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Optimum Heat Transfer
and Network Programming

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Abstract

Of concern is a network in which the conductance of certain branches are variable. The problem posed is the maximization of the input conductance of the network under the constraint that the sum of the branch conductances has a fixed bound. It is shown that at the optimum state the conductance of the variable branches should be proportional to the current through them. This property leads to inequalities which serve to give a numerical estimate of the maximum input conductance.

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Optimum Heat Transfer and Network Programming

1. Introduction

A general problem of heat transfer is the design of machinery so that the structure can carry away excess heat without exceeding weight limits. For example cooling fins are used on the cylinders of air cooled engines and the design problem is to determine the optimum shape of the fins. This problem was studied in two previous papers [2] and [7]. In this paper the machine is treated as a lumped network having a finite number of conducting branches. Certain branches are allowed to vary but the total weight must not exceed a given limit. The problem is to maximize the joint conductance Γ of the network subject to the constraints.

The resulting mathematical problem may be characterized as a maximizing problem of non-linear programming. This suggests that there is a dual minimizing problem. Following such a line of investigation leads to inequalities giving upper and lower bounds for the joint conductance Γ . These bounding inequalities are suitable for giving an estimate of Γ .

A continuous system may be regarded as a limiting case of a lumped network and there should be analogies between the properties of the continuous systems and the properties of the discrete systems. For example it was shown in reference [2] and [7] that the optimum cooling fin has the property that the magnitude of the temperature gradient is a constant. In this paper an analogous property is found to hold for the optimum networks. The analogy is that the ~~temperature~~ difference across the variable branches is constant.

Voltage

Analogy is a two-edged sword and so there is expectation that network analysis should bring to light new properties of the continuous system. This question is examined in the last section of the paper. It is found that the bounding inequalities for the maximum conductance of the network can be carried over to give bounding inequalities for the maximum efficiency of a cooling fin.

The mathematical model for the steady flow of electricity is essentially the same as the model for the steady flow of heat. However, the concepts and terminology are better developed for electrical flow. For this reason electrical terminology will now be adhered to.

2. Some network properties.

An electrical network may be depicted as a graph diagram with m nodes and n arcs. An 'arc' corresponds to a 'branch' of the network. Suppose each arc connects two distinct nodes. Let the nodes be designated by the integers $(1, \dots, m)$ and let the arcs be designated by the integers $(1, \dots, n)$. A direction is assigned to each arc. Suppose arc s connects nodes i and j and that the positive direction is from i to j . If u_i and u_j are the potentials of node i and node j then the branch voltage v_s of branch s is defined as the potential difference

$$(1) \quad v_s = u_i - u_j$$

Thus, assigning potentials to the nodes automatically assigns voltages to the branches.

Let currents y_1, \dots, y_n be assigned to the branches. Then the current source w_i at node i is defined as

$$(2) \quad w_i = \sum_1 y_s - \sum_2 y_s .$$

Here \sum_1 denotes a sum over the arcs starting at node i , and \sum_2 denotes a sum over the arcs terminating at node i . Thus, assigning currents to the branches uniquely assigns current sources to the nodes. The following lemma is seen to be a direct consequence of equations (1) and (2).

Lemma 1. Let $\{u_i\}$ be an arbitrary assignment of node potentials and let $\{y_i\}$ be an arbitrary assignment of branch currents. Then

$$(3) \quad \sum_1^m w_i u_i = \sum_1^n y_s v_s .$$

The common value is termed the power.

The network inequality stated in the following lemma was proved in [3] but the proof is simple and will be repeated here.

Lemma 2. Let the voltages (v_1, v_2, \dots, v_n) of the branches arise from an arbitrary assignment of node potentials (u_1, u_2, \dots, u_m) . Let the current sources (w_1, w_2, \dots, w_m) at the nodes arise from an arbitrary assignment of currents (y_1, y_2, \dots, y_n) through the branches. Then

$$(4) \quad \left(\sum_1^m w_i u_i \right)^2 \leq \sum_1^n g_s v_s^2 \sum_1^n g_s^{-1} y_s^2$$

where (g_1, \dots, g_n) is a set of positive constants. This is an equality if and only if y_s and $g_s v_s$ are proportional.

Proof. Making use of Lemma 1 and the Cauchy inequality gives

$$\left(\sum_1^m w_i u_i \right)^2 = \left(\sum_1^n g_s^{1/2} v_s g_s^{-1/2} y_s \right)^2 \leq \sum_1^n g_s v_s^2 \sum_1^n g_s^{-1} y_s^2.$$

There is an equality under the condition stated so the proof is complete.

It is to be noted that the inequality (4) is valid if some of the g_s vanish provided the corresponding y_s also vanish. The constants $\{g_s\}$ are termed branch conductances. Ohm's law is satisfied if

$$(5) \quad y_s = g_s v_s.$$

If this relation holds for all branches we shall say that there is an equilibrium state. In an equilibrium state it is seen that relation (4) is an equality rather than an inequality. Moreover, it is a corollary of Lemma 1 that

$$(6) \quad \sum_1^m w_i u_i = \sum_1^n g_s v_s^2 = \sum_1^n g_s^{-1} y_s^2$$

at equilibrium.

In the equilibrium state it follows from (2) and (5) that

$$(7) \quad w_i = u_i \sum_s g_s - \sum_s u_{is} g_s$$

where \sum_s denotes a sum over these arcs s meeting node i and u_{is} is the potential at the other node of such an arc. If $w_i = 0$ then it is said that node i is insulated. It follows from (7) that if node i is insulated and $\sum_s g_s \neq 0$ then

$$(8) \quad u_i = \frac{\sum_s u_{is} g_s}{\sum_s g_s}$$

In other words the potential at an insulated node is an arithmetic mean of the potentials of other nodes.

A situation of central interest in this paper arises when the potential of nodes 1 and 0 are given the values $u_1 = 1$ and $u_0 = 0$ and all other nodes are insulated. Then the input conductance between nodes 1 and 0 is defined as $\gamma = w_1$. It then follows from (6) that

$$(9) \quad \gamma = \sum_1^n g_s v_s^2$$

The solution of the input conductance problem can be obtained from a minimum principle of Maxwell which is stated here as the following lemma.

Lemma 3. Suppose that some of the nodes of a network have pre-scribed potentials and that other nodes are insulated. Then there is an equilibrium state in which the potentials of the insulated nodes take on values to minimize the power function

$$(10) \quad E = \sum_1^n g_s v_s^2$$

Moreover, the branch power $g_s v_s^2$ is uniquely determined for each branch.

Proof. Clearly E has a minimum and at the minimum relation (8) is satisfied. The uniqueness statement can be deduced from Lemma 2.

Lemma 4. The input conductance $\gamma(g)$ as a function of the branch conductance g_j satisfies the following conditions:

- (a) It is a homogeneous function of degree 1.
- (b) It is a concave function.
- (c) It is non-decreasing.

Proof. Property (a) is a direct corollary of Lemma 3.

To prove (c) let $\gamma(g')$ and $\gamma(g'')$ be the joint conductances corresponding to branch conductances g' and g'' respectively.

Let $g = pg' + qg''$ where $p + q = 1$, $p \geq 0$, and $q \geq 0$. Let u be an equilibrium state of potential such that $u_0 = 0$ and $u_1 = 1$, then the joint conductance $\gamma(g)$ is given by $\gamma(g) = E(g, v)$. Thus

$$\gamma(g) = E(pg' + qg'', v) = pE(g', v) + qE(g'', v).$$

But $E(g', v) \geq \gamma(g')$ and $E(g'', v) \geq \gamma(g'')$ by virtue of Lemma 3. Thus

$$\gamma(pg' + qg'') \geq p\gamma(g') + q\gamma(g'')$$

and this proves that $\gamma(g)$ is concave.

To prove that $\gamma(g)$ is non-decreasing the above proof is repeated for $p = q = 1$. This gives

$$\gamma(g' + g'') \geq \gamma(g') + \gamma(g'') .$$

Since $\gamma(g'') \geq 0$ for arbitrary $g'' \geq 0$ the proof is complete.

Two nodes of a network are said to be positively connected if it is possible to travel from one to the other along a chain of branches with positive conductance. A network is said to be positively connected if any two nodes are positively connected.

Lemma 5. If a network is positively connected

$$(11) \quad \frac{\partial \gamma(g)}{\partial g_s} = (v_s)^2$$

where v is an equilibrium voltage corresponding to unit input voltage.

Proof. Let $u_0 = 0$ then it is clear that E is a positive definite quadratic form in the potentials u_1, \dots, u_n if the network is positively connected. Thus equations (8) may be solved giving the potentials of the insulated nodes as rational functions of the branch conductances. In particular, it follows that $\partial v_s / \partial g_t$ exists. So

$$\frac{\partial \gamma}{\partial g_t} = v_t^2 + 2 \sum_1^n g_s v_s \frac{\partial v_s}{\partial g_t}$$

The second summation vanishes by virtue of Lemma 3 because $\partial v_s / \partial g_t$ may be regarded as a perturbation of voltage which does not change the input voltage.

It is worth noting that since v_s is given as a rational function of g it follows from Lemma 5 that $\partial \gamma / \partial g_s$ is a 'perfect square'. This leads to interesting algebraic questions [1].

3. Maximization of the input conductance.

The central question of this note may be phrased as follows.

Problem I. Let $\gamma(g)$ be the joint conductance of a network with n branches. Let the branch conductances be fixed for a set of the branches denoted by A . Let B be the set complementary to A .

Find

$$(12) \quad \Gamma = \max \gamma(g)$$

subject to the constraints $g_s \geq 0$ and

$$(13) \quad \sum_B g_s \leq K$$

where K is a positive constant.

The constraints of this problem restrict g to a compact convex set S . By virtue of Lemma 4b the function $\gamma(g)$ is continuous and so there is a point g' where $\gamma(g)$ takes on the maximum value in S . This proves the first part of Theorem 1 to follow.

In the remainder of the paper it shall be assumed that the network is positively connected by the A branches alone. This simplifies the discussion but entails no essential loss of generality because disconnected networks can be treated a posteriori by continuity arguments.

Theorem 1. (a) Problem I always has a solution g' .

(b) If there are two solutions g' and g'' then $g = g'/2 + g''/2$ is also a solution.

(c) Branch voltages are unique.

Proof. To prove statement (b) observe that g is in the convex set S . Then by virtue of Lemma 4b

$$\gamma(g'/2 + g''/2) \geq \gamma(g')/2 + \gamma(g'')/2 = \Gamma$$

and so g is a solution.

To prove statement (c) let v, v' , and v'' be the equilibrium voltages corresponding to g, g' , and g'' respectively. Then

$$\begin{aligned} \Gamma &= E(g'/2 + g''/2, v) = E(g', v)/2 + E(g'', v)/2 \\ &\geq E(g', v')/2 + E(g'', v'')/2 = \Gamma. \end{aligned}$$

Hence it is clear

that $E(g', v) = E(g', v')$. Let $u_0 = 0$ then E is a positive definite quadratic form in the potentials u_1, u_2, \dots, u_n if the network is positively connected. Thus $v = v'$. Similarly, we find $v = v''$ so $v' = v''$ and the proof is complete.

Theorem 2. If g' solves Problem I then there exists a constant λ such that for the equilibrium voltage v'

$$(14) \quad |v'_s| \leq \lambda \quad \text{for } s \in B.$$

Moreover this is an equality if $g'_s > 0$.

Proof. Suppose that p and q are in the set B and that $g'_p > 0$ and $q'_q \geq 0$. Let $g_s = g'_s$ for $s \neq p$ or q . Let $g_p = g'_p - t$ and $g_q = q'_q + t$. Then by use of Lemma 5

$$\frac{d\gamma}{dt} = (v_q)^2 - (v_p)^2.$$

But γ is a maximum for $t = 0$ so $d\gamma/dt \leq 0$. This shows that

$$|v_q| \leq |v_p|. \quad \text{If } g'_q > 0 \text{ it follows by symmetry that } |v_p| = |v_q|.$$

This proves the theorem with $\lambda = |v_p|$.

Theorem 3. Let Γ be the maximum input conductance and let v be a voltage resulting from a potential which is arbitrary except for a unit potential difference at the input. Then

$$(15) \quad \Gamma \leq \sum_A g_s v_s^2 + K \max_B v_s^2 .$$

Moreover this becomes an equality for the equilibrium voltage.

Proof. Let g' be a solution of Problem I then according to Lemma 3

$$\Gamma \leq \sum_1^n g'_s v_s^2 = \sum_A g_s v_s^2 + \sum_B g'_s v_s^2$$

But $\sum_B g'_s v_s^2 \leq (\sum_B g'_s) \max_B v_s^2 \leq K \max_B v_s^2$. This proves the inequality of the theorem.

The case of equality is a direct consequence of Theorem 2.

Corollary 1. If g is an arbitrary set of conductances which satisfy the constraints then

$$(16) \quad \Gamma \leq \gamma(g) - \sum_B g_s \frac{\partial \gamma}{\partial g_s} + K \max_B \frac{\partial \gamma}{\partial g_s} .$$

This is an equality if $g = g'$, an optimum solution.

Proof. In relation (15) let v be the equilibrium solution according to the choice g . Then

$$(17) \quad \gamma = \sum_A g_s v_s^2 + \sum_B g_s v_s^2 \quad \text{and}$$

$$(18) \quad \frac{\partial \gamma}{\partial g_s} = v_s^2$$

Substituting (17) and (18) into (15) proves (16). The case of equality follows as in Theorem 3 and the proof is complete.

4. Application of electrical duality

As is well known there is a duality in the properties of electrical networks which comes about when current and voltage interchange roles. For this duality conductance and resistance must also interchange roles because conductance and resistance are reciprocals.

The previous section was concerned with upper bounds for input conductance. In this section the duality principle is applied to obtain upper bounds for input resistance. Of course this is equivalent to obtaining lower bounds for the conductance which is the main goal.

The following lemma is the electrical dual of Lemma 3.

Lemma 6. Let $\{w_i\}$ be prescribed current sources of a network. Then the equilibrium branch currents $\{y_s\}$ are uniquely determined by requiring that the power function

$$(19) \quad H = \sum_1^{n-1} g_s^{-1} y_s^2$$

be a minimum over the class of branch currents having the same currents sources $\{w_i\}$.

Proof. Of course it is required that $y_s = 0$ if $g_s = 0$. The proof that H is a minimum follows from the fact that inequality (4) of Lemma 2 becomes an equality if $y_s = g_s v_s$. The proof of uniqueness is also deduced from Lemma 2.

Theorem 4. Let Γ be the maximum joint conductance in Problem I.

Then

$$(20) \quad \Gamma^{-1} \leq \sum_A g_s^{-1} y_s^2 + K^{-1} (\sum_B |y_s|)^2$$

where $\{y_s\}$ is any set of branch currents such that the current source has unit magnitude at the input nodes and vanishes at the other nodes. Moreover this becomes an equality for the equilibrium state.

Proof. It follows from Lemma 6 that

$$(21) \quad \Gamma^{-1} \leq \sum_A g_s^{-1} y_s^2 + \sum_B g_s^{-1} y_s^2$$

for any choice of g_s and y_s consistent with the constraints.

In particular, choose g_s to satisfy

$$(22) \quad g_s \sum_B |y_s| = K |y_s|$$

with the understanding that $g_s = 0$ if $y_s = 0$. Summing (22) gives

$$\sum_B g_s \sum_B |y_s| = K \sum_B |y_s|.$$

This shows that $\sum_B g_s = K$ or $\sum_B g_s = 0$ so the constraint is satisfied. Substituting (22) in (21) proves the inequality (20).

If g and y solve Problem I it follows from Theorem 2 that (22) is an equality. Thus (20) is an equality and the proof is completed.

Corollary 2. If g is an arbitrary set of conductances which satisfies the constraints then

$$(23) \quad \Gamma^{-1} \leq \gamma^{-1} - \gamma^{-2} \sum_B g_s \frac{\partial \gamma}{\partial g_s} + K^{-1} \gamma^{-2} \left(\sum_B g_s \left(\frac{\partial \gamma}{\partial g_s} \right)^2 \right)^{1/2}.$$

This becomes an equality if g solves Problem I.

Proof. In relation (20) let y be the equilibrium solution corresponding to the choice g . Then

$$(24) \quad \gamma^{-1} = \sum_A g_s^{-1} y_s^2 + \sum_B g_s^{-1} y_s^2 \quad \text{and}$$

$$(25) \quad \frac{g_s^2}{\gamma^2} \frac{\partial \gamma}{\partial g_s} = y_s^2.$$

Substituting (24) and (25) into (20) proves (23). The case of equality follows as in Theorem 4 and the proof is complete.

There is another application of the duality principle which is worth noting. Thus it is to be expected that the relationships developed for Problem I will have analogs for the following problem.

Problem II. Find the maximum input resistance of a network subject to the constraint that the sum of the branch resistances in a certain set is bounded by a constant.

A general treatment of the electrical duality principle is given in a paper by Bott and the writer [1]. Presumably that paper gives the necessary machinery to extend the relationships developed here to networks which do not obey Kirchhoff's laws. The work of Dennis [4] and Minty [5] is also pertinent in this connection.

5. Cooling fins.

The lumped network problems just discussed are analogous to conduction problems for continuous systems. In particular this analogy will be developed here for cooling fins. Such fins are used to conduct heat away from machines to the ambient media.

The cooling fin problem is to maximize the conductance of a fin of limited weight K . This is to be accomplished by suitably tapering the fin. This problem has been treated rigorously in two previous papers [2], [7]. Here it is proposed to give a heuristic treatment based on the network model. This gives further insight which would be of value in a numerical estimation.

Again it is found convenient to employ electrical rather than thermal terminology and to treat the equivalent electrical problem. Consider a thin conducting plate R in the plane. Then the power input to the plate is

$$(26) \quad E = \iint_R [p(\nabla u)^2 + qu^2] ds.$$

where u is the electric potential, p is the specific conductance, and q is the leakage conductance to ground. It is supposed that ground is at zero potential. The boundary conditions are that $u = 1$ on the part ∂R_1 of the boundary of R and $p\partial u/\partial n = 0$ on the complementary part ∂R_2 of the boundary. Then E is equal to the conductance γ of the plate. Of course this assumes that u is the equilibrium potential which satisfies the differential equation

$$(27) \quad \nabla(p\nabla u) - qu = 0.$$

Problem III. Maximize the conductance Γ subject to the constraint

$$(28) \quad \iint_R p ds \leq K.$$

Without loss of generality it may be assumed that the variation in p is due to a variation in thickness of the plate. Thus we may interpret K as a measure of the weight of the plate. It is assumed that q may be a function of position but is not subject to variation.

Reasoning by analogy from Theorem 2 the optimum plate should be tapered so that

$$(29) \quad |\nabla u| = \lambda \quad \text{if } p > 0$$

for some constant λ . By analogy with Theorem 3 the optimum conductance will have the following upper bound,

$$(30) \quad \Gamma \leq \iint_R q u^2 ds + K \max_R |\nabla u|^2.$$

Here the shape of the region R is arbitrary except that its boundary includes the part ∂R_1 . The function u is arbitrary except that $u = 1$ on ∂R_1 .

A lower bound for Γ is furnished by the following analog of Theorem 4

$$(31) \quad \Gamma^{-1} \leq \iint_R \frac{(\nabla \cdot \mathbf{y})^2}{q} ds + K^{-1} \left(\iint_R |\mathbf{y}| ds \right)^2.$$

Again R is an arbitrary region and \mathbf{y} is a vector field corresponding to a current flow. The net flow across boundary ∂R_1 is unity and the current flow across ∂R_2 vanishes at all points. Otherwise \mathbf{y} is an arbitrary vector field. Presumably rigorous proofs of (30) and (31) could be given by methods employed in references [2] and [7].

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