The Discrete Analogue of a Class of Entire Functions

by

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Abstract

Discrete analytic functions are complex valued functions defined at points of the z-plane with integer coordinates. The real and imaginary parts of these functions are required to satisfy difference equations analogous to the Cauchy-Riemann equations. The pseudo power $z^{(n)}$ is a discrete analytic function which is asymptotic to the ordinary power z^n for large z. The central topic of this paper is the correspondence between the pseudo power series $f = \sum c_n z^{(n)}$ and the ordinary power series $F = \sum c_n z^n$. The coefficients are restricted by the relation $\limsup |n!\, c_n|^{1/n} < 2$ and this insures that both series converge at all points. The correspondence defines a linear transformation T such that f = TF. It is shown that T can be expressed as a contour integral and that T has a unique inverse. By virtue of the transformation various operations on the entire function class (F) induce corresponding operations on the discrete function class (f). In particular a ring of discrete analytic functions can be formed by defining the product of the functions f and g as T(FG).

The Discrete Analogue of a Class of Entire Functions

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1. Introduction. This paper is concerned with complex functions defined at the nodes of a square mesh $\mathcal{M}_h$ that lies in the complex plane and is depicted in Figure 1.1. The theory of these functions was initiated by R. Isaacs [1, 2, 3], J. Ferrand [4], and A. Terracini [5, 6], and has been further developed by R. J. Duffin [7], C. S. Duris [8, 9], E. L. Peterson [10], A. Washburn [11], and J. Rohrer [12, 13]. The functions investigated are said to be "discrete analytic" and are discrete analogues of analytic functions of a continuous complex variable. A complex function f is discrete analytic on a square S belonging to $\mathcal{M}_h$ if the difference quotient of f across one diagonal of S is equal to the difference quotient of f across the other diagonal of S .

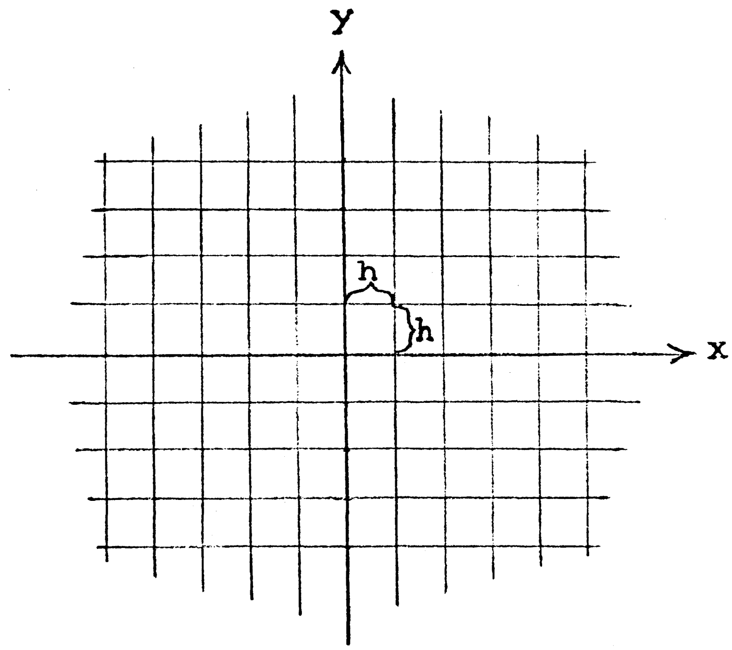

Figure 1.1

G. J. Kurowski [14, 15, 16, 17] has defined and investigated a class of complex functions that have domain consisting of parallel lines in the complex plane. Functions in this class are said to be "semi-discrete analytic" and are semi-discrete analogues of analytic functions of a continuous complex variable. Semi-discrete analytic functions have many properties in common with discrete analytic functions, but they will not be considered in this paper.

The results obtained in [1 - 13] include discrete analogues of the Cauchy Riemann equations, Laplace's equation, the maximum principle, conjugate harmonic functions, differentiation, integration, the residue theorem, Cauchy's theorem, Morera's theorem, Cauchy's integral formula, polynomials, the Laplace transform, analytic continuation, and differential equations. Results concerned with letting the mesh width h approach zero have also been obtained. Some of the concepts studied, including those of "duality" and "bipolynomials", have no direct analogy in the classical continuous theory.

This paper developes a theory for a family $\mathcal{J}$ of "entire discrete analytic functions" that have "discrete Maclaurin series representation". Each function f in $\mathcal{J}$ has an analogue f_c in a family $\mathcal{J}_c$ of entire functions of a continuous complex variable. A mapping T that takes $\mathcal{J}_c$ onto $\mathcal{J}$ in one-to-one fashion and preserves most of the operations involved is found. The image f under T of an arbitrary f_c belonging to $\mathcal{J}_c$ shows properties similar to those of f_c.

It is easy to show that pointwise multiplication does not always preserve discrete analyticity. R. J. Duffin and R. Isaacs have investigated operators that correspond to multiplication of analytic functions of a continuous complex variable, but these operators preserve discrete analyticity only if one of the factors is a polynomial. The approach taken here is to use the transformation T to define a "product" for functions belonging to $\mathcal{J}$. This "product", which is analogous to the usual product for functions in $\mathcal{J}_c$, is commutative, associative, and distributive, and preserves discrete analyticity.

Other "products" have been defined and studied by R. J. Duffin and C. S. Duris [8] , A. Washburn [11] , and R. J. Duffin and J. Rohrer [13] . These "products" are commutative, associative, and distributive, and preserve discrete analyticity. Precise connections between all of the various products have not been completely determined.

In Section 2 we review the basic concepts and theory needed for studying functions in $\mathcal{J}$. Detailed proofs for most theorems are omitted because they can be found in [7] . In Section 3 we state and prove the main results of this paper.

2. Basic Concepts and Theory. The mesh $\mathcal{M}_h$, as shown in Figure 1.1, consists of a union of squares, each of whose sides has length h . The lattice $\mathcal{L}_h$ (or discrete complex plane) is composed of the points that fall on a vertex of a member of $\mathcal{M}_h$. Each member z of $\mathcal{L}_h$ can be represented as $z = xh + iyh$ where x and y are appropriate integers. The even lattice $\mathcal{E}_h$ consists of those lattice points for which $x + y$ is even, and the odd lattice $\mathcal{O}_h$ consists of those lattice points for which $x + y$ is odd.

A chain is an oriented set of lattice points $z_0, z_1, \ldots, z_n$ with the property that $|z_i - z_{i-1}| = h$ for $i = 1, 2, \ldots, n$. A chain $z_0, z_1, \ldots, z_n$ for which $z_0 = z_n$ is called a simple closed chain when its elements, exclusive of z_0 and z_n, are distinct.

A discrete region in the discrete complex plane $\mathcal{L}_h$ consists of the nodes of a union of squares from $\mathcal{M}_h$. The discrete region is said to be constructed from these squares, and the union of the closed planar sets bounded by these squares is called the associated region. The boundary of a discrete region consists of those lattice points that belong to the discrete region and lie on the boundary of its associated region. The union of all lattice points that belong to the discrete region but do not belong to its boundary is called the interior of the discrete region.

A discrete region is said to be finite if it consists of a finite number of lattice points. A simple discrete region is a finite discrete region whose boundary can be represented by a simple closed chain.

We begin our discussion of complex lattice functions by defining a finite difference operator L .

Definition 2.1. Let f be a complex lattice function and suppose that S is a square belonging to $\mathcal{M}_h$. Then the residue $L(f,S)$ of f at the square S is defined by

$$L(f,S) = f(z_0) + if(z_1) + i^2 f(z_2) + i^3 f(z_3) ,$$

where z_0, z_1, z_2, _and_ z_3 _have the orientation depicted in Figure 2.1 and_ $i = \sqrt{-1}$.

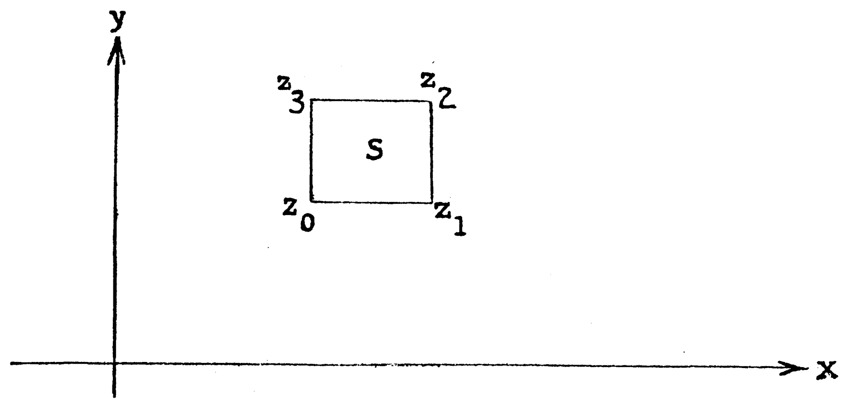

Figure 2.1

The quantity $L(f,S)$ is termed the residue of f at S , because in "discrete contour integration" it plays a role analogous to the role played by the residue of a complex function of a continuous complex variable in the classical theory of contour integration.

Discrete analyticity can be defined in terms of the operator L as follows.

Definition 2.2. _A lattice function_ f _is discrete analytic on a square_ S _belonging to_ $\mathcal{M}_h$ _if its residue_ $L(f,S)$ _at_ S _is zero. A lattice_ function f is discrete analytic in a discrete region $\mathcal{R}$ if it is discrete analytic on each square used in the construction of $\mathcal{R}$. The reader should have no trouble showing that this definition of discrete analyticity is equivalent to the one given in the first paragraph of Section 1.

We shall adopt the following definition for "discrete line integrals".

Definition 2.3. Let $C: z_0, z_1, \ldots, z_m$ be a chain and suppose that f is a lattice function. Then the discrete line integral $(C) \int_{z_0}^{z_m} f(z)\,\delta z$ of f from z_0 to z_m over C is defined by

$$(C) \int_{z_0}^{z_m} f(z)\,\delta z = \sum_{j=1}^{m} \frac{[f(z_j) + f(z_{j-1})]}{2} [z_j - z_{j-1}] .$$

The following "residue theorem" is an elementary consequence of the preceding definition. Its proof can be found in [7] and will not be repeated here.

Theorem 2.1. If f is a lattice function defined on a simple discrete region $\mathcal{R}$ and the chain $\partial\mathcal{R}$ representing the boundary of $\mathcal{R}$ is oriented so that $\mathcal{R}$ is on the left as the chain is traversed, then

$$(\partial\mathcal{R}) \oint f(z)\,\delta z = \frac{(1-i)\,h}{2} \sum_{j=1}^{s} L(f, S^j)$$

where $\{S^j\}_{j=1}^{s}$ *is the set of squares used to construct* $\mathcal{R}$.

"Cauchy's theorem" and "Morera's theorem" are immediate consequences of the "residue theorem". They are stated in the following corollary.

Corollary 2.1.1 *Suppose that* f *is a lattice function defined on a simple discrete region* $\mathcal{R}$. *Then* f *is discrete analytic on* $\mathcal{R}$ *if, and only if,* $(C) \oint f(z)\,\delta z = 0$ *for each simple closed chain* C *lying in* $\mathcal{R}$.

As in the classical theory of a continuous complex variable, "Cauchy's theorem" leads to the generation of discrete analytic functions by discrete line integration. The main theorem follows.

Theorem 2.2. *If* f *is discrete analytic in a simple discrete region* $\mathcal{R}$ *and* $a \in \mathcal{R}$ *then*

$$F(z) = \int_a^z f(t)\,\delta t$$

is single-valued and discrete analytic on $\mathcal{R}$.

The function F is single-valued on $\mathcal{R}$ by virtue of Corollary 2.1.1. The proof that F is discrete analytic on $\mathcal{R}$ follows from computing

$L(F,S)$ in terms of $L(f,S)$. The result is $L(F,S) = \frac{h}{2} L(f,S)$.

The analogy between continuous complex variable theory and discrete complex variable theory breaks down in the study of anti-integration. Given a discrete analytic function F, it is not generally true that

$$F(z) = \int_a^z f(t)\,\delta t + F(a)$$

for some discrete function f obtained by taking difference quotients of F. The next definition and theorem characterize, in terms of F, those functions f that satisfy the preceding functional equation.

Definition 2.4. _Let_ γ _be the lattice function defined by_

$$\gamma(z) = \begin{cases} 1 \text{ \underline{if} } z \text{ \underline{is an even lattice point} } & (z \in \mathcal{E}_h) \\ -1 \text{ \underline{if} } z \text{ \underline{is an odd lattice point} } & (z \in \mathcal{O}_h) \end{cases}$$

Then the dual f_D _of a lattice function_ f _is defined by_

$$f_D(z) = \gamma(z)\, f^*(z)$$

where $f^*(z)$ _is the complex conjugate of_ $f(z)$.

A lattice function f is discrete analytic if, and only if, its dual f_D is discrete analytic. This can be seen by computing $L(f_D, S)$ in terms of $L(f,S)$. The result is $L(f_D, S) = \gamma(z_0)[L(f,S)]^*$ where z_0 has the position shown in Figure 2.1. The notion of duality is important because of the following theorem.

Theorem 2.3. <u>Suppose that</u> F <u>is discrete analytic on a simple discrete region</u> $\mathcal{R}$ <u>and let</u> a <u>and</u> b <u>be fixed points of</u> $\mathcal{R}$. <u>If</u> k <u>is an arbitrary constant, then the discrete function</u>

$$\frac{\delta F}{\delta z}(z;k) = \left(\frac{4}{h^2}\int_b^z F_D(t)\,\delta t + k\right)_D$$

<u>is discrete analytic on</u> $\mathcal{R}$ <u>and</u>

$$F(z) - F(a) = (C)\int_a^z \frac{\delta F}{\delta z}(t;k)\,\delta t$$

<u>for each chain</u> $C: a = z_0, \ldots, z_n = z$ <u>lying in</u> $\mathcal{R}$. <u>Conversely, if</u> f <u>is a lattice function on</u> $\mathcal{R}$ <u>with the property that</u>

$$F(z) - F(a) = (C)\int_a^z f(t)\,\delta t$$

<u>for each chain</u> $C: a = z_0, \ldots, z_m = z$ <u>lying in</u> $\mathcal{R}$, <u>then</u>

$$f(z) = \frac{\delta F}{\delta z}(z;k)$$

for some complex constant k .

The preceding theorem shows that the discrete derivative, unlike continuous differentiation, is not unique and can be given an arbitrary value at a fixed point of the discrete region $\mathcal{R}$.

We shall give a detailed proof of the second part of Theorem 2.3, because the second part is new; a proof for the first part is given in [7] . Thus suppose that f is a function defined on $\mathcal{R}$ with the property that

$$(2.1) \qquad F(z) - F(a) = \int_a^z f(t)\,\delta t$$

for each z belonging to $\mathcal{R}$. Then $(C) \oint f(t)\,\delta t = 0$ for each simple closed chain C lying in $\mathcal{R}$. By Corollary 2.1.1 we see that f is discrete analytic on $\mathcal{R}$. We infer from the first conclusion of Theorem 2.3 that the function

$$(2.2) \qquad \frac{\delta f_D}{\delta z}(z;F(a)) = \left(\frac{4}{h^2}\int_b^z f(t)\,\delta t + F(a)\right)_D$$

has the property

$$(2.3) \qquad f_D(z) - f_D(b) = \int_b^z \frac{\delta f_D}{\delta z}(t;F(a))\,\delta t\ .$$

Equation (2.2) shows that

$$(2.4) \qquad \frac{\delta f_D}{\delta z}(z;F(a)) = \frac{4}{h^2}\left(\int_b^z f(t)\,\delta t\right)_D + F^*(a)\,\gamma(z)\ ,$$

and manipulation of equation (2.1) gives

$$F_D(z) = \left(\int_b^z f(t)\,\delta t + F(b)\right)_D\ .$$

Hence

$$\left(\int_b^z f(t)\,\delta t\right)_D = F_D(z) - F^*(b)\,\gamma(z)\ ,$$

which, when substituted into equation (2.4), gives

$$\frac{\delta f_D}{\delta z}(z;F(a)) = \frac{4}{h^2}F_D(z) - \frac{4}{h^2}F^*(b)\,\gamma(z) + F^*(a)\,\gamma(z)\ .$$

Substituting this expression into (2.3) shows that

$$f_D(z) - f_D(b) = \frac{4}{h^2}\int_b^z F_D(t)\,\delta t + \left(F^*(a) - \frac{4}{h^2}F^*(b)\right)\int_b^z \gamma(t)\,\delta t\ .$$

It is clear from Definitions (2.3) and (2.4) that $\int_b^z \gamma(t)\,\delta t = 0$. Hence

$$f_D(z) = \frac{4}{h^2} \int_b^z F_D(t)\,\delta t + \gamma(b)\, f^*(b),$$

and thus

$$f(z) = \left(\frac{4}{h^2} \int_b^z F_D(t)\,\delta t + \gamma(b)\, f^*(b)\right)_D = \frac{\delta F}{\delta z}(z;k),$$

where $k = \gamma(b)\, f^*(b)$. This completes our proof of Theorem 2.3.

The following theorem provides a discrete analog $e(z;t)$ of the exponential function e^{zt}. We use a semi-colon rather than a comma between the independent variables z and t in $e(z;t)$ to indicate that z is a discrete complex variable and t is a continuous complex variable.

Theorem 2.4. *Let $z = xh + iyh$ be a discrete variable that is restricted to $\mathcal{L}_h$ (i.e. x and y must be integers), and suppose that t is a continuous complex variable restricted so that $t \neq \pm\frac{2}{h}$ and $t \neq \pm i\frac{2}{h}$. Then the discrete exponential function*

$$(2.5) \qquad e(z;t) = \left(\frac{\frac{2}{h} + t}{\frac{2}{h} - t} \right)^x \left(\frac{\frac{2}{h} + it}{\frac{2}{h} - it} \right)^y$$

<u>is a discrete analytic function of</u> z <u>for fixed</u> t <u>and an analytic function of</u> t <u>for fixed</u> z . <u>Moreover,</u>

$$(2.6) \qquad e(z;t) = 1 + t(C) \int_0^z e(z;t)\, \delta z$$

<u>for each chain</u> $C : 0 = z_0, z_1, \ldots, z_n = z$ <u>lying in</u> $\mathscr{L}_h$.

The validity of (2.6) can be demonstrated directly from the definitions for a discrete integral and $e(z;t)$; the details can be found in [7] . It is clear that $e(z;t)$ is a rational function of t for fixed z , with possible poles only at the points $t = \pm \frac{2}{h}$ and $t = \pm i \frac{2}{h}$; hence $e(z;t)$ is analytic in t for fixed z , except possibly at these points. The discrete analyticity of $e(z;t)$ as a function of z for fixed t is a direct consequence of (2.6) and "Morera's theorem" (Corollary 2.1.1).

The discrete exponential function $e(z;t)$ was introduced by Ferrand [4] and plays a fundamental role in our study of "discrete Maclaurin series". From its defining formula (2.5) it is clear that $e(z;t)$ can be expanded as a Maclaurin series in t , with absolute convergence for $|t| < \frac{2}{h}$. The series coefficients generated by this expansion are, of course, functions of the discrete variable z. These observations justify the following theorem and definition.

Theorem 2.5. *The discrete exponential function* $e(z;t)$ *has a Maclaurin series expansion in* t *given by*

$$(2.7) \qquad e(z;t) = \sum_{n=0}^{\infty} \frac{z^{(n)}}{n!} t^n ,$$

which converges absolutely for each z *in* $\mathcal{L}_h$ *and all* t *in the disk* $|t| < \frac{2}{h}$. *The generated coefficients* $z^{(n)}$, $n = 0,1,2,\ldots$ *are called* ***pseudo-powers***.

The pseudo-powers $z^{(n)}$ are discrete analogs of the powers z^n of a continuous complex variable; the pseudo-powers $z^{(n)}$ are generated by $e(z;t)$, and the powers z^n are generated in the same manner by e^{zt}. The following theorem brings out further analogies between $z^{(n)}$ and z^n.

Theorem 2.6. *The pseudo-powers* $z^{(n)}$, $n = 0,1,2,\ldots$ *are discrete analytic on* $\mathcal{L}_h$. *Moreover,*

$$(2.8) \qquad z^{(0)} \equiv 1$$

and

$$(2.9) \qquad z^{(n)} \equiv n(C) \int_0^z z^{(n-1)} \delta z , \qquad n = 1, 2, \ldots$$

for an arbitrary chain $C : 0 = z_0, z_1, \ldots, z_m = z$ lying in $\mathcal{L}_h$.
Equations (2.8) and (2.9) come from substituting (2.7) into (2.6) and equating the coefficients of t^n that appear on each side of the resulting equation. The discrete analyticity of $z^{(n)}$ follows by induction from (2.8), (2.9), and Theorem 2.2.

In the next section we study a class of pseudo-power series $\sum_{n=0}^{\infty} c_n z^{(n)}$ in which the coefficients c_n are complex constants. It is an elementary consequence of Theorem 2.6 and the definition of discrete analyticity that a convergent pseudo-power series converges to a discrete analytic function f. The series $\sum_{n=0}^{\infty} c_n z^{(n)}$ is said to be a discrete Maclaurin series representation of f.

It is clear from (2.8) that $z^{(0)}$ is equal to its continuous counterpart z^0 for z restricted to $\mathcal{L}_h$. Moreover, it is easily seen from (2.9) that $z^{(1)}$ equals z and $z^{(2)}$ equals z^2 for z in $\mathcal{L}_h$. However, an elementary computation shows that z^3, for z restricted to $\mathcal{L}_h$, is not even discrete analytic on the square with vertices $(0,0)$, $(h,0)$, (h,ih), and $(0,ih)$. Hence $z^{(3)}$ is not identical to z^3 for z in $\mathcal{L}_h$. In [7], for each positive integer n it is shown that there exists a complex polynomial $r_{n-1}(x,y)$ of degree $(n - 1)$ in the real variables x and y such that $z^{(n)} = z^n + r_{n-1}(x, y)$ when $z = xh + iyh$ is restricted to $\mathcal{L}_h$. Thus, $\sum_{n=0}^{\infty} c_n z^{(n)}$ isn't necessarily equal to $\sum_{n=0}^{\infty} c_n z^n$ for z in $\mathcal{L}_h$, but we shall see in the next section that $\sum_{n=0}^{\infty} c_n z^n$ is useful in the study of its "discrete analog" $\sum_{n=0}^{\infty} c_n z^{(n)}$.

3. Discrete Maclaurin Series. In this section we study pseudo-power series

$$(3.1) \qquad f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^{(n)}$$

where z is a discrete complex variable. Corresponding to the pseudo-power series (3.1) is the power series

$$(3.2) \qquad f_c(q) = \sum_{n=0}^{\infty} \frac{a_n}{n!} q^n$$

in which q is a continuous complex variable. Series (3.1) is said to be the discrete analog of series (3.2), and series (3.2) is said to be the continuous analog of series (3.1). In our study of (3.1) and its relation to (3.2) we shall find use for the Borel transform

$$(3.3) \qquad f_B(t) = \sum_{n=0}^{\infty} \frac{a_n}{t^{n+1}}$$

of series (3.2); here t is a continuous complex variable. We shall always use the subscripts c and B to distinguish between a pseudo-power series (3.1), its continuous analog (3.2), and their Borel transform (3.3).

Instead of formulating, in terms of the coefficient sequence $\{a_n\}$, conditions that are both necessary and sufficient for the convergence of (3.1), we place a condition on $\{a_n\}$ that guarantees the convergence of (3.1) for each z in $\mathcal{L}_h$, and then we investigate the relations between series (3.1) and series (3.2) and (3.3). The condition is that

$$(3.4) \qquad a = \limsup \sqrt[n]{|a_n|} < \frac{2}{h},$$

where h is the mesh width of $\mathcal{L}_h$.

Theorem 3.1. If $a = \limsup \sqrt[n]{|a_n|}$ satisfies relation (3.4), then

(I) Series (3.1) converges to a discrete analytic function f on all of $\mathcal{L}_h$.

(II) Series (3.2) converges to an entire function f_c of order $O(e^{p|q|})$ for each $p > a$ (i.e. f_c is analytic on the whole complex plane, and for each $p > a$ there exists a

<u>positive constant</u> A_p <u>such that</u> $|f_c(q)| \leq A_p e^{p|q|}$ <u>for all complex</u> q).

(III) <u>Series</u> (3.3) <u>converges to an analytic function</u> f_B <u>for</u> $|t| > a$.

<u>Proof.</u> From (3.4) we see that

$$a < \frac{a}{2} + \frac{1}{h} < \frac{2}{h}, \tag{3.5}$$

and this implies the existence of a positive integer N such that

$$\sqrt[n]{|a_n|} \leq \frac{a}{2} + \frac{1}{h} \quad \text{for } n \geq N, \tag{3.6}$$

because a is the largest limit point of $\{\sqrt[n]{|a_n|}\}$. It is then a consequence of (3.5) and (3.6) that

$$|a_n| \leq \left(\frac{a}{2} + \frac{1}{h}\right)^n < \left(\frac{2}{h}\right)^n \quad \text{for } n \geq N, \tag{3.7}$$

which implies that $\sum_N^\infty \frac{|a_n|}{n!} |z^{(n)}|$ is majorized by $\sum_N^\infty \frac{\left(\frac{a}{2} + \frac{1}{h}\right)^n}{n!} |z^{(n)}|$. But $\sum_N^\infty \frac{\left(\frac{a}{2} + \frac{1}{h}\right)^n}{n!} |z^{(n)}|$ converges for each z in $\mathcal{L}_h$ by virtue of Theorem 2.5 and relation (3.5), so $\sum_0^\infty \frac{a_n}{n!} z^{(n)}$ converges absolutely for each z in $\mathcal{L}_h$. The limit function f is discrete analytic because $z^{(n)}$ is discrete analytic, finite linear combinations of discrete analytic functions are clearly discrete analytic, and the limit of a sequence of discrete analytic functions is easily seen to be discrete analytic. This completes the proof of conclusion (I).

Conclusions (II) and (III) are well-known results from the theory of entire functions. In fact these two conclusions are known to be valid under the weaker hypothesis that $a = \limsup \sqrt[n]{|a_n|}$ be finite. The proofs are similar to, and no more difficult than, the proof of conclusion (I). The details can be carried out by the reader or can be found in [19] .

The preceding theorem states that series (3.1), (3.2), and (3.3) converge to functions f , f_c , and f_B respectively, when (3.4) is satisfied. We now concentrate on the functions themselves rather than on their series representations. The next two theorems give formulas for $f(z)$ in terms of $f_B(t)$ and $f_c(q)$ respectively.

Theorem 3.2. <u>If</u> $a = \limsup \sqrt[n]{|a_n|}$ <u>satisfies relation</u> (3.4), <u>then</u> $f(z)$, <u>defined by</u> (3.1), <u>can be expressed in terms of</u> $f_B(t)$, <u>defined by</u> (3.3), <u>as</u>

$$f(z) = -\frac{1}{2\pi i} \sum_{j=1}^{4} \left\{ \oint_{\partial D_\rho(t_j)} e(z;t) \left[f_B(t) - \frac{a_o}{t} \right] dt \right\} + a_o \qquad (3.8)$$

where $\partial D_\rho(t_j)$ is the boundary of the closed disk $D_\rho(t_j)$ that is centered at the point t_j with radius ρ. Here, $t_1 = \frac{2}{h}$, $t_2 = i\frac{2}{h}$, $t_3 = -\frac{2}{h}$, $t_4 = -i\frac{2}{h}$, and $\rho = (\frac{2}{h} - a)/2$.

Proof. Fix z, and choose r so that $a < r < \frac{2}{h}$. Then Theorems 2.5 and 3.1 imply that $e(z;t) = \sum_{n=0}^{\infty} \frac{z^{(n)}}{n!} t^n$ and $f_B(t) = \sum_{n=0}^{\infty} \frac{a_n}{t^{n+1}}$ converge uniformly in t for $t \in \partial D_r(0)$. Hence

$$\frac{1}{2\pi i} \oint_{\partial D_r(0)} e(z;t)[f_B(t) - \frac{a_o}{t}]\, dt = \frac{1}{2\pi i} \oint_{\partial D_r(0)} \left(\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{a_m z^{(n)}}{n!} t^{n-m-1} \right) dt$$

$$= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{a_m z^{(n)}}{n!} \left[\frac{1}{2\pi i} \oint_{\partial D_r(0)} t^{n-m-1} dt \right] = \sum_{n=1}^{\infty} \frac{a_n}{n!} z^{(n)} = f(z) - a_o .$$

Since t_1, t_2, t_3, and t_4 are the only possible singular points of $e(z;t)$ and since $f_B(t)$ is regular for $|t| > a$, we can deform the contour $\partial D_r(0)$ to obtain

$$f(z) = -\frac{1}{2\pi i} \sum_{j=1}^{4} \oint_{\partial D_\rho(t_j)} e(z;t)\,[\,f_B(t) - \frac{a_o}{t}\,]\, dt +$$

$$\frac{1}{2\pi i} \oint_{\partial D_R(0)} e(z;t)\,[\,f_B(t) - \frac{a_o}{t}\,]\, dt + a_o$$

where $R \geq \frac{2}{h} + \rho$. Hence, it is sufficient to show that

$$(3.9) \qquad \frac{1}{2\pi i} \oint_{\partial D_R(0)} e(z;t) \left[f_B(t) - \frac{a_o}{t} \right] dt = 0 .$$

Letting $t = Re^{i\theta}$ and making use of defining formula (2.5) for $e(z;t)$, we see that

$$\left| \frac{1}{2\pi i} \oint_{D_R(0)} e(z;t) \left[f_B(t) - \frac{a_o}{t} \right] dt \right| =$$

$$\frac{1}{2\pi} \left| \int_o^{2\pi} \left(\frac{2 + hRe^{i\theta}}{2 - hRe^{i\theta}} \right)^x \left(\frac{2 + ihRe^{i\theta}}{2 - ihRe^{i\theta}} \right)^y \left[f_B(Re^{i\theta}) - \frac{a_o}{Re^{i\theta}} \right] iRe^{i\theta} \, d\theta \right|$$

$$\leq \left| \left[\frac{2 + hR}{2 - hR}\right] \right|^{|x|+|y|} \left(\sum_{n=1}^{\infty} \frac{|a_n|}{R^{n-1}} \right) \frac{1}{R} .$$

It is clear that the last expression approaches zero as R approaches infinity. By Cauchy's theorem we conclude that (3.9) is valid, and this completes the proof of Theorem 3.2.

The following theorem gives a formula for $f(z)$ in terms of $f_c(q)$. This theorem is the main result of this paper.

Theorem 3.3. *If* $a = \limsup \sqrt[n]{|a_n|}$ *satisfies relation* (3.4), *then* $f(z)$, *defined by* (3.1), *can be expressed in terms of* $f_c(q)$, *defined by* (3.2), *as*

(3.10)
$$f(z) = + U(x-1) \sum_{j=0}^{x-1} X_j^{+}(x,y) \int_0^{\infty} [f_c(q) - a_o] e^{-\frac{2}{h}q} q^j \, dq$$

$$+ U(-x-1) \sum_{j=0}^{-x-1} X_j^{-}(x,y) \int_0^{-\infty} [f_c(q) - a_o] e^{\frac{2}{h}q} q^j \, dq$$

$$+ U(y-1) \sum_{j=0}^{y-1} Y_j^{+}(x,y) \int_0^{i\infty} [f_c(q) - a_o] e^{i\frac{2}{h}q} q^j \, dq$$

$$+ U(-y-1) \sum_{j=0}^{-y-1} Y_j^{-}(x,y) \int_0^{-i\infty} [f_c(q) - a_o] e^{-i\frac{2}{h}q} q^j \, dq + a_o ,$$

where U *is the unit step function,*

(3.11)
$$U(\beta) = \begin{cases} 0 & \text{if } \beta < 0 \\ 1 & \text{if } \beta \geq 0 , \end{cases}$$

and X_j^{+}, X_j^{-}, Y_j^{+}, Y_j^{-} *are defined as*

(3.12)

$$X_j^+(x,y) = \frac{(-1)^{j+1}}{j!(x-1-j)!x} \frac{d^{x-1-j}}{dw^{x-1-j}}\left([-\frac{4}{h} - w]^x \left[\frac{-i\frac{2}{h} + \frac{2}{h} + w}{-i\frac{2}{h} - \frac{2}{h} - w}\right]^y \right)\Bigg|_{w=0}$$

$$X_j^-(x,y) = \frac{(-1)^{j+1}}{j!(-x-1-j)!(-x)} \frac{d^{-x-1-j}}{dw^{-x-1-j}}\left([\frac{4}{h} - w]^{-x} \left[\frac{-i\frac{2}{h} - \frac{2}{h} + w}{-i\frac{2}{h} + \frac{2}{h} - w}\right]^y \right)\Bigg|_{w=0}$$

$$Y_j^+(x,y) = \frac{(-1)^{j+1}}{j!(y-1-j)!y} \frac{d^{y-1-j}}{dw^{y-1-j}}\left(\left[\frac{\frac{2}{h} - i\frac{2}{h} + w}{\frac{2}{h} + i\frac{2}{h} - w}\right]^x [i\frac{4}{h} - w]^y \right)\Bigg|_{w=0}$$

$$Y_j^-(x,y) = \frac{(-1)^{j+1}}{j!(-y-1-j)!(-y)} \frac{d^{-y-1-j}}{dw^{-y-1-j}}\left(\left[\frac{\frac{2}{h} + i\frac{2}{h} + w}{\frac{2}{h} - i\frac{2}{h} - w}\right]^x [-i\frac{4}{h} - w]^{-y} \right)\Bigg|_{w=0}$$

Proof. The proof consists of several parts. First, we express $f_B(t)$ in terms of $f_c(q)$, which enables us to use Theorem 3.2 to express $f(z)$ in terms of $f_c(q)$. Then, we interchange the orders of integration in the resulting expression for $f(z)$ and make use of Cauchy's formula for the derivatives of an analytic function. Finally, we apply Leibnitz's formula for the derivative of a product of functions to obtain (3.10).

The first step is accomplished with the following lemma.

Lemma 3.1. *If* $\rho = (\frac{2}{h} - a)/2$ *and if* $a = \limsup \sqrt[n]{|a_n|}$ *satisfies relation* (3.4), *then* $f_B(t)$, *defined by* (3.3), *can be expressed in terms of* $f_c(q)$, *defined by* (3.2), *as*

(I) *The integral* $\int_0^{\infty} e^{-tq}[f_c(q) - a_o]\,dq$ *converges uniformly to* $f_B(t) - \frac{a_o}{t}$ *on the disk* $D_\rho(t_1)$ *where* $t_1 = \frac{2}{h}$.

(II) *The integral* $\int_0^{-\infty} e^{-tq}[f_c(q) - a_o]\,dq$ *converges uniformly to* $f_B(t) - \frac{a_o}{t}$ *on the disk* $D_\rho(t_3)$ *where* $t_3 = -\frac{2}{h}$.

(III) *The integral* $\int_0^{-i\infty} e^{-tq}[f_c(q) - a_o]\,dq$ *converges uniformly to* $f_B(t) - \frac{a_o}{t}$ *on the disk* $D_\rho(t_2)$ *where* $t_2 = i\frac{2}{h}$.

(IV) *The integral* $\int_0^{i\infty} e^{-tq}[f_c(q) - a_o]\,dq$ *converges uniformly to* $f_B(t) - \frac{a_o}{t}$ *on the disk* $D_\rho(t_4)$ *where* $t_4 = -i\frac{2}{h}$.

Proof. To prove conclusion (I), let r be an arbitrary positive number. Keeping Theorem 3.1 and relations (3.2) and (3.3) in mind, we observe that

(3.13)

$$\left|\int_0^r e^{-tq}\left(\sum_{n=1}^{\infty}\frac{a_n}{n!}q^n\right)dq - \sum_{n=1}^{\infty}\frac{a_n}{t^{n+1}}\right| = \left|\sum_{n=1}^{\infty}\frac{a_n}{n!}\int_0^r e^{-tq}q^n\,dq - \sum_{n=1}^{\infty}\frac{a_n}{n!}\frac{n!}{t^{n+1}}\right|$$

$$\leq \sum_{n=1}^{M}\frac{|a_n|}{n!}\left|\int_0^r e^{-tq}q^n\,dq - \frac{n!}{t^{n+1}}\right| + \sum_{n=M+1}^{\infty}\left(\frac{|a_n|}{n!}\left|\int_0^r e^{-tq}q^n\,dq\right| + \frac{|a_n|}{|t|^{n+1}}\right)$$

for each positive integer M. Now suppose that t_x and t_y represent the real and imaginary parts respectively of t. Then $\left| \int_0^r e^{-tq} q^n \, dq \right| \leq \int_0^\infty e^{-t_x q} q^n \, dq = \frac{n!}{t_x^{n+1}}$, which, along with inequality (3.13), gives the inequality

$$(3.14) \qquad \left| \int_0^r e^{-tq} \left(\sum_{n=1}^\infty \frac{a_n}{n!} q^n \right) dq - \sum_{n=1}^\infty \frac{a_n}{t^{n+1}} \right|$$

$$\leq \sum_{n=1}^M \frac{|a_n|}{n!} \left| \int_0^r e^{-tq} q^n \, dq - \frac{n!}{t^{n+1}} \right| + \sum_{n=M+1}^\infty \left(\frac{|a_n|}{t_x^{n+1}} + \frac{|a_n|}{|t|^{n+1}} \right) .$$

Now, given $\epsilon > 0$, choose M so large that $\sum_{n=M+1}^\infty \left(\frac{|a_n|}{t_x^{n+1}} + \frac{|a_n|}{|t|^{n+1}} \right) < \frac{\epsilon}{2}$ for $t \in D_\rho(t_1)$. Integration by parts shows that $\sum_{n=1}^M \frac{|a_n|}{n!} \left| \int_0^r e^{-tq} q^n \, dq - \frac{n!}{t^{n+1}} \right|$ can be made less than $\frac{\epsilon}{2}$ by choosing r large enough. Hence, choosing sufficiently large M and r, we see from (3.14) that $\left| \int_0^r e^{-tq} \left(\sum_{n=1}^\infty \frac{a_n}{n!} q^n \right) dq - \sum_{n=1}^\infty \frac{a_n}{t^{n+1}} \right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for $t \in D_\rho(t_1)$. This completes the proof of conclusion (I). Proofs for the other three conclusions are similar to the proof of conclusion (I), hence are omitted.

In completing the proof of Theorem 3.2 we will assume that $z = xh + iyh$ is in the first quadrant of $\mathcal{L}_h$; that is, x and y

are non-negative integers. Proofs for the other three cases are similar and are left to the reader. Under this assumption we see, from defining formula (2.5), that $e(z;t)$ has possible singularities only at $t = \frac{2}{h}$ and $t = -i\frac{2}{h}$. This fact and conclusion (III) of Theorem 3.1 show that the second and third terms of formula (3.8) are zero, by virtue of Cauchy's theorem. The first and fourth terms of formula (3.8) can be reformulated by applying the first and fourth conclusions of Lemma 3.1. The result is

$$f(z) = -\frac{1}{2\pi i} \oint_{\partial D_\rho(t_1)} e(z;t) \int_0^{\infty} e^{-tq} [\, f_c(q) - a_o \,] \, dq \, dt$$

(3.15)

$$-\frac{1}{2\pi i} \oint_{\partial D_\rho(t_4)} e(z;t) \int_0^{i\infty} e^{-tq} [\, f_c(q) - a_o \,] \, dq \, dt + a_o$$

Now, we show that the order of integration can be reversed in each term of the preceding sum. Letting r be a positive number and considering the first term, we see that

$$\left| \oint_{\partial D_\rho(t_1)} e(z;t) \int_o^\infty e^{-tq} [\, f_c(q) - a_o \,]\, dq dt - \int_o^r [\, f_c(q) - a_o \,] \oint_{\partial D_\rho(t_1)} e(z;t) e^{-tq} dt dq \right|$$

$$= \left| \oint_{\partial D_\rho(t_1)} e(z;t) \left\{ \int_o^\infty e^{-tq} [\, f_c(q) - a_o \,]\, dq - \int_o^r e^{-tq} [\, f_c(q) - a_o \,]\, dq \right\} dt \right|$$

$$\leq 2\pi\rho \max_{t \in \partial D_\rho(t_1)} |e(z;t)| \max_{t \in \partial D_\rho(t_1)} \left| \int_r^\infty e^{-tq} [\, f_c(q) - a_o \,]\, dq \right| .$$

Since $\int_o^\infty e^{-tq} [\, f_c(q) - a_o \,]\, dq$ converges uniformly on $D_\rho(t_1)$, the $\max_{t \in \partial D_\rho(t_1)} \left| \int_r^\infty e^{-tq} [\, f_c(q) - a_o \,]\, dq \right|$ can be made arbitrarily small by making r sufficiently large. Hence the order of integration can be reversed in the first term of (3.15). A similar argument shows that the order of integration can be reversed in the other term. Using these facts with defining formula (2.5) for $e(z;t)$, we rewrite equation (3.15) as

(3.16)

$$f(z) = \int_o^\infty [\, f_c(q) - a_o \,]\, \psi_1(q,z)\, dq + \int_o^{i\infty} [\, f_c(q) - a_\rho \,]\, \psi_4(q,z)\, dq + a_o$$

where

$$\psi_1(q,z) = \frac{-1}{2\pi i} \oint_{\partial D_\rho(t_1)} e^{-tq} \left(\frac{2+ht}{2-ht} \right)^x \left(\frac{2+iht}{2-iht} \right)^y dt ,$$

and

$$\psi_4(q,z) = \frac{-1}{2\pi i} \oint_{\partial D_\rho(t_4)} e^{-tq} \left(\frac{2+ht}{2-ht} \right)^x \left(\frac{2+iht}{2-iht} \right)^y dt .$$

Making changes in the integration variables gives

$$\psi_1(q,z) = \frac{-1}{2\pi i} \oint_{\partial D_\rho(0)} e^{-(\frac{2}{h}+w)q} \left(\frac{4+hw}{-hw} \right)^x \left(\frac{2+i2+ihw}{2-i2-ihw} \right)^y dw ,$$

and

$$\psi_4(q,z) = \frac{-1}{2\pi i} \oint_{\partial D_\rho(0)} e^{-(-i\frac{2}{h}+w)q} \left(\frac{2-i2+hw}{2+i2-hw} \right)^x \left(\frac{4+ihw}{-ihw} \right)^y dw .$$

Hence

$$\psi_1(q,z) = \frac{-e^{-\frac{2}{h}q}}{2\pi i} \oint_{\partial D_\rho(0)} e^{-qw} \left(\frac{-\frac{4}{h}-w}{w} \right)^x \left(\frac{-i\frac{2}{h}+\frac{2}{h}+w}{-i\frac{2}{h}-\frac{2}{h}-w} \right)^y dw ,$$

and

$$\psi_4(q,z) = \frac{-e^{i\frac{2}{h}q}}{2\pi i} \oint_{\partial D_\rho(0)} e^{-qw} \left(\frac{\frac{2}{h}-i\frac{2}{h}+w}{\frac{2}{h}+i\frac{2}{h}-w} \right)^x \left(\frac{i\frac{4}{h}-w}{w} \right)^y dw .$$

Using the unit step function U and Cauchy's formula for the derivatives of an analytic function gives

$$\psi_1(q,z) = -U(x-1)\,\frac{e^{-\frac{2}{h}q}}{x!}\,\frac{d^{x-1}}{dw^{x-1}}\left(e^{-qw}\left[-\frac{4}{h}-w\right]^{x}\left(\frac{-i\frac{2}{h}+\frac{2}{h}+w}{-i\frac{2}{h}-\frac{2}{h}-w}\right)^{y}\right)\Bigg|_{w=0},$$

and

$$\psi_4(q,z) = -U(y-1)\,\frac{e^{i\frac{2}{h}q}}{y!}\,\frac{d^{y-1}}{dw^{y-1}}\left(e^{-qw}\left(\frac{\frac{2}{h}-i\frac{2}{h}+w}{\frac{2}{h}+i\frac{2}{h}-w}\right)^{x}\left[i\frac{4}{h}-w\right]^{y}\right)\Bigg|_{w=0}.$$

Making use of Leibnitz's formula for the derivative of a product, we see that

$$\psi_1(q,z)=-U(x-1)\frac{e^{-\frac{2}{h}q}}{x!}\sum_{j=0}^{x-1}\frac{(x-1)!}{(x-1-j)!\,j!}(-q)^j\,\frac{d^{x-1-j}}{dw^{x-1-j}}\left(\left[-\frac{4}{h}-w\right]^{x}\left(\frac{-i\frac{2}{h}+\frac{2}{h}+w}{-i\frac{2}{h}-\frac{2}{h}-w}\right)^{y}\right)\Bigg|_{w=0},$$

and

$$\psi_4(q,z)=-U(y-1)\frac{e^{i\frac{2}{h}q}}{y!}\sum_{j=0}^{y-1}\frac{(y-1)!(-q)^j}{j!(y-1-j)!}\,\frac{d^{y-1-j}}{dw^{y-1-j}}\left(\left(\frac{\frac{2}{h}-i\frac{2}{h}+w}{\frac{2}{h}+i\frac{2}{h}-w}\right)^{x}\left[i\frac{4}{h}-w\right]^{y}\right)\Bigg|_{w=0}.$$

Substitution of these two expressions into (3.16) establishes equation (3.10) in the case that x and y are non-negative integers. This completes our proof of Theorem 3.3.

If (3.4) is satisfied, we know from Theorem 3.1 that $f(z)$, defined by (3.1), is discrete analytic on $\mathcal{L}_h$ and that $f_c(q)$, defined by (3.2), is entire and of order $O(e^{r|q|})$ for some $r < \frac{2}{h}$. Moreover, Theorem 3.3 shows that $f(z)$ can be expressed in terms of $f_c(q)$ by (3.10). The next theorem is a converse of these statements.

Theorem 3.4. <u>If $f_c(q)$ is an arbitrary entire function of order $O(e^{r|q|})$ for some $r < \frac{2}{h}$, then</u>

(I) <u>Formula (3.10) defines a function $f(z)$ that is discrete analytic on $\mathcal{L}_h$ and can be expanded in a pseudo-Maclaurin series (3.1) such that (3.4) is satisfied.</u>

(II) <u>Series (3.2) converges to $f_c(q)$ and f_c is of order $O(e^{p|q|})$ for each $p > a$.</u>

(III) <u>Series (3.3) converges for $|t| > a$ to the Borel transform f_B of f_c.</u>

<u>Proof.</u> It is a well-known result of the theory of entire functions that f_c has a Maclaurin expansion (3.2) such that $a = \limsup \sqrt[n]{|a_n|} \leq r$ and such that conclusions (II) and (III) are valid. In fact this is known to be true under the weaker hypothesis that f_c be entire and of order $O(e^{r|q|})$ for some finite r. A proof can be based on Cauchy's integral formula and Stirling's formula for $n!$, and can be found in [19]. Since $a \leq r$, it follows that $a < \frac{2}{h}$ because $r < \frac{2}{h}$ by hypothesis. The proof of Theorem 3.4

can now be easily completed by applying Theorems 3.1 and 3.3.

The preceding theorem justifies the following definition.

Definition 3.1. <u>The analog mapping</u> $T : \mathcal{F}_c \longrightarrow \mathcal{F}$ <u>has domain</u>

$$(3.17)\qquad \mathcal{F}_c = \{ f_c \mid f_c(q) \ \underline{\text{is entire and}} \quad f_c(q) = O(e^{r|q|}) \ \underline{\text{for some}}\ r < \frac{2}{h} \}$$

<u>and range</u>

$$(3.18)\qquad \mathcal{F} = \{ f \mid f(z) \ \underline{\text{has a pseudo-Maclaurin}} \quad \underline{\text{expansion}}\ (3.1)\ \underline{\text{for which}}\ (3.4)\ \underline{\text{is valid}} \} ,$$

<u>and is defined by</u>

$$(3.19)\qquad f = T(f_c)$$

<u>where</u> $f(z)$ <u>is given by</u> (3.10) <u>in terms of</u> $f_c(q)$.

It is clear from (3.17) that $\mathcal{F}_c$ forms a vector space, and it is a well-known fact that the powers $\{q^n\}$ are a basis for $\mathcal{F}_c$. A close inspection of (3.10) shows that T is a linear transformation on

$\mathcal{J}_c$; hence $\mathcal{J}$ is a vector space. Moreover, it is clear from Theorem 3.3 that $z^{(n)} = T(q^n)$ for $n = 0, 1, 2, \ldots$; hence $\{z^{(n)}\}$ spans $\mathcal{J}$. Among other things, the next theorem shows that $\{z^{(n)}\}$ is actually a basis for $\mathcal{J}$.

Theorem 3.5. <u>The family $\mathcal{J}_c$ forms a vector space with basis $\{q^n\}$, and the family $\mathcal{J}$ forms a vector space with basis $\{z^{(n)}\}$. Moreover, the analog mapping T is a one-to-one linear transformation from $\mathcal{J}_c$ onto $\mathcal{J}$ such that $z^{(n)} = T(q^n)$ for $n = 0, 1, 2, \ldots$. Furthermore, a function f_c that is in $\mathcal{J}_c$ and is identically zero on the non-negative real axis is identically zero on the whole complex plane, and a function f that is in $\mathcal{J}$ and is identically zero on the non-negative real discrete axis is identically zero on the whole discrete complex plane $\mathcal{L}_h$.</u>

<u>Proof.</u> To show that T is one-to-one and that $\{z^{(n)}\}$ is a basis for $\mathcal{J}$, we need only prove that the kernel of T contains only the identically zero function. Thus suppose that $f(z) \equiv 0$. Then $a_o = f(0) = 0$, and hence equation (3.10) reduces to

$$f(x) = \sum_{j=0}^{x-1} X_j(x, 0) \int_o^\infty f_c(q)\, e^{-\frac{2}{h}q}\, q^j\, dq \quad \text{for } x > 0 ,$$

where

$$X_j(x,0) = \frac{(-1)^{x+j+1}(x-1)!}{j!(x-1-j)!(j+1)!} \left(\frac{4}{h}\right)^{j+1}, \quad 0 \le j \le x-1 .$$

In particular, we have

$$f(1) = \frac{4}{h} \int_0^{\infty} f_c(q)\, e^{-\frac{2}{h}q}\, dq ,$$

which means that

$$\int_0^{\infty} f_c(q)\, e^{-\frac{2}{h}q}\, dq = 0 .$$

Employing the second principle of induction, we assume that

$$\int_0^{\infty} f_c(q)\, e^{-\frac{2}{h}q}\, q^j\, dq = 0, \quad 0 \le j \le k .$$

Then, by (3.10), we have

$$f(k+2) = X_{k+1}(k+2,0) \int_0^{\infty} f_c(q)\, e^{-\frac{2}{h}q}\, q^{k+1}\, dq ,$$

or

$$f(k+2) = \frac{1}{(k+2)!} \left(\frac{4}{h}\right)^{k+2} \int_0^{\infty} f_c(q)\, e^{-\frac{2}{h}q}\, q^{k+1}\, dq ,$$

and hence

$$\int_0^{\infty} f_c(q)\, e^{-\frac{2}{h}q}\, q^{k+1}\, dq = 0 .$$

Thus we have shown that

$$(3.20) \qquad \int_0^{\infty} f_c(q)\, e^{-\frac{2}{h}q}\, q^j\, dq = 0 , \qquad j = 0,1,\ldots .$$

Because $f_c(q)$ is $O(e^{p|q|})$ for some $p < \frac{2}{h}$ (see Theorem 3.4), the improper integral $\int_0^{\infty} e^{-sq} f_c(q)\, e^{-\frac{2}{h}q}\, dq$ converges uniformly to a function $F(s)$ on the interval $[\frac{a}{2} - \frac{1}{h}, +\infty)$. Equation (3.20) states that all derivatives of F vanish at the origin; hence $F(s) \equiv 0$. From the uniqueness theory for the Laplace transform it now follows that $f_c(q)\, e^{-\frac{2}{h}q} \equiv 0$, and hence $f_c(q) \equiv 0$. Thus the kernel of T contains only the identically zero function, which implies that T is one-to-one and that $\{z^{(n)}\}$ is a basis for $\mathcal{J}$.

It is a well-known result of function theory that f_c is identically zero on the whole complex plane when it is identically zero on the non-negative real axis. In fact this is known to be true under the weaker condition that f_c be zero on an infinite set of distinct points.

To prove that f is identically zero on the whole discrete complex plane when it is identically zero on the non-negative real discrete axis, we first repeat the reasoning of the preceding paragraph to show that its continuous analog f_c is identically zero. It then follows that $f = T(f_c)$ is identically zero because T is linear. This completes the proof of Theorem 3.5.

The following corollary is easily proved from Theorem 3.5.

Corollary 3.5.1. <u>The pseudo-powers</u> $\{z^{(n)}\}$ <u>are linearly independent, and the pseudo-Maclaurin series representation</u> (3.1) <u>for an arbitrary function</u> f <u>in</u> $\mathcal{J}$ <u>is uniquely determined by the values of</u> f <u>on the non-negative real discrete axis.</u>

The analog transformation T leads quite naturally to a definition for multiplication of functions in $\mathcal{J}$. The following notation will prove useful for defining multiplication and stating the remaining theorems of this paper:

$$(3.21) \qquad f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^{(n)} \quad \text{where } a = \limsup \sqrt[n]{|a_n|} < \frac{2}{h}$$

and

$$(3.22) \qquad g(z) = \sum_{n=0}^{\infty} \frac{b_n}{n!} z^{(n)} \quad \text{where } b = \limsup \sqrt[n]{|b_n|} < \frac{2}{h} ;$$

here h is the mesh width of $\mathcal{L}_h$. As the following definition and theorem shows, multiplication is defined only on a proper subset $\mathcal{P}$ of $\mathcal{J} \times \mathcal{J}$.

Theorem 3.6. _Let_ f _in_ $\mathcal{J}$ _be given by_ (3.21), _and let_ g _in_ $\mathcal{J}$ _be given by_ (3.22). _If_ (f, g) _is in_

(3.23) $$\mathcal{P} = \{(f,g) \mid f, g \in \mathcal{J} \text{ and } a + b < \frac{2}{h}\},$$

then

(I) _The pointwise product function_ $f_c \times g_c$, _where_ $f_c = T^{-1}(f)$ _and_ $g_c = T^{-1}(g)$, _is in_ $\mathcal{J}_c$.

(II) _The discrete product function_ $f \otimes g = T(f_c \times g_c)$ _of_ f _and_ g _is in_ $\mathcal{J}$.

(III) _The discrete product function_ $f \otimes g$ _has a discrete Maclaurin series representation with coefficient sequence_ $\{\sum_{i+j=n} \frac{a_i b_j}{i!\, j!}\}$, _where_ $\{a_n\}$ _and_ $\{b_n\}$ _are the coefficient sequences of_ f _and_ g _respectively_.

(IV) _The discrete product_ $\otimes$ _is commutative, associative, and distributive over point-wise addition_.

Proof. Using (3.21), (3.22), and (3.23), we deduce from conclusion (II) of Theorem 3.1 that $f_c \times g_c$ is $O(e^{p|q|})$ for each $p > a + b$. Hence, by virtue of (3.23), the function $f_c \times g_c$ is $O(e^{r|q|})$ for some $r < \frac{2}{h}$, and it is then a consequence of (3.17) that $f_c \times g_c$ is in $\mathcal{J}_c$. This proves conclusion (I) and establishes conclusion (II). Conclusion (III) results from multiplying the Maclaurin expansions for f_c and g_c, and then applying T to the resulting function in $\mathcal{J}_c$. Commutativity, associativity, and distributivity of $\otimes$ are inherited from the corresponding properties of the pointwise product $\times$ for functions in $\mathcal{J}_c$. This completes the proof of Theorem 3.6.

The analog transformation T can also be used to define differentiation on $\mathcal{J}$.

Theorem 3.7. *If f is in $\mathcal{J}$, then*

(I) $\frac{df_c}{dq}$ *is in* $\mathcal{J}_c$, *where* $f_c = T^{-1}(f)$.

(II) *The discrete derivative* $\frac{\delta f}{\delta z} = T\left(\frac{df_c}{dq}\right)$ *of f is in* $\mathcal{J}$.

(III) $\frac{\delta}{\delta z}$ *is a linear operator on* $\mathcal{J}$.

(IV) $\frac{\delta f}{\delta z} = \sum_{n=1}^{\infty} \frac{a_n}{(n-1)!} z^{(n-1)}$, *where* $\{a_n\}$ *is the coefficient sequence of* f.

(V) $f(z) = \sum_{n=0}^{\infty} \frac{\left.\frac{\delta^n f}{\delta z^n}\right|_0}{n!} z^{(n)}$, *where* $\frac{\delta^n}{\delta z^n}$ *is the composition of* $\frac{\delta}{\delta z}$ *with itself* n *times.*

(VI) $\frac{\delta(f \otimes g)}{\delta z} = \frac{\delta f}{\delta z} \otimes g + f \otimes \frac{\delta g}{\delta z}$, *provided that* (f, g) *is in* $\mathcal{P}$ *as given by* (3.23).

Proof. Conclusion (I) is a well-known result from the theory of entire functions. Its proof can be based on the Cauchy integral formula and won't be given here. Conclusion (II) is justified by conclusion (I), and conclusion (III) follows from the linearity of $\frac{d}{dq}$ and T. Conclusion (IV) results from applying T to the Maclaurin expansion of $\frac{df_c}{dq}$, and conclusion (V) comes from repeated application of conclusion (IV). Conclusion (VI) is an immediate consequence of the linearity of T and the corresponding formula for functions in $\mathcal{J}_c$. This completes the proof of Theorem 3.7.

The following theorem relates discrete integration, as defined by Definition 2.3, to continuous integration.

Theorem 3.8. *If* f *is in* $\mathcal{J}$, *then*

(I) $\int_0^q f_c(q)\,dq$ *is in* $\mathcal{J}_c$, *where* $f_c = T^{-1}(f)$.

(II) $\int_0^z f(z)\,\delta z$ *is in* $\mathcal{J}$ *and equals* $T(\int_0^q f_c(q)\,dq)$.

(III) $\int_0^z \quad \delta z$ *is a linear operator on* $\mathcal{J}$.

(IV) $\int_0^z f(z)\,\delta z = \sum_{n=0}^{\infty} \frac{a_n}{(n+1)!} z^{(n+1)}$, *where* $\{a_n\}$ *is the coefficient sequence of* f.

Proof. The proof of conclusion (I) is a well-known application of Cauchy's integral formula and won't be given here. To prove conclusion (II), first establish conclusion (IV) by "integrating" (3.21) and applying Theorem 2.6. Then let T operate on the integral of (3.2) and use conclusion (IV) to show that $\int_o^z f(z)\,\delta z = T(\int_o^q f_c(q)\,dq)$. The function $T(\int_o^q f(q)\,dq)$ is in $\mathcal{F}$ by virtue of conclusion (I). Finally, $\int_o^z \quad \delta z$ is a linear operator on the family of all discrete analytic functions, as can be seen from its Definition 2.3. This completes the proof of Theorem 3.8.

The preceding theorem provides the machinery needed to relate the operator $\frac{\delta}{\delta z}$ to the operator $\frac{\delta}{\delta z}(\ \ ;k)$ defined by Theorem 2.3.

Theorem 3.9. *If* f *is in* $\mathcal{F}$, *then* $\frac{\delta f}{\delta z} = \frac{\delta f}{\delta z}(\ \ ;a_1^*)$ *where* a_1^* *is the complex conjugate of the coefficient* a_1 *appearing in* (3.21).

Proof. Using Definition 2.3 for discrete integration, with conclusion (II) of Theorem 3.8 and with the fact that T is linear, we see that for arbitrary lattice points a and z

$$\int_a^z \frac{\delta f}{\delta z}\,\delta z = \int_o^z \frac{\delta f}{\delta z}\,\delta z - \int_o^a \frac{\delta f}{\delta z}\,\delta z$$

$$= T\left[\int_o^q \frac{df_c}{dq}\,dq\right](z) - T\left[\int_o^q \frac{df_c}{dq}\,dq\right](a)$$

$$= T[\,f_c(q) - f_c(0)\,](z) - T[\,f_c(q) - f_c(0)\,](a)$$

$$= T[\,f_c(q)\,](z) - T[\,f_c(0)\,](z) - T[\,f_c(q)\,](a) + T[\,f_c(0)\,](a)$$

$$= f(z) - a_o - f(a) + a_o$$

$$= f(z) - f(a)\,.$$

This relation and the second conclusion of Theorem 2.3 show that

$$\frac{\delta f}{\delta z}(z) = \frac{\delta f}{\delta z}(z;k) = \left(\frac{4}{h^2}\int_o^z f_D(t)\,\delta t + k\right)_D$$

for some complex number k. This identity and Definition 2.4 for the dual of a lattice function show that $k^* = \frac{\delta f}{\delta z}(0)$. But, according to conclusion (IV) of Theorem 3.7, we know that $\frac{\delta f}{\delta z}(0) = a_1$; hence $k = a_1^*$ and the proof of Theorem 3.9 is complete.

In concluding this paper we discuss the discrete analog of integration by parts.

Theorem 3.10. *If* $(f, g) \in \mathcal{P}$ *as given by* (3.23), *and if* a *and* b *are lattice points, then*

$$\int_a^b f \otimes \frac{\delta g}{\delta z}\,\delta z = f \otimes g\,\Big]_a^b - \int_a^b g \otimes \frac{\delta f}{\delta z}\,\delta z\,.$$

Proof. According to conclusion (VI) of Theorem 3.7

$$\frac{\delta(f \otimes g)}{\delta z} = \frac{\delta f}{\delta z} \otimes g + f \otimes \frac{\delta g}{\delta z}\,.$$

Hence

$$\int_a^b \frac{\delta(f \otimes g)}{\delta z}\,\delta z = \int_a^b \frac{\delta f}{\delta z} \otimes g\,\delta z + \int_a^b f \otimes \frac{\delta g}{\delta z}\,\delta z\,,$$

and thus

$$f \otimes g\,\Big]_a^b = \int_a^b \frac{\delta f}{\delta z} \otimes g\,\delta z + \int_a^b f \otimes \frac{\delta g}{\delta z}\,\delta z$$

by virtue of Theorems 3.9 and 2.3. This completes the proof of Theorem 3.10.

There are, no doubt, many more interesting properties of the family $\mathcal{J}$ that can be found by employing the analog transformation T .

4. References.

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