An Existence Theorem for a Class of Nonlinear Integral Equations[1]

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1. Introduction. In this note we use variational methods to prove the existence of a non-trivial solution to an integral equation of the form

$$(1) \qquad y(x) = \int K(x,t)y(t)F(y^2(t),t)dt,$$

where Q is a bounded region in Euclidean space and K(x,,t) is symmetric., square integrable over *Q* x R, and positive definite: The kernal K(x,t) need not be bounded but beyond square integrability, a further, and fairly strong, restriction on its singularities is assumed here. The conditions on F(?7,x) are set down in detail in Section 2; here we mention at least that F is assumed to be non-negative and a strictly increasing function of *rj* for fixed x. Thus we are imposing a condition of strict non-linearity on the problem (1)*y* and it is obvious that some such condition is necessary for the sort of existence theorem obtained here.

Theorems 1 and 2 below are suggested by results of Nehari, [3], for an integral equation of the same form with a continuous kernel. Except where we deal with the difficulties resulting from the unboundedness of the kernel, the proofs of our results parallel those of the analogous results in [3]. In particular we have followed Nehari in the choice of the variational problem to be used in the investigation of (1). This variational problem is not an analogue of the variational problem used to treat the linear case, in fact the functional which we minimize here (with respect to a certain side condition) is identically zero in the linear case. Finally we remark that we impose a

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polynomial growth condition (see (6), (7) below) on F which is not required in [3]. While we know that some polynomial growth condition on P is necessary in Theorem 1 we do not know whether the limit on *y* in (7) can be increased.

In Section 6 we apply our results to the boundary value problem

$$Ay + y F(y^2,x) = 0 \quad \text{in} \quad 0, \qquad y|\wedge_Q = 0,$$

where A is the Laplace operator and *Q* is a region in the plane. The polynomial growth condition (6)*3* (7) limits the applicability of our results essentially to the case where fi is a plane region. We mention that an existence theorem for eigenfunctions of the eigenvalue problem

$$(3) \qquad Ay + AyF(y^2,x) = 0 \quad \text{in} \quad Q, \qquad \wedge d\, 0^{5 * 0}{}'$$

is contained in results of Berger [1]. A boundary value problem of the form (3) is also treated by Levinson, [2] and by Pohožaev [4]. Berger gives conditions in [1] under which the positive spectrum of (3) has a cluster point at oo and conditions under which it has a cluster point at zero. The theorems of Section 6 establish conditions under which the positive spectrum of (3)fills an interval.

2. <u>Statement of the Theorem</u> . Let Q be a bounded region in Euclidean n-space and let K(x,t) be a real valued symmetric function of (x.,t) defined on £2 x fi which is measurable in t for almost all fixed xefi, square integrable over *Q* X *Q,* and positive definite. Assume furthermore that for some number q: 2 < q < oo

$$(4) \qquad M = \text{ess sup}_{x \in n} \left(\int_\Omega |K(x,t)|^q dt \right)^{rr\ V q\ '} < oo .$$

Let F(77,x) be a function defined for real 77 > 0, xe£J$_r$ and which satisfies the following conditions i) the Carathéodory hypothesis (i.e. for almost all fixed xefi; F(rj^x) is continuous in *t]* for T|_> 0, and *F(r)jK)* is measurable in x for each fixed 77 >^ 0, ii) there is a positive constant *e* such that for almost every *xedj* when x is fixed,,

(5) $0 < 7^{\epsilon}F(T?,x) \leq r?2^{\epsilon}F(T?_2,x)$, for $0 < 7\wedge < T]\wedge$

iii) there exist positive numbers c_o, c_1, y such that for almost all **x∊Ω,** when x is fixed,

(6) $F(7?,x) < c_o rj^y + c_{ls}$ for $0 < r\backslash < 00.$

Theorem 1. Let fi,K(x,,t) and F(?7_,t) be as above. If

(7) $7 < (q - 2)/2,$

then the integral equation (1) has at least one non-trivial essentially bounded solution«

3. Formulation of the Variational Problem. We define a function G(rj,x) with the same domain as that of *F(r),x)*

(8) $G(Th,x) = \int_0 F(s,x)ds.$

The variational problem is formulated in terms of functionals J(u,v)j N(y), H(y) which are defined/ for u,v,y€L°° *(Q ,* as follows

(9) $J(u,v) = \int_0 \int_0 K(x,t)u(x)F(u^2(x),x)v(t)F(v^2(t),t)dxdt,$

(10) $H(y) = \int [y^2(x)F(y^2(x),x) - G(y^2(x),x)]dx,$

(11) $N(y) = \int_Q y^2(x)F(y^2(x),x)dx - J(y,y).$

The functionals H and N are continuous on $L^{\infty}(\Omega)$, in fact the following stronger result is valid.

(*) The functionals H and N are continuous on bounded subsets of $L^{\infty}(\Omega)$ with respect to the L^2 topology.

For the proof of (*) we shall require the following.

Lemma 1. Let $f(y,x)$ be a Carathéodory function on $R^1 \times \Omega$. If B is a subset of $L^{\infty}(\Omega)$ and if there is a constant p such that for $y \in B$

(12) $$|f(y(x),x)| \le p \qquad \text{a.e. in } \Omega,$$

then the mapping $y(x) \to f(y(x),x)$ is continuous from B to $L^2(\Omega)$ with respect to the relative L^2 topology on B. Moreover, $\int f(y(x),x)\,dx$ is continuous on B with respect to the L^2 topology.

Proof. With respect to the relative $L^2(\Omega)$ topology on B suppose that the mapping in question is discontinuous at $y_o \in B$. Then there exists a sequence $\{y_n\}$ in B such that $\lim_{n\to\infty} \int_{\Omega} |y_n(x) - y_o(x)|^2 dx = 0$ while

(13) $$\limsup_{n\to\infty} \int |f(y_n(x),x) - f(y_o(x),x)|^2 dx > 0.$$

We can assume that $\{y_n\}$ converges to y_o almost everywhere in Ω, therefore because of the Carathéodory hypothesis and (12) it follows from the Lebesgue bounded convergence theorem that (13) is impossible. The last assertion follows from Schwarz's inequality and the first part of the Lemma.

Proof of (*). Because of $(6)_3$ $f(y,x) = yF(y^2/x)$ satisfies the hypothesis of Lemma 1 for any bounded subset B of $L^{\infty}(U)$. The operator K defined by

(14) $$[Ku](x) = \int K(x,t)u(t)\,dt,$$

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is continuous on L (f2)$_3$ so it follows from the continuity of the inner product that J(u,v) is a continuous function on B X B with respect to the L topology for any bounded subset B of L (ft). The continuity of J y (x) F (y (x), x)dx and of I G (y (x), x)dx on bounded subsets of i/^fft) with respect to the L topology follows from (6) and the last assertion of Lemma 1. This completes the proof of (*).

It is clear from (4) and the Holder inequality that K (defined by (14)) is a bounded operator on L (0). We shall say that a function y€L *(Q)* is admissible if it is not almost everywhere equal to zero and can be represented in the form

(15) **y** = Xu, UGL°° (fi).

Observe that^ because of (6), $y(x)F(y^2(x),x)$ is essentially bounded if y(x) is. Thus from the positive definiteness of K(x,t) and from (5) it follows that if y€L°° (flj > y is not almost everywhere equal to zero and $v(x) = \int_\Omega K(x,t)y(t)F(y^2(t),t)dt$, then v is admissible.

We now summarize certain properties of the functionals H(y) and N(y) which are derived in [3] and whose derivations there remain valid under the hypotheses of this paper.

(**) If **y** is an admissible function then there is a positive **constant** *ex.* such that

(16) **$N(ay_Q) = 0$.**

If y is admissible and satisfies

(17) N(y) = 0,

and if

(18) $v(x) = a \int_\Omega K(x,t)y(t)F(y^2(t),t)dt$.

where $a > 0$ is chosen so that

(19) $$N(v) = 0$$

then (v is admissible and)

(20) $$\int_\Omega y^2(x)F(y^2(x),x)\,dx \le \int_\Omega v^2(x)F(y^2(x),x)\,dx,$$

and

(21) $$H(v) \le H(y).$$

Equality holds in (21) if and only if y is a solution of (1).

Finally, if $y \in L^\infty(\Omega)$, then

(22) $$H(y) \ge e(1+e)^{-1}\int_\Omega y^2(x)F(y^2(x),x)\,dx$$

where e is the constant in (5).

For proofs of the assertions in (**) see [3].

The variational problem which we shall now consider is that of minimizing the functional $H(y)$ within the class of admissible functions and subject to the side condition $N(y) = 0$. It is clear from *(**)* that a solution y of this variational problem must be a solution of the integral equation (1).

4. Solution of the Variational Problem. In this section we prove the existence of a solution to the variational problem posed above. We assume throughout that the hypotheses of Theorem 1 are satisfied.

Lemma 2. There is a positive constant m such that if y is an admissible function satisfying (17) then

(23) $$\int_\Omega y^2(x)F(y^2(x),x)\,dx \ge m.$$

Moreover there are positive constants k_1, k_2 such that

(24) $$\left|\int_\Omega K(x,t)y(t)F(y^2(t),t)dt\right| \le (k_1 k_2 \int_\Omega y^2(t)F(y^2(t),t)dt)^{p},$$

where $\frac{1}{p}+\frac{1}{q}=1$; q is the number in (4).

Proof. By the Schwarz inequality it follows that when (17) holds then

(25) $$\int_\Omega y^2(x)F(y^2(x),x)dx \le \left(\int_\Omega y^2(x)F(y^2(x),x)dx\right)^{\frac{1}{2}}\left(\int_\Omega\left(\int_\Omega K(x,t)y(t)F(y^2(t),t)dt\right)^2 F(y^2(x),x)dx\right)^{\frac{1}{2}}.$$

By Hölders inequality

(26) $$\left|\int_\Omega K(x,t)y(t)F(y^2(t),t)dt\right| \le M\left(\int_\Omega |y(t)F(y^2(t),t)|^p dt\right)^{\frac{1}{p}},$$

where M is the constant in (4). Thus we have

(27) $$\left(\int_\Omega y^2(x)F(y^2(x),x)dx\right)^{\frac{1}{2}} \le M\left(\int_\Omega |y(t)F(y^2(t),t)|^p dt\right)^{\frac{1}{p}}\left(\int_\Omega F(y^2(t),t)dt\right)^{\frac{1}{2}}$$

By another application of Hölder's inequality we obtain

(28) $$\int_\Omega |y(t)F(y^2(t),t)|^p dt \le \left(\int_\Omega |F(y^2(t),t)|^r dt\right)^{\frac{1}{2r}}\left(\int_\Omega y^2(t)F(y^2(t),t)dt\right)^{\frac{1}{2}},$$

where $r = q/(q-2)$. Combining (27) and (28) we get

(29) $$1 \le M\left(\int_\Omega |F(y^2(t),t)|^r dt\right)^{\frac{1}{2r}}\left(\int_\Omega F(y^2(x),x)dx\right)^{\frac{1}{2}},$$

(it follows from (5) and the definition of admissibility that the term on left in (27) is positive). For simplicity we shall assume that the measure of fl is 1^ we then have

$$\int_\Omega F(y^2(x),x)dx \leq \left(\int_\Omega |F(y^2(x),x)|^r dx\right)^{\frac{1}{r}},$$

and using this in (29) we get

$$(30) \qquad 1 \leq M\left(\int_\Omega |F(y^2(x),x)|^r dx\right)^{\frac{1}{r}} .$$

The proof of the first assertion now follows by contradiction. Suppose that $\{y_n(x)\}$ is a sequence of admissible functions satisfying (17) and

$$\int_\Omega y_n^2(x)F(y_n^2(x),x)dx \to 0 \qquad \text{as } n \to \infty .$$

We can then conclude, using (5), that a subsequence of $\{y_n(x)\}$ which can be assumed to be the full sequence converges almost everywhere to zero. Let A_n denote the subset of Ω where $|y_n(x)| \geq 1$. Since $(r-1)\gamma = 2\gamma/(q-2) \leq 1$, we have on A_n,

$$|F(y_n^2(x),x)|^r = |F(y_n^2(x),x)|\,|F(y_n^2(x),x)|^{r-1} \leq c_2 y_n^2(x)F(y_n^2(x),x)$$

where $c_2 = (c_0 + c_1)^{r-1}$. Thus

$$\int_{A_n} |F(y_n^2(x),x)|^r dx \to 0, \qquad \text{as } n \to \infty.$$

If $B_n = \Omega \setminus A_n$ then since $y_n(x) \to 0$ almost everywhere in Ω, it follows from (6) and the bounded convergence theorem that

$$\int_{B_n} |F(y_n^2(x),x)|^r dx \to 0, \qquad \text{as } n \to \infty .$$

Thus our supposition has led to a contradiction of (30) and (23) is proved.

Let y be an arbitrary function in $L^\infty(\Omega)$, then, by an argument similar to that used above, we obtain the following inequality for almost all $x\in\Omega$,

$$|F(y^2(x),x)|^r \leq c_2 y^2 F(y^2(x),x) + (c_Q + c_j)^r .$$

By integrating this inequality over Q and using the resulting in-equality in (28) we obtain (24) from (26). This completes the proof of Lemma 2.

We now show that the problem

(31) $$H(y) = \min., \qquad N(y) = 0,$$

has a non-trivial solution in the class of admissible functions. Let y^ v and OL be as in (**) . The function $F(77\_,x)$ is increasing in $7]$ for almost all x therefore

$$0 \leq \int_\Omega (v^2(x)-y^2(x))(F(v^2(x),x)-F(y^2(x),x))dx$$

and this implies, in view of (21) and (22),

$$\int y^2(x)F(y^2(x),x)dx \leq \int (v^2(x)F(v^2(x),x)+y^2(x)F(y^2(x),x))dx$$

$$\leq 2e^{-1}(1+e)H(y) .$$

Using (20) this gives

$$a^2 \int_\Omega y^2(x)F(y^2(x),x)dx \leq 2e^{-1}(1+e)H(y) .$$

From (23) follows

(32) $$\alpha^2 \leq CH(y), \qquad C = 2(me)^{-1}(1+e),$$

and finally from (18), (22), (29) and (32) we have

(33) $$\|v\|^\wedge \leq C_o(1+H(y))^{(2+p)/2p} .$$

Let B be a set of admissible functions such that (17) holds for all $y\in B$. Let $\theta(B)$ denote the set of all functions v of the form (18) where $y \in B$ and a is chosen so that (19) holds. Then $\theta(B)$ also consists of admissible functions satisfying (17). It follows from (**) that for B as above

(34) $\quad \inf\{H(y) : y \in \theta(B)\} \leq \inf\{H(y) : y \in B\}$.

Again, if B is as above, and $H(y)$ remains bounded for $y \in B$ then from (33) it follows that $\theta(B)$ is bounded in $L^{\infty}(\Omega)$ norm; from (**), $H(y)$ is bounded on $\theta(B)$ if it is bounded on B. Suppose finally B is as above, $H(y)$ is bounded on B and B is bounded in $L^{\infty}(\Omega)$ norm. Then from (32) and condition iii) on P it follows that $\theta(B)$ is of the form $K B_1$ where B_1 is a bounded set in $L^{\infty}(\Omega)$. Hence, in this last case, $\theta(B)$ regarded as a subset of $L^2(\Omega)$ has compact closure in $L^2(\Omega)$.

From (22) and (23) it follows that

$$A = \inf\{H(y) : y \text{ admissible}, N(y) = 0\} > 0.$$

Take $A_1 > A$ and let $B = \{y : y \text{ admissible}, N(y) = 0, H(y) \leq A_1\}$. Then from the results of the paragraph above, $\theta^2(B) = \theta(\theta(B))$ is conditionally compact in $L^2(\Omega)$. Since $\theta^2(B)$ is bounded in the $L^{\infty}(\Omega)$ norm so is B_2, the closure of $\theta^2(B)$ in $L^2(\Omega)$. Consequently by (*), the functionals H and N are continuous on $\bar{B}_2$, in particular N vanishes identically on B_2. Since B_2 is a compact subset of $L^2(\Omega)$ there is an element $y_0 \in B_2$ such that $H(y_0) = \min\{H(y) : y \in B_2\}$. From the definition of B_2 and the continuity of H, $H(y_0) = \inf\{H(y) : y \in \theta^2(B)\}$. Since $\theta^2(B)$ consists entirely of admissible functions satisfying (17) we must therefore have $H(y_0) \geq A$. On the other hand $\inf\{H(y) : y \in \theta^2(B)\} \leq \inf\{H(y) : y \in B\} = A$. Thus $H(y_0) = A$ and since $A > 0$, y_0 is not almost everywhere equal to zero. It remains to show that y_0 is admissible. This is seen as follows, $\theta(B)$ has the form $K B_1$, where B_1 is bounded in L^{∞} norm. Since

B_1 is therefore a conditionally weakly compact set in $L^2(\Omega)$ and as K is weakly continuous, $\bar{B}_2 = K\bar{B}_3$ where $\bar{B}_3$ is the weak closure of B_1 in $L^2(\Omega)$. It follows that y^0 is admissible. Thus we have proved that under the hypothesis of Theorem 1 the problem

$$H(y) = \min., \quad y \text{ admissible}, \quad N(y) = 0, \tag{35}$$

has a solution. As has already been pointed out a solution of (35) must satisfy (1). Since an admissible function is not almost everywhere equal to zero this completes the proof of Theorem 1.

Remarks. 1. Let C denote the cone of almost everywhere non-negative functions, or some other closed convex cone in $L^\infty(\Omega)$, and suppose that K and F are such that $\int_\Omega K(x,t)y(t)F(y^2(t),t)\,dt$ is in C whenever y is. Then one can add to the definition of admissibility the condition that $y \in C$; with this definition of admissibility the argument given above implies that (1) has a non-trivial solution in C.

2. Solutions of the integral equation studied in [3] are obtained as cluster points (in the topology of uniform convergence) of a certain sequence. We could have used such a sequence here to obtain a solution of (1). The construction is as follows. Let $y = y_1$ be any admissible function satisfying (17) and, for each n, let $y_{n+1}(x) = \alpha_n \int K(x,t)y_n(t)F(y_n^2(t),t)\,dt$ where α_n is chosen so as to make $y = y_{n+1}$ satisfy (17). It is clear from the proof of Theorem 1 that any such sequence lies in a compact subset of $L^2(\Omega)$ and that each of its cluster points is a solution of (1).

5. A More General Equation. The following theorem is the analogue of Theorem II of [3].

Theorem 2. *Let K, F, Ω be as in Section 2 and assume that the* hypothesis of Theorem 1 is^ satisfied. If P(x) is real valued, measurable, non-negative and essentially bounded on Q and if the least eigenvalue A of the symmetrizable linear integral equation

$$(37) \qquad u(x) = -k\int_{\Omega} K(x,t)P(t)u(t)dt$$

is larger than 1, then the integral equation

$$(38) \qquad y(x) = \int_{\Omega} K(x,t)y(t)\,(P(t)+F(y^2(t),t))dt,$$

has a^ non-trivial essentially bounded solution.

The only place in the proof of Theorem 1 where the argument can break down when $F(y^2,x)$ is replaced by $F_1(y^2,x) = P(x)+F(y^2,x)$ is in the demonstration (for which the reader was referred to [3]) that the normalization (16) is possible for any admissible function y_0. However if $F(y^2,x)$ is replaced by $F_1(y^2,x) = P(x)+F(Y^2,x)$ in (9) and (11) then the normalization (17) is still possible provided the least eigenvalue of (37) exceeds 1. For a proof of this we again refer the reader to [3]. All of the rest of the arguments used above remain valid as they stand when F is replaced by F_1. It should be noted that H(y) remains unchanged when F is replaced by P_x.

6. Application to a Non-linear Elliptic Boundary Value Problem. We consider the boundary value problem

$$(39) \qquad \frac{\partial^2 y}{\partial x_1^2} + \frac{\partial^2 y}{\partial x_2^2} + y^2F(y^2,x) = 0 \qquad \text{in } \Omega, \quad y|_{\partial\Omega} = 0.$$

where $x = (x_1,x_2)$ and Q is a bounded region in the x-plane for which the Diriclit problem is solvable. If G(x^t) denotes Green's function for the Diriclit problem in *Q,* we have then for suitable positive contants a^b,

$$0 < G(x,t) \le -a \log|x-t| + b, \quad x.teQ,$$

Thus for any q > 2 there is a constant M such that

$$\left(\int_\Omega |G(x,t)|^q dt\right)^{\frac{1}{q}} < M, \qquad \text{for all } x\in\Omega.$$

Using Theorem 1 and Remark 1 following that theorem we therefore obtain the following result.

Theorem 3. Let F(y,x) satisfy conditions (i), (ii) and (iii) of Section 2, where y iri (6)i^ any positive number. Suppose also that *F(T),x)* satisfies <a local Holder condition in (^?^x) cm the region { *(r),x) iV* > 0^ *xeCij* . Then (39) has a^ solution y(x) which is positive and of class C^2 rn *Of* and is continuous in *Q.*

We have also the following theorem.

Theorem 4. Let F(7^x) be as in Theorem 3. Let P(x) satisfy <* local Holder condition in *Q,*. If the least eigenvalue of the problem

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \lambda P(x)u = 0 \quad \text{in } Q, \qquad u|_{\partial\Omega} = 0,$$

is larger than 1, then the problem

$$\frac{\partial^2 y}{\partial x_1^2} + \frac{\partial^2 y}{\partial x_2^2} + p(x)y + F(y,x) = 0 \text{ in } \Omega, \qquad y|_{\partial\Omega} = 0,$$

has a solution y(x) which is positive and of class C^2 in *Q* and is continuous in O.

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