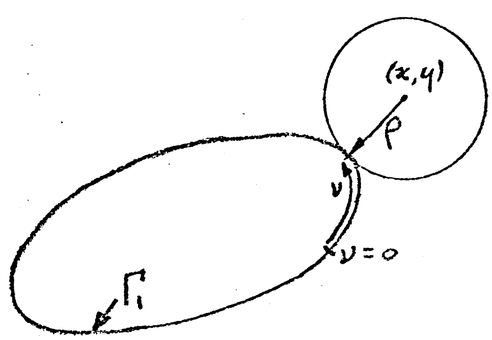

Figure 3

Given a point (x,y) in the exterior of Γ_1 the corresponding (ρ,ν) coordinates can be determined by constructing the circle with center (x,y) and minimum radius which intersects Γ_1. The intersection point will be unique because Γ_1 is convex. Then the value of ν is the arc length from the point $\nu = 0$ on Γ_1 to the point of intersection, and the value of ρ is the radius of the constructed circle. Analytically, the validity is shown by the fact that the Jacobian is never zero. When the calculations are carried out we obtain

(5a) $$\frac{\partial(x,y)}{\partial(\rho,\nu)} = (f'g'' - f''g')\rho + 1.$$

Noting that $(f'g'' - f''g')$ is the curvature of Γ_1, and that this curvature is always positive for a convex curve with the arc length increasing in the counterclockwise direction, we conclude that $\frac{\partial(x,y)}{\partial(\rho,\nu)} \neq 0$ in the exterior of Γ_1. Hence the change of variables $(x,y) \rightarrow (\rho,\nu)$ is valid and transforms the exterior of Γ_1 in the $x - y$ plane into $R = \{(\rho,\nu) \mid \rho \geq 0,\ 0 \leq \nu \leq L\}$ in the $\rho - \nu$ plane.

From (3a) and (4a) it is easily shown that a line of constant ν is a straight line perpendicular to Γ_1, and that a line of constant ρ is a curve at a distance ρ from Γ_1.

Consider the function

(6a) $$u_o(\rho,\nu) = \begin{cases} T - C\rho, & 0 \leq \rho \leq TC^{-1} \\ 0 \quad , & \rho > TC^{-1} \end{cases}$$

The function u_o is continuous, piecewise continuously differentiable, satisfies (1a) for $0 < \rho < TC^{-1}$, and satisfies the boundary condition $u_o = T$ on Γ_1.

We have a constant gradient temperature function, and we seek an admissible thickness function p_o such that u_o minimizes $E(p_o,u)$. Since u_o has continuous second partial derivatives for $0 < \rho < TC^{-1}$, it is a standard result of the calculus of variations that the Euler equation of the functional $E(p_o,u)$ will be satisfied by u_o and p_o, and that the natural boundary condition, $p_o \frac{\partial u_o}{\partial n} = 0$, will be satisfied on Γ_2, the curve $\rho = TC^{-1}$, where n is the outward normal to Γ_2. For the given geometry $\frac{\partial u_o}{\partial n} = \frac{\partial u_o}{\partial \rho} = -C \neq 0$, so that $p_o = 0$ on Γ_2. But the Euler equation is just the differential equation (1), and therefore, we are looking for a function p_o satisfying

(7a) $$\frac{\partial}{\partial x}\left(p_o \frac{\partial u_o}{\partial x}\right) + \frac{\partial}{\partial y}\left(p_o \frac{\partial u_o}{\partial y}\right) - qu_o = 0$$

(8a) $$p_o = 0 \quad \text{on} \quad \Gamma_2.$$

Changing variables $(x,y) \longrightarrow (\rho,\nu)$ transforms (7a) and (8a) into

(9a) $$\frac{\partial p_o}{\partial \rho} + \frac{F}{F\rho + 1} p_o = -q(TC^{-1} - \rho)$$

(10a) $$p_o = 0 \quad \text{on the curve} \quad \rho = TC^{-1}$$

where $F(\nu) = f'(\nu)g''(\nu) - f''(\nu)g'(\nu)$, the curvature of Γ_1. It can be shown that $\frac{F}{F\rho + 1}$ is the curvature of the curve which is a distance ρ from Γ_1 so that (9a) agrees with (21) which was derived heuristically. Solving (9a) subject to (10a) gives

(11a) $$p_o(\rho,\nu) = (F\rho + 1)^{-1} \int_{\rho}^{TC^{-1}} q(TC^{-1} - \tau)(F\tau + 1)\,d\tau$$

for $0 \leq \rho \leq TC^{-1}$. For $\rho > TC^{-1}$ we take $p_o = 0$.

From the theory of ordinary differential equations the solution of (9a) is unique for a given value of C. Hence we have found a thickness function p_o, unique up to the choice of C, which has the property that u_o minimizes $E(p_o,u)$. C must be chosen so that the total mass of the fin is K, i.e.,

$$\int_{\nu=0}^{L} \int_{\rho=0}^{TC^{-1}} p(\rho,\nu)\, dA = K. \tag{12a}$$

The question of a unique solution for C for any given positive number K is discussed in Appendix B.

Appendix B: The Determination of the Gradient Constant C.

Equation (11a) gives a thickness function for the constant gradient fin and it is unique up to the choice of the constant C. The existence of a unique C determined by

$$\iint_S p_o \, dA = K, \tag{5}$$

where S is the support of p_o, i.e., $S = \{(\rho,\nu) \mid 0 \leq \rho < TC^{-1}, 0 \leq \nu \leq L\}$, is now shown.

LEMMA 1B: At each point of S, $p_o(\rho,\nu)$ is a decreasing function of C.

Proof: Differentiating (11a) w.r.t. C gives

$$\frac{dp_o}{dC} = \frac{-T}{C^2(F\rho + 1)} \int_{\rho}^{TC^{-1}} q(F\tau + 1)\,d\tau < 0 \quad \text{in} \quad S \tag{1b}$$

since $T > 0$, F is non-negative, and q is positive everywhere. Therefore, at each point of S, p_o is a decreasing function of C.

LEMMA 2B: C is a decreasing function of K.

Proof: If K is increased, $\iint_S p_o \, dA$ must increase, and by Lemma 1B and the definition of S this can occur only if C is decreased.

LEMMA 3B: As $C \to \infty$, $p_o(\rho,\nu) \to 0$ for all (ρ,ν).

Proof: $p_o(0,\nu) > p_o(\rho,\nu)$ for $\rho > 0$.

$$p_o(0,\nu) = \int_0^{TC^{-1}} q(TC^{-1} - \tau)(F\tau + 1)\,d\tau$$

$$< \int_0^{TC^{-1}} qTC^{-1}(FTC^{-1} + 1)\,d\tau$$

and this last integral approaches zero as $C \to \infty$. Hence $p_o(0,\nu) \to 0$ as $C \to \infty$ which implies $p_o(\rho,\nu) \to 0$ as $C \to \infty$.

<u>LEMMA</u> 4B: <u>As</u> $C \to 0$, $p_o(\rho,\nu) \to \infty$ <u>for all</u> (ρ,ν).

<u>Proof</u>: Choose an arbitrary point $(\bar{\rho},\bar{\nu})$. For $C < T\rho^{-1}$, $(\bar{\rho},\bar{\nu}) \in S$, and $p_o(\bar{\rho},\bar{\nu})$ is given by (11a). As $C \to 0$ the region of integration and the integrand becomes arbitrarily large. Hence $p_o(\bar{\rho},\bar{\nu}) \to \infty$. Since $(\bar{\rho},\bar{\nu})$ is an arbitrary point, it follows that $\lim_{C \to 0} p_o(\rho,\nu) = \infty$ for all (ρ,ν).

<u>THEOREM</u> 1B: <u>For every positive number</u> K <u>there exists a unique</u> C <u>such that</u>

$$\iint_S p_o \, dA = K.$$

<u>Proof</u>: As $C \to \infty$ area $S \to 0$. From this and Lemma 3B, it follows that $\lim_{C \to \infty} \iint_S p_o \, dA = 0$. From Lemma 4B, it follows that $\lim_{C \to 0} \iint_S p_o \, dA = \infty$. Hence for any positive K there exists at least one C such that $\iint_S p_o \, dA = K$. By Lemma 2B, this C is unique.

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