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ON THE EXISTENCE, UNIQUENESS, AND STABILITY OF SOLUTIONS OF THE EQUATION $\sigma'(u_x)u_{xx} + \lambda u_{xtx} = \rho_o u_{tt}$

by

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1. Introduction

This paper presents a study of the mixed boundary initial value problem for the third-order partial differential equation

$$\rho_o u_{tt} = \sigma'(u_x)u_{xx} + \lambda u_{xtx}, \qquad (x,t) \in (0,1)\times(0,\infty), \qquad (E')^{\#,\#\#}$$

where λ and ρ_o are positive constants. It will be shown that if $\sigma'(\xi) > 0$ the problem is well set in the sense that there exists a unique solution which is stable with respect to perturbations in the initial data. Moreover, it will be shown that the solution decays to zero as t tends to infinity.

A physical prototype of the problem studied here arises when one considers purely longitudinal motions of a homogeneous bar of uniform cross-section and unit length. If we denote by x the position of a cross-section (which is assumed to move as a vertical plane section) in the homogeneous rest configuration of the bar, by u(x,t) the displacement at time t of the section from its rest position, by $\tau(x,t)$ the stress on the section at time t, and by $\rho_o > 0$ the constant density of points in the rest position, then the equation of motion becomes

$$\rho_o u_{tt} = \tau_x, \qquad (x,t) \in (0,1)\times(0,\infty). \qquad (1.1)$$

If one takes the ends of the bar (in our case the points 0 and 1) to be clamped for all times $t > 0$, then the displacement u must satisfy the

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#In the future (') will denote differentiation with respect to u_x.
##Arguments of functions, if not stated explicitly, will be x and t.

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auxiliary conditions

$$u(0,t) = u(1,t) = 0, \qquad t > 0. \tag{A}$$

One may now obtain a differential equation for the displacement u by making a specific assumption about the dependence of the stress τ on the displacement u. The simplest such assumption is Hooke's law which asserts that

$$\tau = E_o u_x, \tag{1.2}$$

where E_o is a positive constant. In this case (1.1) becomes the linear wave equation.

It would appear somewhat more realistic to allow for a nonlinear dependence of the stress τ on u_x. Thus one would assume that

$$\tau = \sigma(u_x). \tag{1.3}$$

Equation (1.1) then yields the nonlinear wave equation

$$\rho_o u_{tt} = \sigma'(u_x)u_{xx}, \qquad (x,t) \in (0,1) \times (0,\infty). \tag{1.4}$$

For a bar with clamped ends it would seem reasonable to seek a solution of (1.4) satisfying (A) together with the initial conditions

$$u(x,0) = f(x), \qquad 0 \leq x \leq 1, \tag{B}$$

$$u_t(x,0) = g(x), \qquad 0 \leq x \leq 1. \tag{C}$$

The problem (1.4), (A), (B), and (C) were considered in [1]. It was assumed there that σ satisfies the physically reasonable conditions

$$\sigma(0) = 0, \quad \text{and} \quad \sigma'(\xi) > 0, \quad \xi \in (-\infty,\infty). \tag{*}$$

Moreover, σ' was taken to be monotone decreasing in $|\xi|$. Rather surprisingly, the result was that the problem can have a global, smooth solution only if σ is a linear function as in (1.2). Otherwise, some second derivative of the solution must somewhere become infinite after a finite time.

In this paper we assume the material to be a nonlinear Kelvin solid; that is we assume a stress relation of the following form:

$$\tau = \sigma(u_x) + \lambda u_{xt}, \tag{1.5}$$

where λ is a positive constant which may be interpreted as a viscosity coefficient. Then (1.1) yields equation (E'). The conditions (A), (B), and (C) remain the same.

There are two considerations which suggest the modification (1.5). First, the inclusion of the strain rate term λu_{xt} begins to reflect the past history of the strain u_x. Thus (1.5) can be considered as a move toward the more general memory theories encountered in rational mechanics (see for example [2]). It appears to be the simplest possible model having this feature. Second, one can hope that (E') will lead to a "viscosity method" approach to equation (1.4). There is a conjecture concerning non-linear hyperbolic equations such as (1.4). This is that although a smooth solution may not be possible, under certain conditions there will always be a uniquely determined weak solution, that is one containing shocks. The problem is how to find this preferred weak solution. One idea that has been suggested is to add an artificial higher order derivative multiplied by a small parameter λ, solve the problem for $0 < \lambda < \lambda_1$, and then let λ tend to 0^+. The conjecture is that the limit function will be the appropriate weak solution. Here, we simply solve the problem for fixed $\lambda > 0$. We emphasize, however, that many of our estimates would break down if we maintained the constant as λ and then let λ tend to zero.

One can see that the stress law (1.5) has certain features which are more desirable than (1.3) by making a very simple computation. Consider the linearized version of (1.5); that is assume that the nonlinear term $\sigma(u_x)$ in (1.5) is replaced by $E_o u_x$. Then the equation becomes

$$\rho_o u_{tt} = E_o u_{xx} + \lambda u_{xtx}, \qquad (x,t) \in (0,1) \times (0,\infty). \qquad (1.6)$$

Let us seek solutions by separation of variables in the form $T_n(t)\sin n\pi x$. These clearly satisfy (A). The functions $T_n(t)$ must satisfy the equations

$$\rho_o T_n''(t) = -n^2\pi^2(E_o T_n + \lambda T_n').$$

It is easy to see that these T_n's satisfy the relations

$$T_n(t) = 0(e^{\beta_n t}), \qquad \beta_n = \frac{-\lambda n^2\pi^2 + \sqrt{\lambda^2 n^4\pi^4 - 4E_o\rho_o n^2\pi^2}}{2\rho_o} \qquad \text{as } t \to \infty. \qquad (1.7)$$

Thus the solution of (1.6) with conditions (A), (B), and (C) can be approximated by functions which vanish exponentially in t. This is in contrast to solutions of the linear wave equation which do not vanish as t tends to ∞.

The calculation of the preceding paragraph strongly suggests that solutions of (1.6), (A), (B), and (C) tend to zero as t tends to infinity. Hence, the introduction of the term λu_{xt} in (1.5) appears to add a damping mechanism to the process. We shall see later that the presence of λu_{xt} does indeed damp solutions of the general problem.[#]

[#] See Theorem 1, equation (2.4).

It will be observed in the argument above that the sign of the constant λ is crucial. If λ were negative, solutions of the linear problem would grow exponentially in time. Throughout the paper it will be seen that our results depend heavily on the fact that λ is positive. It is important to remark that once one adopts the stress law (1.5) it is possible to show that the second law of thermodynamics requires that λ be positive. The proof of this fact can be obtained by specializing the results of [3].

For ease in writing we now set both ρ_o and λ equal to one. Hence we replace (E') by

$$u_{tt} = \sigma'(u_x)u_{xx} + u_{xtx}, \qquad (x,t) \in (0,1) \times (0,\infty). \qquad \text{(E)}$$

2. Statement of the Main Results

Throughout this paper we shall assume that the function σ is $C^3(-\infty,\infty)$ and that its derivative σ' satisfies (*). We shall also assume that the functions f and g appearing in (B) and (C) are, respectively, C^4 and C^2 on [0,1] and vanish together with their second derivatives at zero and one.

Let T be any positive number. For functions U which are C^2 on the strip $S_T = \{x,t \mid 0 \le x \le 1,\ t \in [0,T]\}$ we let

$$|||U|||(t) = \sum_{i=0}^{2} \sum_{k=0}^{i} \max_{x \in [0,1]} \left| \frac{\partial^i U}{\partial x^{i-k} \partial t^k} \right| , \qquad t \in [0,T], \tag{2.1}$$

and for functions Λ and Ω which are C^2 on [0,1] we let

$$J(\Lambda,\Omega) = \sum_{i=0}^{2} \left(\max_{x \in [0,1]} |\overset{(i)}{\Lambda}(x)| + \max_{x \in [0,1]} |\overset{(i)}{\Omega}(x)| \right). \tag{2.2}$$

An (f,g) displacement in S_T will be any function u such that:

(i) all derivatives appearing in (2.1) are continuous on S_T,

(ii) $u_{txx} = u_{xtx} = u_{xxt}$ in $(0,1) \times (0,T]$, and

(iii) u satisfies (E) in $(0,1) \times (0,T]$ and conditions (A), (B), and (C).

An (f,g) displacement on S_∞ is defined analogously.

The principal results of the paper are contained in the following two theorems.

Theorem 1. If u is an (f,g) displacement on S_∞, then there exists a constant M, which depends on J(f,g) and tends to zero as J(f,g) tends to zero, such that

$$|||u|||(t) \le M, \qquad t \ge 0. \tag{2.3}$$

Moreover,

$$\lim_{t\to\infty} |||u|||(t) = 0. \tag{2.4}$$

Theorem 2. For any f and g there exists a unique (f,g) displacement on S_∞.

We here give a brief outline of the proofs of Theorems 1 and 2; details appear in the following sections.

There are two main ideas used in the proof of Theorem 1. The first involves viewing equation (E) as two different inhomogeneous equations. That is if u is an (f,g) displacement on S_∞, then:

(i) for any τ and $\alpha \ge 0$ the velocity field $V(x,t) \overset{\text{def}}{=} u_t(x,t)$ satisfies the following linear heat equation:

$$\left.\begin{aligned} V_t(x,t) - V_{xx}(x,t) &= \phi(u)(x,t), && (x,t) \in (\tau,\tau+\alpha], \\ V(x,\tau) &= u_t(x,\tau), && x \in [0,1], \\ V(0,t) = V(1,t) &= 0, && t \in (\tau,\tau+\alpha], \end{aligned}\right\} \tag{2.5}$$

where

$$\phi(u)(x,t) \stackrel{\text{def}}{=} \sigma'(u_x(x,t))u_{xx}(x,t); \text{ and} \tag{2.6}$$

(ii) for each $(x,\tau) \in [0,1] \times [0,\infty)$ the function $W(t) \stackrel{\text{def}}{=} u_{xx}(x,t)$ satisfies the ordinary differential equation

$$\begin{aligned} \frac{dW(t)}{dt} + \alpha(t)W(t) &= \beta(t), \qquad t \geq \tau, \\ W(\tau) &= u_{xx}(x,\tau), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} \alpha(t) &\stackrel{\text{def}}{=} \sigma'(u_x(x,t)) > 0, \quad \text{and} \\ \beta(t) &= u_{tt}(x,t). \end{aligned} \tag{2.8}$$

Now (2.5) and (2.6) imply the existence of a functional M such that

$$u_t(x,t) = M(\phi(u)(\cdot,\cdot); u_t(\cdot,\tau)), \tag{2.9}$$

while (2.7) and (2.8) guarantee the existence of another functional N such that

$$u_{xx}(x,t) = N(u_x(x,\cdot),\ u_{tt}(x,\cdot);\ u_{xx}(x,\tau)). \tag{2.10}$$

The second key idea arises as follows. In Section 4 we make use of energy inequalities to derive uniform bounds for the spatial L_2 norms of u_t and u_{xx}. Then formulas (2.9) and (2.10) allow us to obtain pointwise bounds for all necessary derivatives. The latter process is described in Section (5).

We establish Theorem 2 in the following way. Setting $\tau = 0$ and using (2.9), (2.10), and the initial data we are able to show that the derivatives u_x and u_t of an (f,g) displacement u must satisfy a certain pair of nonlinear functional equations. For sufficiently small t (say $\leq \alpha$) these equations are uniquely solvable. Moreover, they do indeed give rise to a unique (f,g) displacement u^1 in $[0,1] \times [0,\alpha]$

Next, setting $\tau = \alpha$ and using (2.9), (2.10), and $u^1(\cdot,\alpha)$ a new set of functional equations for the first derivatives of an (f,g) displacement are derived. Making use of the fact that u^1 satisfies the a priori estimates of Theorem 1, we may conclude that these new equations have a unique solution in $[0,1] \times [\alpha, 2\alpha]$. Using this last pair of solutions we may extend the function u^1 to an (f,g) displacement u^2 on $[0,1] \times [0,2\alpha]$.

Proceeding inductively, we then extend the domain of existence and uniqueness, in increments of α, to $[0,1] \times [0,\infty)$.

3. Results for the Linear Heat Equation

Our primary goal in this section is to establish certain properties of the solution operator M for the inhomogeneous linear heat equation (see (2.9)); thus, for any $\alpha > 0$, we seek properties of the function V satisfying

$$\left.\begin{aligned} V_t(x,t) - V_{xx}(x,t) &= \phi(x,t), & (x,t) &\in (0,1)\times(\tau,\tau+\alpha], \\ V(0,t) = V(1,t) &= 0, & t &\in (\tau,\tau+\alpha], \\ V(x,t) &= \psi(x), & x &\in [0,1]. \end{aligned}\right\} \text{(H)}$$

It is well known that if ϕ satisfies a Hölder condition jointly in x and t and if ψ is sufficiently smooth in x, then there exists a unique representation of the solution of (H) in terms of the Green's function $G(x,\xi,t-\tau)$ (see Friedman [4]); this solution can be written as

$$V(x,t) = P(x,t,\tau;\psi) + Q(x,t,\tau;\phi), \tag{3.1}$$

where

$$P(x,t,\tau;\psi) = \int_0^1 G(x,\xi,t-\tau)\psi(\xi)d\xi \tag{3.2}$$

and

$$Q(x,t,\tau;\phi) = \int_\tau^t\int_0^1 G(x,\xi,t-\eta)\phi(\xi,\eta)d\xi d\eta. \tag{3.3}$$

In Sections 5 and 6 we shall have need of (3.1) under two different sets of hypotheses, neither of which is quite standard. We shall also need a set of bounds for the potentials P and Q, some of which are non-standard. In essence, the remainder of this section is devoted to verifying that under two different sets of hypotheses solutions of (H) are given by (3.1) and that P and Q possess certain boundedness properties.

Throughout this section we shall assume

(A-1) ψ is C^2 on $[0,1]$ and vanishes at zero and one; and either

(A-2) (i) ϕ is continuous on $[0,1]\times[\tau,\tau+\alpha]$,

(ii) $\phi(0,t) = \phi(1,t) = 0$, $t \in [\tau,\tau+\alpha]$, and

(iii) ϕ_x and ϕ_{xx} are continuous on $[0,1]\times[\tau,\tau+\alpha]$; or

(A-2)' (A-2)' is simply (A-2) with (iii) replaced by

(iii)' there exists a constant $K < \infty$ such that

$$|\phi(x,t)-\phi(x,\eta)| \le K|t-\eta|^{1/2}, \qquad (x,t,\eta) \in [0,1]\times[\tau,\tau+\alpha]\times[\tau,\tau+\alpha]. \tag{3.4}$$

For functions h defined on $[0,1]\times[\tau,\tau+\alpha]$ we let

$$\left.\begin{aligned} |h|(t) &= \max_{x\in[0,1]} |h(x,t)|, \qquad |h|_{\tau,\alpha} = \max_{t\in[\tau,\tau+\alpha]} |h|(t), \\ \|h\|(t) &= \left(\int_0^1 h^2(x,t)\,dx\right)^{1/2}, \quad \text{and} \quad \|h\|_{\tau,\alpha} = \max_{t\in[\tau,\tau+\alpha]} \|h\|(t); \end{aligned}\right\} \tag{3.5}$$

and for functions Ω defined on $[0,1]$ we let

$$|\Omega| = \max_{x\in[0,1]} |\Omega(x)|, \quad \text{and} \quad \|\Omega\| = \left(\int_0^1 \Omega^2(x)\,dx\right)^{1/2}. \tag{3.6}$$

We can now state the main results of this section. Throughout this section α_o will denote an arbitrary but fixed positive scalar.

Theorem 3. If $\alpha \leq \alpha_o$ and if ψ and ϕ satisfy (A-1) and (A-2) respectively, then there exists a unique solution of (H) which can be represented by (3.1). Moreover, there exists a constant C, independent of α, ϕ, and ψ, such that the potentials P and Q satisfy the following inequalities:

$$C^{-1}|P|(t) \leq \begin{cases} |\psi|, \\ (t-\tau)^{-1/4}\|\psi\| \end{cases} ; \tag{3.7}$$

$$C^{-1}|P_x|(t) \leq \begin{cases} |\psi_x|, \\ (t-\tau)^{-3/4}\|\psi\| \end{cases} ; \tag{3.8}$$

$$C^{-1}|P_{xx}|(t) = C^{-1}|P_t|(t) \leq \begin{cases} |\psi_{xx}|, \\ (t-\tau)^{-1/2}|\psi_x| \end{cases} ; \tag{3.9}$$

$$C^{-1}|Q|(t) \leq \begin{cases} (t-\tau)|\phi|_{\tau,\alpha}, \\ (t-\tau)^{3/4}\|\phi\|_{\tau,\alpha} \end{cases} ; \tag{3.10}$$

$$C^{-1}|Q_x|(t) \leq \begin{cases} (t-\tau)|\phi_x|_{\tau,\alpha}, \\ (t-\tau)^{1/2}|\phi|_{\tau,\alpha}, \\ (t-\tau)^{1/4}\|\phi\|_{\tau,\alpha}, \end{cases} \tag{3.11}$$

$$C^{-1}|Q_{xx}|(t) \leq (t-\tau)|\phi_{xx}|_{\tau,\alpha} ; \tag{3.12}$$

and

$$C^{-1}|Q_t|(t) \leq (t-\tau)|\phi_{xx}|_{\tau,\alpha} + C^{-1}|\phi|_{\tau,\alpha} \tag{3.13}$$

Theorem 4. If $\alpha \leq \alpha_o$ and if ψ and ϕ satisfy (A-1) and (A-2)', then there exists a unique solution of (H) which can be represented by (3.1). Moreover, the inequalities (3.7)-(3.10), $(3.11)_2$ and $(3.11)_3$ of Theorem 4 remain valid, and, in addition, Q_{xx} and Q_t satisfy

$$C^{-1}|Q_{xx}|(t) \leq K(t-\tau)^{1/2} + |\phi|_{\tau,\alpha}, \tag{3.14}$$

and

$$C^{-1}|Q_t|(t) \leq K(t-\tau)^{1/2} + (1+C^{-1})|\phi|_{\tau,\alpha}; \tag{3.15}$$

where K is the constant appearing in (3.4).

An immediate consequence of the Theorems 3 and 4 is

Corollary 1. If $\alpha \leq \alpha_o$, if ψ satisfies (A-1), and if ϕ satisfies both (A-2) and (A-2)', then all the bounds of Theorems 3 and 4 are valid.

Remark. The L_2 bounds of Theorem 3 appear to be new. These bounds, when combined with energy estimates of the type to be derived in section 5, should be of some use in the discussion of the existence and uniqueness of solutions of semilinear parabolic partial differential equations.

Our first task in proving Theorems 3 and 4 is to obtain some information about the Green's function $G(x,\xi,t-\tau)$. We define functions

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Γ and K by

$$\Gamma(a,b) = \frac{1}{2\sqrt{\pi}\, b^{1/2}} e^{-a^2/4b} \tag{3.16$_1$}$$

and

$$K(a,b) = \sum_{m=1}^{\infty} \Big(\Gamma(a+2m,b) + \Gamma(a-2m,b)\Big). \tag{3.16$_2$}$$

Then $\Gamma(x-\xi,t-\tau)$ is the fundamental solution for the linear heat equation and the function

$$G(x,\xi,t-\tau) = \Gamma(x-\xi,t-\tau) - \Gamma(x+\xi,t-\tau) + K(x-\xi,t-\tau) - K(x+\xi,t-\tau) \tag{3.17}$$

is the Green's function for problem (H). On occasion we shall use the following notation:

$$\left.\begin{aligned} G^1(x-\xi,t-\tau) &= \Gamma(x-\xi,t-\tau) + K(x-\xi,t-\tau), \quad G^2(x+\xi,t-\tau) = -\Gamma(x+\xi,t-\tau) - K(x+\xi,t-\tau), \\ \text{and} \quad G(x,\xi,t-\tau) &= G^1(x-\xi,t-\tau) + G^2(x+\xi,t-\tau). \end{aligned}\right\} \tag{3.18}$$

We first note some elementary properties of G.

(i) For each $(\xi,\tau) \in [0,1] \times [0,\infty)$ and each positive α

$$G_{xx} - G_t = 0, \qquad (x,t) \in (0,1) \times (\tau,\tau+\alpha], \tag{3.19$_1$}$$

$$G(0,\xi,t-\tau) = G(1,\xi,t-\tau) = 0, \qquad t \in (\tau,\tau+\alpha], \tag{3.19$_2$}$$

and

$$G(x,\xi,0) = 0, \qquad x \in [0,1] \text{ with } x \neq \xi. \tag{3.19$_3$}$$

(ii) For each $(x,\xi,t-\tau) \in [0,1] \times [0,1] \times (0,\alpha]$

$$G_t = -G_\tau \quad \text{and} \quad G_{xx} = G_{\xi\xi}, \tag{3.20}$$

(iii) For each $(x,t) \in [0,1] \times [0,\infty)$ and each positive α

$$G_{\xi\xi} + G_\tau = 0, \qquad (\xi,\tau) \in (0,1) \times [t-\alpha,t), \tag{3.21$_1$}$$

$$G(x,0,t-\tau) = G(x,1,t-\tau) = 0, \qquad \tau \in [t-\alpha,t), \tag{3.21$_2$}$$

and

$$G(x,\xi,0) = 0, \qquad \xi \in [0,1] \text{ with } \xi \neq x. \tag{3.21$_3$}$$

We now give some estimates for G.

Lemma 3.1. Suppose $\alpha \leq \alpha_o$. Then there exists a constant C, independent of α, such that

$$C^{-1}|G(x,\xi,t-\tau)| \leq \Gamma(x-\xi,t-\tau) + \Gamma(x+\xi,t-\tau), \tag{3.22$_1$}$$

$$C^{-1}|G_x(x,\xi,t-\tau)| \leq |\Gamma_x(x-\xi,t-\tau)| + |\Gamma_x(x+\xi,t-\tau)|, \tag{3.22$_2$}$$

$$C^{-1}|G_\xi(x,\xi,t-\tau)| \leq |\Gamma_\xi(x-\xi,t-\tau)| + |\Gamma_\xi(x+\xi,t-\tau)|, \tag{3.22$_3$}$$

and

$$C^{-1}|(t-\tau)^{1/2}G_\tau(x,\xi,t-\tau)| \leq \frac{\Gamma(x-\xi,t-\tau)}{(t-\tau)}\left[1+\frac{|x-\xi|^2}{(t-\tau)}\right] + \frac{\Gamma(x+\xi,t-\tau)}{(t-\tau)}\left[1+\frac{|x+\xi|^2}{(t-\tau)}\right]. \tag{3.22$_4$}$$

for all $(x,\xi,t-\tau) \in [0,1] \times [0,1] \times [0,\alpha]$.

Remark. It will be evident from the proof that the same bounds apply to G^1 and G^2 separately.

Proof of Lemma 3.1. Since

$$\Gamma(x-\xi\pm 2m,t-\tau) = \Gamma(x-\xi,t-\tau)e^{\frac{-m^2\mp m(x-\xi)}{(t-\tau)}}, \qquad m \geq 0,$$

and since $m^2 \pm m(x-\xi) \geq (m-1)^2$ for all $x-\xi$ in $[-1,1]$ we have

$$\sum_{m=1}^{\infty} e^{\frac{-m^2\mp m(x-\xi)}{(t-\tau)}} \leq 2\left(1+\alpha^{1/2}\int_0^{\infty} e^{-z^2}dz\right) \leq 2+\alpha_o^{1/2}\sqrt{\pi} \overset{def}{=} k. \quad (3.23)$$

It now follows from $(3.16)_2$ and $(3.18)_1$ that

$$|G^1(x-\xi,t-\tau)| \leq (2k+1)\Gamma(x-\xi,t-\tau). \qquad (3.24)_1$$

A similar calculation yields

$$|G^2(x+\xi,t-\tau)| \leq (2k+1)\Gamma(x+\xi,t-\tau), \qquad (3.24)_2$$

and hence $(3.21)_1$ follows from $(3.18)_3$, $(3.24)_1$, and $(3.24)_2$.

We now observe that for $m \geq 0$

$$\Gamma_x(x-\xi\pm 2m,t-\tau) = \frac{-(x-\xi\pm 2m)}{4\sqrt{\pi}(t-\tau)^{3/2}}e^{\frac{-(x-\xi\pm 2m)^2}{4(t-\tau)}} = \Gamma_x(x-\xi,t-\tau)\left(1\pm\frac{2m}{(x-\xi)}\right)e^{\frac{-m^2\mp m(x-\xi)}{(t-\tau)}}.$$

A direct consequence of the last formula and $(3.18)_1$ is the identity

$$G_x^1(x-\xi,t-\tau) = \Gamma_x(x-\xi,t-\tau)\left\{1+\sum_{m=1}^{\infty}\left(e^{\frac{-m^2-m(x-\xi)}{(t-\tau)}}+e^{\frac{-m^2+m(x-\xi)}{(t-\tau)}}\right)-\sum_{m=1}^{\infty}\frac{4me^{\frac{-m^2}{(t-\tau)}}}{(x-\xi)}\sinh m\left(\frac{x-\xi}{t-\tau}\right)\right\}. \qquad (3.25)$$

We now show that for all $(x,\xi,t-\tau) \in [0,1] \times [0,1] \times [0,\alpha]$ with $x \neq \xi$ and $\alpha \leq \alpha_o$, the series $\sum_{m=1}^{\infty} \frac{me^{\frac{-m^2}{(t-\tau)}}}{(x-\xi)} \sinh m\left(\frac{x-\xi}{t-\tau}\right)$ is uniformly bounded. We observe that

$$\left| \frac{m}{(x-\xi)} e^{\frac{-m^2}{(t-\tau)}} \sinh m\left(\frac{x-\xi}{t-\tau}\right) \right| = \frac{m^2}{(t-\tau)} e^{\frac{-m^2}{(t-\tau)}} \left| \frac{t-\tau}{m(x-\xi)} \sinh m\left(\frac{x-\xi}{t-\tau}\right) \right| .$$

Noting that

$$|z|e^{-|z|} \leq \epsilon' e^{\frac{-|z|}{2}} \quad \text{and} \quad |w|^{-1}|\sinh w| \leq \epsilon'' e^{|w|}$$

for all z and w, we have

$$\left| \frac{m}{(x-\xi)} e^{\frac{-m^2}{(t-\tau)}} \sinh m\left(\frac{x-\xi}{t-\tau}\right) \right| \leq Ce^{\frac{-m^2+2m}{2(t-\tau)}} . \tag{3.26}$$

Since $(t-\tau) \in [0,\alpha]$ and $\alpha \leq \alpha_o$, (3.26) implies that

$$\left| \sum_{m=2}^{\infty} \frac{me^{\frac{-m^2}{(t-\tau)}}}{(x-\xi)} \sinh m\left(\frac{x-\xi}{t-\tau}\right) \right| \leq k' < \infty ; \tag{3.27}$$

and hence incorporating the term with m = 1 we obtain

$$\left| \sum_{m=1}^{\infty} \frac{m}{(x-\xi)} e^{\frac{-m^2}{(t-\tau)}} \sinh m\left(\frac{x-\xi}{t-\tau}\right) \right| \leq \left| \frac{1}{2(x-\xi)} e^{\frac{-1}{(t-\tau)}} \left(e^{\frac{(x-\xi)}{(t-\tau)}} - e^{\frac{-(x-\xi)}{(t-\tau)}} \right) \right| + k'. \tag{3.28}$$

It is easily verified that for $(x,\xi) \in [0,1] \times [0,1]$

$$\left| \frac{1}{(x-\xi)} \left(e^{\frac{(x-\xi)}{(t-\tau)}} - e^{\frac{-(x-\xi)}{(t-\tau)}} \right) \right| \leq \left| e^{\frac{1}{(t-\tau)}} - e^{\frac{-1}{(t-\tau)}} \right| \tag{3.29}$$

and hence it follows that the first term on the right hand side of (3.28) is uniformly bounded for all $(t-\tau) \geq 0$. Equations (3.25), (3.23), (3.28), and (3.29) now yield

$$|G_x^1(x-\xi,t-\tau)| \leq k''|\Gamma_x(x-\xi,t-\tau)|, \qquad (3.30)_1$$

A similar calculation shows that

$$|G_x^2(x+\xi,t-\tau)| \leq k''|\Gamma_x(x-\xi,t-\tau)|; \qquad (3.30)_2$$

and hence (3.18) and (3.30) establish $(3.22)_2$ and $(3.22)_3$.

Similar arguments yield $(3.22)_4$.

Our next result concerns integrals of G and its derivative.

Lemma 3.2. Suppose $\alpha \leq \alpha_o$. Then there exists a constant C, independent of α, such that

$$\int_0^1 |G(x,\xi,t-\tau)|\,d\xi \leq C, \qquad (3.31)_1$$

$$\int_0^1 |G(x,\xi,t-\tau)|^2 d\xi \leq C(t-\tau)^{-1/2}, \qquad (3.31)_2$$

$$\int_0^1 |G_x(x,\xi,t-\tau)|\,d\xi \leq C(t-\tau)^{-1/2}, \qquad (3.31)_3$$

$$\int_0^1 |G_\xi(x,\xi,t-\tau)|\,d\xi \leq C(t-\tau)^{-1/2}, \qquad (3.31)_4$$

$$\int_0^1 |G_x(x,\xi,t-\tau)|^2 d\xi \leq C(t-\tau)^{-3/2}, \qquad (3.31)_5$$

$$\int_0^1 |G_\xi(x,\xi,t-\tau)|^2 d\xi \leq C(t-\tau)^{-3/2}, \tag{3.31$_6$}$$

and

$$\int_0^1 |(t-\tau)^{1/2} G_\tau(x,\xi,t-\tau)| d\xi \leq C(t-\tau)^{-1/2}. \tag{3.31$_7$}$$

Remark. The remark following the statement of Lemma 3.1 implies that the conclusions of Lemma 3.2 hold with G^1 and G^2 replacing G.

Proof of Lemma 3.2. The lemma is a direct consequence of equation (3.22) and the boundedness of all moments of the function e^{-z^2}.

We turn now to the proofs of Theorems 3 and 4. We begin with Theorem 3. We observe first that the usual arguments (see for example Friedman [4]) allow us to establish the following facts concerning P under the assumption that ψ satisfies (A-1):

(i) P is continuous on $[0,1] \times [\tau,\tau+\alpha]$,

(ii) $\lim_{t \to \tau^+} P(x,t,\tau;\psi) = \psi(x)$ uniformly in x on [0,1],

(iii) $P(0,t) = P(1,t) = 0 \qquad t \in (\tau,\tau+\alpha]$, and

(iv) for each $(x,t) \in (0,1) \times (\tau,\tau+\alpha]$, P_x, P_{xx}, and P_t exist, are continuous and are given by

$$P_x(x,t) = \int_0^1 G_x(x,\xi,t-\tau)\psi(\xi)d\xi$$

$$P_{xx}(x,t) = \int_0^1 G_{xx}(x,\xi,t-\tau)\psi(\xi)d\xi \qquad (3.32)$$

$$P_t(x,t) = \int_0^1 G_t(x,\xi,t-\tau)\psi(\xi)d\xi.$$

One can also obtain, from standard arguments, the following results for Q:

(i) Q is continuous on $[0,1] \times [\tau,\tau+\alpha]$,

(ii) $\lim_{t\to\tau^+} Q(x,t) = 0$,

(iii) $Q(0,t) = Q(1,t) = 0, \quad t \in (\tau,\tau+\alpha]$, and

(iv) for each $(x,t) \in (0,1)\times(\tau,\tau+\alpha]$ Q_x exists, is continuous, and is given by

$$Q_x(x,t) = \int_\tau^t\int_0^1 G_x(x,\xi,t-\eta)\phi(\xi,\eta)d\xi d\eta. \qquad (3.33)$$

Properties (i), (ii), (iii), and (iv) for Q are true under either (A-2) or (A-2)'; in fact they require only the continuity of ϕ. If, however, one wishes to calculate Q_t or Q_{xx}, then additional conditions must be placed on ϕ. As we have already noted the usual condition is that ϕ is jointly Hölder continuous in x and t. Our task is to show that this condition can be replaced by either (iii) of (A-2) or (iii)' of (A-2)'.

The results we want are the following:

$$Q_{xx} \text{ and } Q_t \text{ are continuous in } (0,1)\times(\tau,\tau+\alpha), \tag{3.34}$$

and

$$Q_t = Q_{xx} + \phi. \tag{3.35}$$

Once these formulas are established it is clear that the function V(x,t), defined by (3.1), does indeed yield a solution of (H). The uniqueness of this solution is again a standard result.

<u>Completion of the Proof of Theorem</u> 3. We begin by obtaining the estimates for P. We have by $(3.31)_1$

$$|P|(t) \le \int_0^1 |G(x,\xi,t-\tau)||\psi(\xi)|d\xi \le C|\psi|. \qquad (3.36)_1$$

By $(3.31)_2$ and Schwarz's inequality we obtain

$$|P|(t) \le \left(\int_0^1 |G(x,\xi,t-\tau)|^2 d\xi\right)^{1/2}\|\psi\| \le C(t-\tau)^{-1/4}\|\psi\|. \qquad (3.36)_2$$

Next we observe that $(3.32)_1$ implies that

$$|P_x|(t) \le \int_0^1 |G_x(x,\xi,t-\tau)||\psi(\xi)|d\xi. \tag{3.37}$$

Equation (3.37) and $(3.31)_3$ now yield

$$|P_x|(t) \le C(t-\tau)^{-1/2}|\psi|, \qquad (3.38)_1$$

while (3.37), Schwarz's inequality, and $(3.31)_5$ give

$$|P_x|(t) \leq C(t-\tau)^{-3/4}\|\psi\|. \qquad (3.38)_2$$

If we note that $G_x = G^1_x + G^2_x = -G^1_\xi + G^2_\xi$ and make use of the fact that ψ satisfies (A-1), then we obtain the following representation for P_x:

$$P_x(x,t) = \int_0^1 [G^1(x-\xi,t-\tau) - G^2(x+\xi,t-\tau)]\psi_\xi(\xi)d\xi. \qquad (3.39)$$

Equations (3.39), (3.24), and $(3.31)_1$ with G replaced by G^1 and G^2 now yield

$$|P_x|(t) \leq C|\psi_\xi|. \qquad (3.40)$$

The above formulas establish equations (3.7)-(3.9). Equations (3.10), $(3.11)_2$, and $(3.11)_3$ are established in the same way except that now one must integrate with respect to η over (τ,t).

We have now reduced the proof of Theorem 3 to the verification that (3.34), (3.35), $(3.11)_1$, (3.12), and (3.13) are valid under the assumption that (ii) and (iii) of (A-2) hold. Making use of the vanishing of ϕ at zero and one (see (ii) of (A-2)) for all t in $[\tau,\tau+\alpha]$, we can use the arguments employed in establishing (3.39) to obtain

$$Q_x(x,t) = \int_\tau^t\int_0^1 [G^1(x-\xi,t-\eta) - G^2(x+\xi,t-\eta)]\phi_\xi(\xi,\eta)d\xi d\eta. \qquad (3.41)$$

We can now differentiate Q_x again with respect to x to obtain

$$Q_{xx}(x,t) = \int_\tau^t\int_0^1 [G^1_x(x-\xi,t-\eta) - G^2_x(x+\xi,t-\eta)]\phi_\xi(\xi,\eta)d\xi d\eta.$$

Since $G^1_x = -G^1_\xi$, $G^2_x = G^2_\xi$, and $G = G^1 + G^2$, we have $G^1_x - G^2_x = -G_\xi$. Integrating the last equation by parts and making use of $(3.21)_2$, we find that

$$Q_{xx}(x,t) = \int_\tau^t \int_0^1 G(x,\xi,t-\eta)\phi_{\xi\xi}(\xi,\eta)\,d\xi\,d\eta. \tag{3.42}$$

Equations $(3.11)_1$ and (3.12) now follow from (3.41), (3.42), $(3.31)_3$, and $(3.31)_1$.

The treatment of Q_t is more involved. We calculate the difference quotient $\Delta_h = h^{-1}[Q(x,t+h) - Q(x,t)]$, $h > 0$. We have

$$\begin{aligned}\Delta_h &= h^{-1}\left\{\int_\tau^{t+h}\int_0^1 G(x,\xi,t+h-\eta)\phi(\xi,\eta)\,d\xi\,d\eta - \int_\tau^t\int_0^1 G(x,\xi,t-\eta)\phi(\xi,\eta)\,d\xi\,d\eta\right\}\\ &= h^{-1}\left\{\int_t^{t+h}\int_0^1 G(x,\xi,t+h-\eta)\phi(\xi,\eta)\,d\xi\,d\eta + \int_\tau^t\int_0^1\int_t^{t+h} G_\lambda(x,\xi,\lambda-\eta)\phi(\xi,\eta)\,d\lambda\,d\xi\,d\eta\right\}.\end{aligned} \tag{3.43}$$

Consider the first integral on the right hand side of (3.43). The inner integral is $P(x,t+h,\eta;\phi(\cdot,\eta))$ (see equation (3.2)). We write

$$P(x,t+h,\eta;\phi(\cdot,\eta)) = P(x,t+h,\eta;\phi(\cdot,t)) + P(x,t+h,\eta;\phi(\cdot,\eta)-\phi(\cdot,t)).$$

Property (ii) for the potential P implies that

$$P(x,t^+,t;\phi(\cdot,t)) = \phi(x,t),$$

and thus we find

$$h^{-1}\int_t^{t+h} P(x,t+h,\eta;\phi(\cdot,t))\,d\eta = \phi(x,t) + h^{-1}\int_t^{t+h}\left\{P(x,t+h,\eta;\phi(\cdot,t)) - P(x,t^+,t;\phi(\cdot,t))\right\}d\eta.$$

Noting that $P(x,t+h,\eta;\phi(\cdot,t)) = P(x,t+h-\eta,0;\phi(\cdot,t))$ and that the function $\lambda \to P(x,\lambda,0;\phi(\cdot,t))$ is continuous in λ for $\lambda \geq 0$, we may conclude that given any $\epsilon > 0$

$$\left| h^{-1}\int_t^{t+h} P(x,t+h,\eta;\phi(\cdot,t))d\eta - \phi(x,t)\right| \leq \epsilon/2$$

provided h is sufficiently small. On the other hand, we note that

$$I = h^{-1}\int_t^{t+h} P(x,t+h,\eta;\phi(\cdot,\eta)-\phi(\cdot,t))d\eta = h^{-1}\int_t^{t+h}\int_0^1 G(x,\xi,t+h-\eta)(\phi(\xi,\eta)-\phi(\xi,t))d\xi d\eta.$$

Since $0 \leq t+h-\eta \leq h$, $(3.7)_1$ allows us to conclude that for all $h \leq \alpha_o$

$$|I| \leq C \max_{\substack{0\leq \xi \leq 1 \\ t\leq \eta \leq t+h}} |\phi(\xi,\eta)-\phi(\xi,t)|.$$

Moreover, the continuity of ϕ implies that for h sufficiently small

$$\max_{\substack{0\leq \xi \leq 1 \\ t\leq \eta \leq t+h}} |\phi(\xi,\eta)-\phi(\xi,t)| \leq \epsilon/2c.$$

Collecting all our estimates we deduce that

$$\left| h^{-1}\int_t^{t+h} P(x,t+h,\eta;\phi(\cdot,\eta))d\eta - \phi(x,t)\right| \leq \epsilon$$

and hence

$$\lim_{h\to 0^+} h^{-1}\int_t^{t+h}\int_0^1 G(x,\xi,t+h-\eta)\phi(\xi,\eta)d\xi d\eta = \phi(x,t). \qquad (3.44)$$

Now consider the second term on the right side of (3.43). Since

$$G_\lambda(x,\xi,\lambda-\eta) = -G_\eta(x,\xi,\lambda-\eta) = G_{\xi\xi}(x,\xi,\lambda-\eta),$$

(ii) and (iii) of (A-2) allow us to integrate by parts with respect to ξ and obtain for the second term

$$h^{-1}\int_t^{t+h}\int_\tau^t\int_0^1 G(x,\xi,\lambda-\eta)\phi_{\xi\xi}(\xi,\eta)d\xi d\eta d\lambda .$$

Making use of property (ii) for P we can now pass to the limit in this expression and obtain

$$\int_\tau^t\int_0^1 G(x,\xi,t-\eta)\phi_{\xi\xi}(\xi,\eta)d\xi d\eta. \qquad (3.45)$$

It follows that Δ_h has a limit as h tends to 0^+. Moreover if we compare (3.44) and (3.45) with (3.42) we see that (3.35) holds. The case $h < 0$ can be treated in the same way and the proof of Theorem 3 is now complete.

Completion of the Proof of Theorem 4. We shall now assume that (ii) and (iii)' of (A-2)' hold. Our goal is to show that under these hypotheses equations (3.34), (3.35), (3.14), and (3.15) are valid.

We shall first establish that Q_{xx} exists and is continuous. In the process we shall derive an explicit representation for Q_{xx}. We take as our starting point the formula

$$Q_x(x,t) = \int_\tau^t\int_0^1 G_x(x,\xi,t-\eta)\phi(\xi,\eta)d\xi d\eta, \qquad (3.46)$$

and we consider the difference quotient $\Delta = h^{-1}[Q(x+h,t) - Q(x,t)]$. It is easily verified that

$$\Delta = h^{-1}\int_{\tau}^{t}\int_{0}^{1}\int_{x}^{x+h} G_{\lambda\lambda}(\lambda,\xi,t-\eta)\phi(\xi,\eta)\,d\lambda\, d\xi\, d\eta .$$

By $(3.19)_1$ and (3.20) this can be written as

$$\begin{aligned}\Delta &= -h^{-1}\int_{\tau}^{t}\int_{0}^{1}\int_{x}^{x+h} G_{\eta}(\lambda,\xi,t-\eta)\phi(\xi,\eta)\,d\lambda\, d\xi\, d\eta \\ &= -h^{-1}\int_{\tau}^{t}\int_{0}^{1}\int_{x}^{x+h} G_{\eta}(\lambda,\xi,t-\eta)(\phi(\xi,\eta)-\phi(\xi,t))\,d\lambda\, d\xi\, d\eta \\ &\quad - h^{-1}\int_{\tau}^{t}\int_{0}^{1}\int_{x}^{x+h} G_{\eta}(\lambda,\xi,t-\eta)\phi(\xi,t)\,d\lambda\, d\xi\, d\eta . \qquad (3.47)\end{aligned}$$

We can integrate the second term on the right side of (3.47) with respect to η to obtain

$$h^{-1}\left[\int_{0}^{1}\int_{x}^{x+h} G(\lambda,\xi,t-\tau)\phi(\xi,t)\,d\lambda\, d\xi - \lim_{\eta\to t^-}\int_{0}^{1}\int_{x}^{x+h} G(\lambda,\xi,t-\eta)\phi(\xi,t)\,d\lambda\, d\xi\right]. \qquad (3.48)$$

Observe that the second integral in (3.48) is simply

$$\int_{x}^{x+h} P(\lambda,t,\eta;\phi(\cdot,t))\,d\lambda$$

where again P is defined in (3.2). By property (ii) of P we have $\lim_{\eta\to t^-} P(\lambda,t,\eta;\phi(\cdot,t)) = \phi(x,t)$ uniformly in x, and hence it follows that

$$\lim_{\eta\to t^-}\int_{0}^{1}\int_{x}^{x+h} G(\lambda,\xi,t-\eta)\phi(\xi,t)\,d\lambda\, d\xi = \int_{x}^{x+h}\phi(\lambda,t)\,d\lambda.$$

We now have the second term on the right of (3.47) converging to

$$\int_0^1 G(x,\xi,t-\tau)\phi(\xi,t)d\xi - \phi(x,t) \tag{3.49}$$

as h tends to zero.

We now turn to the first term on the right side of (3.47). Since

$$|\phi(\xi,\eta)-\phi(\xi,t)| \leq K|t-\eta|^{1/2}, \qquad \xi \in [0,1],$$

we have

$$\left|h^{-1}\int_\tau^t\int_0^1\int_x^{x+h} G_\eta(\lambda,\xi,t-\eta)(\phi(\xi,\eta)-\phi(\xi,t))d\lambda d\xi d\eta\right|$$

$$\leq K|h^{-1}|\int_\tau^t\int_0^1\int_x^{x+h}|t-\eta|^{1/2}|G_\eta(\lambda,\xi,t-\eta)|d\lambda d\xi d\eta. \tag{3.50}$$

Equation $(3.31)_7$ implies that the right side of (3.50) is dominated by

$$CK|h^{-1}|\left|\int_\tau^t\int_x^{x+h}(t-\eta)^{-1/2}d\lambda d\eta\right| \leq CK(t-\tau)^{1/2}. \tag{3.51}$$

It now follows from Lebesgue's dominated convergence theorem that we can pass to the limit as h tends to zero in (3.47) and obtain

$$Q_{xx}(x,t) = -\int_\tau^t\int_0^1 G_\eta(x,\xi,t-\eta)(\phi(\xi,\eta)-\phi(\xi,t))d\xi d\eta + \int_0^1 G(x,\xi,t-\tau)\phi(\xi,t)d\xi - \phi(x,t). \tag{3.52}$$

Finally we have to verify the existence of Q_t and the validity of (3.35). Again we work with the difference quotient $h^{-1}[Q(x,t+h)-Q(x,t)]$. We have

$$h^{-1}[Q(x,t+h)-Q(x,t)] = -h^{-1}\left\{\int_\tau^t\int_0^1\int_t^{t+h} G_\eta(x,\xi,\lambda-\eta)(\phi(\xi,\eta)-\phi(\xi,t))\,d\lambda\,d\xi\,d\eta\right.$$

$$-\int_\tau^t\int_0^1\int_t^{t+h} G_\eta(x,\xi,\lambda-\eta)\phi(\xi,t)\,d\lambda\,d\xi\,d\eta$$

$$\left.+\int_t^{t+h}\int_0^1 G(x,\xi,t+h-\eta)\phi(\xi,\eta)\,d\xi\,d\eta\right\}. \tag{3.53}$$

As in earlier calculations we deduce that the limit of the first term as h tends to zero is given by

$$-\int_\tau^t\int_0^1 G_\eta(x,\xi,t-\eta)(\phi(\xi,\eta)-\phi(\xi,t))\,d\xi\,d\eta, \tag{3.54}$$

while the limit of the last term is $\phi(x,t)$. As for the middle term we carry out the η integration and find that it is equal to

$$-h^{-1}\int_0^1\int_t^{t+h}[G(x,\xi,\lambda-t) - G(x,\xi,\lambda-\tau)]\phi(\xi,t)\,d\lambda\,d\xi. \tag{3.55}$$

The limit as h tends to zero of the first term in (3.55) is $-\phi(x,t)$, while the limit of the second is

$$\int_0^1 G(x,\xi,t-\tau)\phi(\xi,t)\,d\xi.$$

Collecting all our results we see that the limit of the difference quotient (3.53) exists and equals

$$-\int_{\tau}^{t}\int_{0}^{1} G_{\eta}(x,\xi,t-\eta)(\phi(\xi,\eta)-\phi(\xi,t))d\xi d\eta + \int_{0}^{1} G(x,\xi,t-\tau)\phi(\xi,t)d\xi .$$

If we compare this with (3.52) we see that (3.35) holds.

The bound (3.14) is a direct consequence of (3.53), condition (iii)' of (A-2)', and $(3.31)_1$ and $(3.31)_7$; while (3.15) follows from (3.14) and (3.35).

4. A priori Energy Estimates for the Solutions of (E)

In this section we shall derive certain energy estimates which must be satisfied by (f,g) displacements u. These estimates, when combined with the results of Section 3, will enable us to establish Theorem 1.

We first record some results which will be of use later.

Remark 4.1.

(a) Suppose that Φ is C^2 on $[0,1] \times [0,\infty)$ and vanishes at 0 and 1 for all times $t \geq 0$. Then,

$$\|\Phi\|(t) \leq |\Phi|(t) \leq \|\Phi_x\|(t) \leq |\Phi_x|(t) \leq \|\Phi_{xx}\|(t) \leq |\Phi_{xx}|(t), \qquad t \geq 0, \qquad (4.1)$$

where $\|\cdot\|$ and $|\cdot|$ are defined in (3.13).

(b) Suppose that Ψ is uniformly continuous and integrable on $[0,\infty)$. Then,

$$\lim_{t\to\infty} \Psi(t) = 0. \qquad (4.2)$$

Now, and in the remainder of this section, u will be an (f,g) displacement on S_∞.

If we multiply (E) by u_t, integrate the resulting expression over $(0,1) \times (t_1,t_2)$, and make use of the fact that u satisfies the edge conditions (A), we are led to the following identity:

$$\|u_t\|^2(t_2) + 2\int_0^1 \int_0^{u_x(x,t_2)} \sigma(\lambda)d\lambda + 2\int_{t_1}^{t_2}\|u_{\tau x}\|^2(\tau)d\tau = \|u_t\|^2(t_1) + 2\int_0^1 \int_0^{u_x(x,t_1)} \sigma(\lambda)d\lambda dx. \tag{4.2}$$

Similarly, if we multiply (E) by u_{xx}, integrate over $(0,1)\times(t_1,t_2)$, and note that the assumed smoothness of u and the edge conditions (A) imply that

$$u_{tt}(0,t) = u_{tt}(1,t) = 0, \qquad t \geq 0, \tag{4.3}$$

we find that u_{xx} must satisfy

$$\|u_{xx}\|^2(t_2) + 2\int_{t_1}^{t_2}\int_0^1 \sigma'(u_x(x,\tau))u_{xx}^2(x,\tau)dxd\tau$$

$$= \|u_{xx}\|^2(t_1) + 2\int_{t_1}^{t_2}\int_0^1 u_{xx}u_{\tau\tau}(x,\tau)dxd\tau$$

$$= \|u_{xx}\|^2(t_1) + 2\int_0^1 (u_{xx}u_t(x,t_2) - u_{xx}u_t(x,t_1))dx + 2\int_{t_1}^{t_2}\|u_{\tau x}\|^2(\tau)d\tau. \tag{4.4}$$

Lemma 4.1. There exists a constant M(J(f,g)) which tends to zero as J(f,g) tends to zero[#] such that

[#] In the remainder of the paper all constants M, M_*, $M_\#$, etc. will be functions of J(f,g) and will tend to zero as J(f,g) tends to zero.

$$\|u_t\|^2(t) \leq M^2 \text{ and } \int_0^t \|u_{\tau x}\|^2(\tau)d\tau \leq M^2, \qquad t \geq 0. \tag{4.5}$$

Proof. Since σ satisfies the monotonicity condition (*) given in the Introduction, it follows that $\int_0^1 \int_0^{u_x(x,\tau)} \sigma(\lambda)d\lambda dx \geq 0$, and hence (4.2), with $t_1 = 0$, implies

$$\|u_t\|^2(t) + 2\int_0^t \|u_{\tau x}\|^2(\tau)d\tau \;\leq\; \|g\|^2 + 2\int_0^1 \int_0^{f^{(1)}(x)} \sigma(\lambda)d\lambda \;\leq\; M^2, \qquad t \geq 0, \tag{4.6}$$

where

$$M^2 \;\underline{\underline{\mathrm{def}}}\; |||g|||^2 + E_1(|||f|||)\,|||f|||^2, \tag{4.7}$$

and where for any $a > 0$ $E_1(a)$ is defined by

$$E_1(a) \;=\; \sup_{\{\lambda \mid |\lambda| \leq a\}} \sigma'(\lambda). \tag{4.8}$$

Corollary 1. $\|u_t\|^2(\cdot)$ is integrable on $[0,\infty)$ and

$$\int_0^\infty \|u_t\|^2(\tau)d\tau \;\leq\; M^2. \tag{4.9}$$

Proof. The assertion is a direct consequence of $(4.5)_2$ and (4.1) with Φ equal to u_t.

Lemma 4.2. There exists a constant M_* such that

$$\|u_{xx}\|^2(t) \leq M_*^2 \quad \text{and} \quad \int_0^t \|u_{xx}\|^2(\tau)d\tau \leq M_*^2, \qquad t \geq 0. \tag{4.10}$$

Proof. If we set $t_1 = 0$ in (4.4), apply Schwarz's inequality to $\int_0^1 u_{xx}u_t(x,t_2)dx$, and make use of (4.5) and the fact that σ' is positive on $(-\infty,\infty)$, we see that

$$\|u_{xx}\|^2(t) \leq 2M\|u_{xx}\|(t) + M_1^2, \qquad t \geq 0 \tag{4.11}$$

where

$$M_1^2 = |||f|||^2 + 2|||f|||\,|||g||| + M^2. \tag{4.12}$$

It now follows from (4.11) that

$$\|u_{xx}\|^2(t) \leq \left(M + \sqrt{M^2 + M_1^2}\right)^2 \stackrel{\text{def}}{=\!=} M_2^2, \qquad t \geq 0, \tag{4.13}$$

and hence (4.13) and (4.1) with $\Phi = u$ yields

$$|u|(t) \leq |u_x|(t) \leq M_2, \qquad t \geq 0. \tag{4.14}$$

Equation (4.13), (4.14), and (4.4) then imply that

$$\int_0^t \|u_{xx}\|^2(\tau)d\tau \leq \frac{2}{E_o(M_2)}(M^2 + MM_1 + M_1^2) \stackrel{\text{def}}{=\!=} M_3^2, \qquad t \geq 0, \tag{4.15}$$

where for any $a > 0$

$$E_o(a) = \inf_{\{\lambda \,|\, |\lambda| \leq a\}} \sigma'(\lambda) > 0. \tag{4.16}$$

The lemma now follows with $M_* = \max(M_2, M_3)$.

Corollary 2.

$$|u|(t) \le |u_x|(t) \le M_*, \qquad t > 0. \tag{4.17}$$

Proof. Equation (4.17) follows from $(4.10)_1$ and (4.1) with $u = \Phi$.

Lemma 4.3.

$$\lim_{t\to\infty} \|u_t\|^2(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \|u_{xx}\|^2(t) = 0. \tag{4.18}$$

Proof. We shall first show that the function $\|u_t\|^2(\cdot)$ is uniformly continuous on $[0,\infty)$. It will then follow from Corollary 1 and Remark 4.1 (b) that $\lim_{t\to\infty} \|u_t\|^2(t) = 0$.

Equation (4.2) implies that

$$\left| \|u_t\|^2(t_2) - \|u_t\|^2(t_1) \right| \le 2\int_{t_1}^{t_2} \|u_{\tau x}\|^2(\tau)\,d\tau + \left| 2\int_0^1 \int_{u_x(x,t_1)}^{u_x(x,t_2)} \sigma(\lambda)\,d\lambda\,dx \right|. \tag{4.19}$$

Since $\|u_{\tau x}\|^2(\cdot)$ is integrable on $[0,\infty)$, it follows that given any $\epsilon > 0$ there exists a $\delta > 0$ such that t_1 and t_2 in $[0,\infty)$ and $|t_2 - t_1| < \delta$ implies $\int_{t_1}^{t_2} \|u_{\tau x}\|^2(\tau)\,d\tau < \epsilon$; hence it suffices to show that the second term in (4.19) is uniformly continuous. We note that

$$\left| \int_0^1 \int_{u_x(x,t_1)}^{u_x(x,t_2)} \sigma(\lambda)\,d\lambda\,dx \right| = \left| \int_{t_1}^{t_2} \int_0^1 \sigma(u_x(x,\tau))\,u_{\tau x}(x,\tau)\,dx\,d\tau \right|. \tag{4.20}$$

Equation (4.17) implies that

$$|\sigma(\lambda)| \leq E_1(M_*)|\lambda| \tag{4.21}$$

where again $E_1(\cdot)$ is defined by (4.8). Therefore equations (4.20) and (4.21) imply that

$$\left|\int_0^1 \int_{u_x(x,t_1)}^{u_x(x,t_2)} \sigma(\lambda)d\lambda dx\right| \leq E_1(M_*)\int_{t_1}^{t_2} \left\{\|u_x\|^2(\tau) + \|u_{\tau x}\|^2(\tau)\right\}d\tau. \tag{4.22}$$

The uniform continuity of $\|u_t\|^2(\cdot)$ now follows from integrability of $\|u_{\tau x}\|^2$ and uniform boundedness of $\|u_x\|^2(\cdot)$ on $[0,\infty)$.

We shall now show that $\lim_{t\to\infty} \|u_{xx}\|^2(t) = 0.$ Since $\|u_{xx}\|^2(\cdot)$ is integrable on $[0,\infty)$, we are guaranteed the existence of a sequence of time points $\{t_i\}$ such that $\lim_{i\to\infty} t_i = +\infty$ and $\lim_{i\to\infty} \|u_{xx}\|^2(t_i) = 0$. Equation (4.4) now tells us that

$$\|u_{xx}\|^2(t) \leq 2\int_{t_i}^{\infty} \|u_{\tau x}\|^2(\tau)d\tau + \|u_{xx}\|^2(t_i) + 2\|u_{xx}\|(t)\|u_t\|(t)$$
$$+ 2\|u_{xx}\|(t_i)\|u_t\|(t_i), \qquad t \geq t_i. \tag{4.23}$$

The assertion that $\lim_{t\to\infty} \|u_{xx}\|^2(t) = 0$ now follows from (4.23), Lemmas 4.1 and 4.2, and the fact that $\lim_{t\to\infty} \|u_t\|^2(t) = 0$.

5. Completion of the Proof of Theorem 1

Throughout this section u will be an (f,g) displacement on S_∞ and δ will be some preassigned number in $(0,\alpha_o]$.#

#We take $\delta \in (0,\alpha_o]$ so that we may later apply the results of Section 3.

Let us now treat equation (E) as an ordinary differential equation for $u_{xx}(x,\cdot)$. If we multiply (E) by the factor $\exp\left(\int_{t_1}^{t} \sigma'(u_x(x,\eta))d\eta\right)$ and integrate the resulting expression over (t_1,t_2), we see that the derivative u_{xx} of an (f,g) displacement u must satisfy the following identity:

$$u_{xx}(x,t_2) = u_t(x,t_2) + \exp\left(-\int_{t_1}^{t_2} \sigma'(u_x(x,\eta))d\eta\right)[u_{xx}(x,t_1) - u_t(x,t_1)]$$
$$- \exp\left(-\int_{t_1}^{t_2} \sigma'(u_x(x,\eta))d\eta\right)\left\{\int_{t_1}^{t_2} u_\tau(x,\tau)\sigma'(u_x(x,\tau))\exp\left(\int_{t_1}^{\tau} \sigma'(u_x(x,\eta))d\eta\right)d\tau\right\} \quad (5.1)$$

for all (x,t_1,t_2) in $[0,1]\times[0,\infty)\times[0,\infty)$ with $t_1 \le t_2$.

Setting $t_1 = 0$ in (5.1), noting that f and g are respectively C^4 and C^2 on $[0,1]$ and that σ is C^3 on $(-\infty,\infty)$, and making use of the fact that since u is an (f,g) displacement the derivatives u_{tx} and u_{txx} are by hypothesis continuous on $[0,1]\times[0,\infty)$ we obtain

<u>Lemma</u> 5.1. u_{xx} has two continuous space derivatives on $[0,1] \times [0,\infty)$. Moreover

$$u_{xx}(0,t) = u_{xx}(1,t) = 0, \qquad t \geq 0. \tag{5.2}$$

Our next result is

<u>Lemma</u> 5.2.

(a) The function $\phi(u)$ defined by

$$\phi(u)(x,t) = \sigma'(u_x(x,t))u_{xx}(x,t), \qquad (x,t) \in [0,1] \times [0,\infty), \tag{5.3}$$

satisfies hypothesis (A-2) of Theorem 3.

(b) The velocity field u_t admits the following representation:

$$u_t(x,t) = \begin{cases} \displaystyle\int_{t-\delta}^{t}\int_0^1 G(x,\xi,t-\eta)\phi(u)(\xi,\eta)\,d\xi\,d\eta + \int_0^1 G(x,\xi,\delta)u_t(\xi,t-\delta)\,d\xi, & (x,t) \in [0,1] \times [\delta,\infty), \\ \displaystyle\int_0^t\int_0^1 G(x,\xi,t-\eta)\phi(u)(\xi,\eta)\,d\xi\,d\eta + \int_0^1 G(x,\xi,t)g(\xi)\,d\xi, & (x,t) \in [0,1] \times [0,\delta], \end{cases} \tag{5.4}$$

where of course G is the Green's function for (H) (see (3.16)-(3.18)), and g is the initial velocity field.

<u>Proof</u>. Assertion (a) follows immediately from Lemma 5.1.

Assertion (b) depends on the fact that we can regard (E) as the heat equation (H) for u_t. Equation $(5.4)_1$ is an immediate consequence of part (a) of the lemma, the fact that by hypothesis u_t has two continuous space derivatives on $[0,1] \times [0,\infty)$ and vanishes at 0 and 1 for all times $t \geq 0$, and Theorem 3; while $(5.4)_2$ follows from part (a), the smoothness of the initial data, and Theorem 3.

Equations (5.4), (3.7), (3.8), (3.10), and (3.11) now tell us that

$$C^{-1}|u_t|(t) \leq \begin{cases} \delta^{3/4}\|\phi(u)\|_{t-\delta,\delta} + \delta^{-1/4}\|u_t\|(t-\delta), & t \geq \delta, \\ t^{3/4}\|\phi(u)\|_{0,t} + |g|, & t \in [0,\delta], \end{cases} \tag{5.5}$$

and

$$C^{-1}|u_{tx}|(t) \leq \begin{cases} \delta^{1/4}\|\phi(u)\|_{t-\delta,\delta} + \delta^{-3/4}\|u_t\|(t-\delta), & t \geq \delta, \\ t^{1/4}\|\phi(u)\|_{0,t} + |g_x|, & t \in [0,\delta], \end{cases} \tag{5.6}$$

where $\|\cdot\|_{t-\delta,\delta}$, $\|\cdot\|(t-\delta)$, and $|\cdot|$ are defined in (3.5). It follows from equation (5.3) that

$$\|\phi(u)\|_{t-\delta,\delta} \leq E_1\left(|u_x|_{t-\delta,\delta}\right)\|u_{xx}\|_{t-\delta,\delta}, \tag{5.7}$$

and

$$\|\phi(u)\|_{0,t} \leq E_1\left(|u_x|_{0,t}\right)\|u_{xx}\|_{0,t}, \tag{5.8}$$

where $|\cdot|_{0,t}$ and $|\cdot|_{t-\delta,\delta}$ are defined in (3. 5), and where $E_1(\cdot)$ is defined in (4.8). Lemmas 4.1 and 4.3 now imply

$$\|u_t\|(t-\delta) \leq M, \qquad t \geq \delta,$$

and (5.9)

$$\lim_{t\to\infty} \|u_t\|(t-\delta) = 0,$$

where M is the constant in (4.5), while Lemmas 4.2 and 4.3 and Corollary 2 yield

$$|u_x|_{t-\delta,\delta} \leq \|u_{xx}\|_{t-\delta,\delta} \leq M_*, \qquad t \geq \delta,$$
$$|u_x|_{0,t} \leq \|u_{xx}\|_{0,t} \leq M_*, \qquad t \in [0,\delta], \tag{5.10}$$

and

$$\lim_{t\to\infty} \|u_{xx}\|_{t-\delta,\delta} = 0,$$

where M^* is the constant in (4.10).

Combining equations (5.5)-(5.10) we obtain

Lemma 5.3. There exists a constant M_{**} such that

$$|u_t|(t) \leq M_{**} \quad \text{and} \quad |u_{tx}|(t) \leq M_{**}, \qquad t \geq 0. \tag{5.11}$$

Moreover,

$$\lim_{t\to\infty} |u_t|(t) = \lim_{t\to\infty} |u_{tx}|(t) = 0. \tag{5.12}$$

We shall now prove

Lemma 5.4. There exists a constant $M_\#$ such that

$$|u_{xx}|(t) \leq M_\#, \qquad t \geq 0. \tag{5.13}$$

Moreover,

$$\lim_{t\to\infty} |u_{xx}|(t) = 0. \tag{5.14}$$

Proof. An immediate consequence of (5.1) is the identity

$$|u_{xx}|(t_2) \le |u_t|(t_2) + e^{-E_o(t_2-t_1)}[|u_{xx}|(t_1) + |u_t|(t_1)] + |u_t|_{t_1,t_2-t_1}[1-e^{-E_1(t_2-t_1)}] \quad (5.15)$$

for $0 \le t_1 \le t_2 < \infty$, where

$$E_o \stackrel{\text{def}}{=\!=} E_o(M_*) \quad \text{and} \quad E_1 \stackrel{\text{def}}{=\!=} E_1(M_*) \qquad (5.16)$$

and M_* is the upper bound for u_x. Setting $t_1 = 0$ in (5.15) and making use of the fact that Lemma 5.3 implies

$$|u_t|_{t_1,t_2-t_1} \le M_{**} \quad \text{for all} \quad 0 \le t_1 < t_2 < \infty,$$

we see that

$$|u_{xx}|(t) \le 3M_{**} + [|f_{xx}| + |g|] \stackrel{\text{def}}{=\!=} M_{\#}.$$

Equations (5.13) and (5.15) now yield

$$|u_{xx}|(t_2) \le 4|u_t|_{t_1,t_2-t_1} + M_{\#}e^{-E_o(t_2-t_1)}; \qquad (5.17)$$

and (5.17) and $(5.12)_1$ establish (5.14).

Our final task shall be to show that the function $\phi(u)$ defined in (5.3) satisfies condition (iii)' of (A-2)'. We shall prove

Lemma 5.5. For any $t \in [0,\infty)$ and any fixed $\delta \in (0,\alpha_o]$ there exists a constant $K'(t,\delta) < \infty$ such that

$$\frac{|\phi(u)(x,\eta) - \phi(u)(x,t)|}{|\eta - t|^{1/2}} \le K'(t,\delta), \qquad (x,\eta) \in [0,1]\times[t,t+\delta]. \qquad (5.18)$$

Moreover,

$$K'(t,\delta) \leq K'' < \infty, \qquad t \in [0,\infty), \tag{5.19}$$

and

$$\lim_{t\to\infty} K'(t,\delta) = 0. \tag{5.20}$$

Granting the validity of Lemma 5.5, it then follows from equations (5.4), (3.9), (3.14), and (3.15) that

$$C^{-1}|u_{txx}|(t) \leq \begin{cases} K'(t-\delta,\delta)\delta^{1/2} + |\phi_u|_{t-\delta,\delta} + \delta^{-1/2}|u_{tx}|(t-\delta), & t \geq \delta, \text{ and} \\ K'(0,t)t^{1/2} + |\phi_u|_{0,t} + |g_{xx}|, & t \in [0,\delta], \end{cases} \tag{5.21}$$

and

$$C^{-1}|u_{tt}|(t) \leq C^{-1}|u_{txx}|(t) + |\phi(u)|(t), \qquad t \geq 0. \tag{5.22}$$

Since

$$\left.\begin{aligned} |\phi(u)|_{t-\delta,\delta} &\leq E_1\left(|u_x|_{t-\delta,\delta}\right)|u_{xx}|_{t-\delta,\delta}, \\ |\phi(u)|_{0,t} &\leq E_1\left(|u_x|_{0,t}\right)|u_{xx}|_{0,t}, \quad \text{and} \\ |\phi(u)|(t) &\leq E_1\left(|u_x|(t)\right)|u_{xx}|(t), \end{aligned}\right\} \tag{5.23}$$

Lemmas 5.3 and 5.4, when combined with equations (5.21)-(5.23), establish

Lemma 5.6. There exists a constant $M_{\#\#}$ such that

$$|u_{tt}|(t) \leq M_{\#} \quad \text{and} \quad |u_{txx}|(t) \leq M_{\#\#}, \qquad t \geq 0. \tag{5.24}$$

In addition,

$$\lim_{t\to\infty} |u_{tt}|(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} |u_{txx}|(t) = 0. \tag{5.25}$$

We now prove Lemma 5.5. Since

$$\begin{aligned} |\phi(u)(x,\eta)-\phi(u)(x,t)| &= |\sigma'(u_x(x,\eta))u_{xx}(x,\eta)-\sigma'(u_x(x,t))u_{xx}(x,t)| \\ &\le |u_{xx}(x,\eta)||\sigma'(u_x(x,\eta))-\sigma'(u_x(x,t))| \\ &\quad + |\sigma'(u_x(x,t))||u_{xx}(x,\eta)-u_{xx}(x,t)|, \end{aligned} \tag{5.26}$$

since

$$|\sigma'(u_x(x,\eta))-\sigma'(u_x(x,t))| = |\sigma''(u_x(x,\alpha))u_{xt}(x,\alpha)(\eta-t)|$$

for some $\alpha \in (\eta,t)$, and since $|u_x|(t)$, $|u_{xx}|(t)$, and $|u_{xt}|(t)$ are uniformly bounded on $[0,\infty)$ and tend to zero as t tends to infinity, we see that the first term of (5.26) gives no trouble, and hence it suffices to show that (5.18)-(5.20) hold with $\phi(u)$ replaced by u_{xx}. Now (5.1) with $t_2 = \eta$ and $t_1 = t$ implies

$$\begin{aligned} |u_{xx}(x,\eta)-u_{xx}(x,t)| &\le |u_t(x,\eta)-u_t(x,t)| \\ &\quad + \left\{|u_{xx}|(t) + |u_t|(t) + |u_t|_{t,\eta-t}\right\} \times \left|1 - \exp\left(-\int_t^\eta \sigma'(u_x(x,\lambda))d\lambda\right)\right| \\ &\le |u_t(x,\eta)-u_t(x,t)| + \left\{|u_{xx}|(t) + |u_t|(t) + |u_t|_{t,\eta-t}\right\}(1-e^{-E_o(\eta-t)}) \end{aligned}$$

for $0 \le t \le \eta < \infty$ where E_o is defined in (5.16). Since the sum $|u_{xx}|(t) + |u_t|(t) + |u_t|_{t,\eta-t}$ is uniformly bounded and tends to zero as t tends to infinity, it now suffices to show that (5.18)-(5.20) hold with $\phi(u)$ replaced by u_t.

We now consider $|u_t(x,\eta)-u_t(x,t)|$ where $t \in [0,\infty)$, $\eta \in [t,t+\delta]$, and $\delta \leq \alpha_o$. Equation (5.4) with η replacing t and t replacing $t-\delta$ and equation (3.10) imply that

$$|u_t(x,\eta)-u_t(x,t)| \leq C|\phi(u)|_{t,\eta-t}|\eta-t| + \left|\int_0^1 G(x,\xi,\eta-t)u_t(\xi,t)d\xi - u_t(x,t)\right|. \quad (5.27)$$

Remembering that

$$\lim_{\alpha\to 0^+}\int_0^1 G(x,\xi,\alpha)u_t(\xi,t)d\xi = u_t(x,t),$$

we have

$$\left|\int_0^1 G(x,\xi,\eta-t)u_t(\xi,t)d\xi - u_t(x,t)\right| = \left|\int_0^1\int_t^\eta G_\lambda(x,\xi,\eta-\lambda)u_t(\xi,t)d\lambda d\xi\right|$$

$$= \left|\int_0^1\int_t^\eta G_{\xi\xi}(x,\xi,\eta-\lambda)u_t(\xi,t)d\lambda d\xi\right|. \quad (5.28)$$

Since u satisfies (A), we then have

$$\left|\int_0^1 G(x,\xi,\eta-t)u_t(\xi,t)d\xi - u_t(x,t)\right| = \left|\int_0^1\int_t^\eta G_\xi(x,\xi,\eta-\lambda)u_{t\xi}(\xi,t)d\lambda d\xi\right|. \quad (5.29)$$

Equations $(3.11)_2$, (5.27), and (5.29) then yield

$$|u_t(x,\eta)-u_t(x,t)| \leq C|\phi(u)|_{t,\eta-t}|\eta-t| + C|u_{tx}|(t)|\eta-t|^{1/2}, \quad (5.30)$$

where again C is the constant of Theorem 3. Equation (5.3) and Lemma 5.4 tell us that $|\phi(u)|_{t,\eta-t}$ is uniformly bounded and tends to zero as t tends to infinity, while Lemma 5.3 tells us the same conclusion is true for $|u_{tx}|(\cdot)$. Hence, (5.30) establishes Lemma 5.5.

Summarizing the results of Sections 4 and 5 we obtain Theorem 1.

6. Existence and Uniqueness of Solutions of (E), (A), (B), and (C)

We introduce the following notation. For each pair of numbers (τ,α) in $[0,\infty)\times(0,\alpha_o]^{\#}$ we define $D_{\tau,\alpha}$, $C(D_{\tau,\alpha})$, and $C(D_{\tau,\alpha})\times C(D_{\tau,\alpha})$ as

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$^{\#}\alpha_o$ is the same constant appearing in Theorems 3 and 4 of Section 3.

---

follows:

$$D_{\tau,\alpha} = \left\{x,t \mid 0\le x\le 1,\ \tau\le t\le\tau+\alpha\right\}, \tag{6.1}$$

$$C(D_{\tau,\alpha}) = \left\{\text{continuous functions } f \text{ on } D_{\tau,\alpha}\right\}, \tag{6.2}$$

$$C(D_{\tau,\alpha})\times C(D_{\tau,\alpha}) = \left\{(f_1,f_2) \mid f_i\in C(D_{\tau,\alpha}),\ i=1,2\right\}. \tag{6.3}$$

It is easily verified that $C(D_{\tau,\alpha})$ and $C(D_{\tau,\alpha})\times C(D_{\tau,\alpha})$ are Banach spaces under the norms

$$\left.\begin{aligned} |f|_{\tau,\alpha} &= \max_{(x,t)\in D_{\tau,\alpha}} |f(x,t)| \quad \text{and} \\ |(f_1,f_2)|_{\tau,\alpha} &= \max\left(|f_1|_{\tau,\alpha};|f_2|_{\tau,\alpha}\right) \end{aligned}\right\} \tag{6.4}$$

For each (τ,α) in $[0,\infty)\times(0,\alpha_o]$ and each pair of functions Λ and Ω in $C^4[0,1]$ and $C^2[0,1]$, respectively, we define the operators h, T_1, and T_2 mapping $C(D_{\tau,\alpha})\times C(D_{\tau,\alpha})$ into $C(D_{\tau,\alpha})$ by

$$\begin{aligned} h(V,W)(x,t) &= \exp\left(-\int_\tau^t \sigma'(W(x,\eta))d\eta\right)(\Lambda_{xx}(x)-\Omega(x)) \\ &- \exp\left(-\int_\tau^t \sigma'(W(x,\eta))d\eta\right)\left[\int_\tau^t V(x,\eta)\sigma'(W(x,\eta))\exp\left(\int_\tau^\eta \sigma'(W(x,\gamma))d\gamma\right)d\eta\right], \end{aligned} \tag{6.5}$$

$$T_1(V,W)(x,t) = \int_\tau^t \int_0^1 G(x,\xi,t-\eta)\sigma'(W(\xi,\eta))\big(V(\xi,\eta)+h(V,W)(\xi,\eta)\big)d\xi d\eta + \int_0^1 G(x,\xi,t-\tau)\Omega(\xi)d\xi, \quad (6.6)$$

and

$$T_2(V,W)(x,t) = \int_0^x T_1(V,W)(y,t)dy - \int_0^1 d\xi \int_0^\xi T_1(V,W)(y,t)dy + \int_0^x h(V,W)(y,t)dy$$
$$- \int_0^1 d\xi \int_0^\xi h(V,W)(y,t)dy, \qquad (x,t) \in D_{\tau,\alpha}, \qquad (6.7)$$

where of course σ is defined in (1.5) and G is the Green's function for problem (H).

If u is an (f,g) displacement in S_∞, then it is easily verified that for any $(\tau,\alpha) \in [0,\infty) \times (0,\alpha_o]$ the derivatives u_x and u_t satisfy

$$u_t(x,t) = T_1(u_t,u_x)(x,t) \quad \text{and} \quad u_x(x,t) = T_2(u_t,u_x)(x,t), \qquad (x,t) \in D_{\tau,\alpha} \qquad (6.8)$$

provided we let $\Lambda(\cdot) \equiv u(\cdot,\tau)$ and $\Omega(\cdot) \equiv u_t(\cdot,\tau)$.

Our first step is to show that for any $\tau \geq 0$ and any pair of functions Λ and Ω the equations

$$V(x,t) = T_1(V,W)(x,t) \quad \text{and} \quad W = T_2(V,W)(x,t), \qquad (x,t) \in D_{\tau,\alpha}, \qquad (6.9)$$

have a unique solution provided $\alpha \leq \alpha_o$ is sufficiently small.

We let L be any fixed positive number satisfying

$$J(\Lambda,\Omega) \leq L \qquad (6.10)^{\#}$$

#See equation (2.2) for a definition of $J(\Lambda,\Omega)$.

and we let (V,W) and (V',W') be any two pairs of functions in $C(D_{\tau,\alpha}) \times C(D_{\tau,\alpha})$

obeying

$$|(V,W)|_{\tau,\alpha} \leq 4L \quad \text{and} \quad |(V',W')|_{\tau,\alpha} \leq 4L \tag{6.11}$$

Then, a straightforward calculation shows that

$$|(T_1(V,W),T_2(V,W))|_{\tau,\alpha} \leq \alpha k(L) \tag{6.12}$$

and

$$|(T_1(V,W)-T_1(V',W'),\ T_2(V,W)-T_2(V',W'))|_{\tau,\alpha} \leq \alpha \ell(L)|(V-V',W-W')|_{\tau,\alpha}. \tag{6.13}$$

It now follows that for any L satisfying (6.10) we can choose $\alpha \leq \alpha_o$ sufficiently small so that the mapping $(V,W) \to (T_1(V,W),T_2(V,W))$ maps the ball $\left\{(V,W) \;\middle|\; |(V,W)|_{\tau,\alpha} \leq 4L\right\}$ into itself and is contracting. Hence, there exists a unique pair of functions (V,W) in $C(D_{\tau,\alpha}) \times C(D_{\tau,\alpha})$ with $|(V,W)|_{\tau,\alpha} \leq 4L$ satisfying (6.9).

In addition, we have

Lemma 6.1. Let (V,W) be the unique pair in $C(D_{\tau,\alpha}) \times C(D_{\tau,\alpha})$ satisfying (6.9) and define U_1 and U_2 by

$$U_1(x,t) = \int_0^x W(\xi,t)d\xi \quad \text{and} \quad U_2(x,t) = \Omega(x) + \int_\tau^t V(x,\eta)d\eta \tag{6.14}$$

for $(x,t) \in D_{\tau,\alpha}$. Then:

(i) $U_1 \equiv U_2$ in $D_{\tau,\alpha}$. Moreover,

$$U(x,\tau) = \Lambda(x), \quad U_t(x,\tau) = \Omega(x), \quad \text{and} \quad U(0,t) = U(1,t) = 0, \tag{6.15}$$

where now $U \equiv U_1 \equiv U_2$.

(ii) The partial derivatives V_x, V_t, V_{xx}, W_x, W_t, W_{xx}, W_{xt}, W_{tx}, and W_{xxx} exist, are in $C(D_{\tau,\alpha})$, and satisfy

$$V_t - V_{xx} = \sigma'(W)W_{xx}, \qquad W_t = V_x, \quad \text{and} \quad V_{xx} = W_{tx} = W_{xt} \quad \text{in } D_{\tau,\alpha}. \tag{6.16}$$

<u>Proof</u>. The assumptions on Λ, Ω, and σ, the fact that V and W satisfy (6.9), and calculations similar to these employed in the proof of Theorems 3 and 4 imply that W has three and V two continuous space derivatives in $D_{\tau,\alpha}$. The arguments used to establish Theorem 3, the smoothness of the spatial derivatives of V and W, and equation (6.9) allow us to conclude that V_t and hence W_t exist and are continuous in $D_{\tau,\alpha}$. That W_{tx} and W_{xt} are in $C(D_{\tau,\alpha})$ now follows from the representations for W_x and W_t. The verification of (6.10)-(6.12) is also a straightforward calculation.

<u>Lemma</u> 6.2. For any (f,g) there exists a number $\alpha \leq \alpha_o$ and there exists a unique (f,g) displacement u^1 defined on $D_{o,\alpha} = S_{o,\alpha}$. Moreover,

$$|||u^1|||(t) \leq M, \qquad t \in [0,\alpha] \tag{6.17}$$

where M = M(J(f,g)) is the a priori bound obtained in Theorem 1.

<u>Proof</u>. The existence of an $\alpha \leq \alpha_o$ and the existence and uniqueness of an (f,g) displacement on $D_{o,\alpha}$ follow from Lemma 6.1 by setting $\Lambda = f$ and $\Omega = g$. The bound (6.17) follows from Theorem 1.

<u>Completion of the Proof of Theorem</u> 2. We extend the interval of existence in the following way. We now take $\Lambda(\cdot) = u^1(\cdot,\alpha)$ and $\Omega(\cdot) = u^1_t(\cdot,\alpha)$ where u^1 denotes the function obtained in Lemma 6.2 and we solve (6.9) in $D_{\alpha,\alpha}$.#

#The fact that $J(f,g) \leq M(J(f,g))$ and equation (6.17) allow us to choose α in Lemma 6.2 so that this may be done.

Using the solutions of (6.9) we employ the construction of Lemma 6.1 and obtain a function U which extends the function u^1 to the strip $D_{\alpha,\alpha}$. The composite function u^2 defined by

$$u^2(x,t) = \begin{cases} u^1(x,t), & (x,t) \in D_{o,\alpha} \\ U(x,t), & (x,t) \in D_{\alpha,\alpha} \end{cases} \tag{6.18}$$

is an (f,g) displacement in $D_{o,2\alpha}$ and hence by Theorem 1 satisfies (6.17) for all t in $[0,2\alpha]$.

Proceeding inductively, we may now extend the domain of existence and uniqueness, increments of α, to $[0,1]\times[0,T]$ for any $T > 0$.

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[3] Coleman, B. D. and W. Noll, "The Thermodynamics of Elastic Materials with Heat Conduction and Viscosity", Arch. Rational Mech. Anal. **13**, 167-178 (1963).

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p.15 $(3.22)_4$: '$\|(t-\tau)^{1/2} G_\tau(x,\xi,t-\tau)\|$' should be '$\|G_\tau(x,\xi,t-\tau)\|$'.

p.16 (3.23): '$2(1+\alpha^{1/2} \int_0^\infty e^{-z^2} dz)$' should be '$1 + \alpha^{1/2} \int_0^D e^{-z^2} dz$'

p.30 delete '(4.2)'

p.33 (4.12): 'M^2' should be '$2M^2$'

(4.15): '$\frac{2}{E_o(M_2)}(M^2+MM_1+M_1^2)$' should be '$\frac{1}{2E_o(M_2)}(2M^2+4MM_2+M_2^2)$'

p.34 line 3: '$(4.10)_1$ and (4.1) with $u = \phi$.' should be '(4.14).'

line -2: 'is uniformly continuous.' should be 'can be made arbitrarily small simply by making the difference $|t_2-t_1|$ small.'

p.41 (5.22): delete 'C^{-1}' twice

(5.24): 'M_*' should be 'M_{**}'

p.42 line -5: '$(1-e^{-E_o(\eta-t)})$' should be '$(1-e^{-E_1(\eta-t)})$'

line -4: 'E_o' should be 'E_1'

p.43 (5.30): '$\|u_{tx}\|(t)$' should be '$\|u_{tx}\|_{t,\eta-t}$'

p.44 line 2: '$\|u_{tx}\|(\cdot)$.' should be '$\|u_{tx}\|_{t,\eta-t}$.'

p.47 (6.11): '4L' should be 'NL' (twice)

(6.12): '$\alpha k(L)$' should be '$\alpha k_1(\alpha_o,N,L) + CL$'

line 5: 'and' should be 'where C is independent of N, L, and α and'

(6.13): '$\alpha \ell(L)$' should be '$\alpha k_2(\alpha_o,N,L)$'

line 7: 'that for' should be 'that if N is chosen to be greater than C then for'

line 9: '4L' should be 'NL'

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(6.14): '$L(x)$' should be '$\Lambda(x)$'

p. 48 (6.16): '$u'(W)W_{xx}$' should be '$u'(W)W_x$'

line 13: 'and there exists' should be 'such that there exists'

p. 49 Footnote: '$J(f,g) \leq M(J(f,g))$ and equation (6.17) allow'
should be 'by equation (6.17) $J(\Lambda,\Omega) \leq M(J(f,g))$ allows'

line -3: 'increments' should be 'in increments'

line -1: 'Research and' should be 'Research under Grants AF-AFOSR-728-66 and AF-AFOSR-647-66 and'
'Foundation.' should be 'Foundation under Grant GP-6173.'

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