POTENTIAL THEORY ON A RHOMBIC

LATTICE

by

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Abstract

Of concern are complex valued .functions defined on. the lattice points of the complex: plane* The lattice can. be the usual lattice of square blocks but of main concern is an irregular lattice with squares replaced by rhomfec A function is defined to be discrete analytic if the difference quotient across *one* diagonal of a rhomb- equals the difference quotient across the other diagonal* Based on this definition discrete analogs of the following concepts in, classical function theory are developed: Laplace eciuation5Cauohy-liic;;:a/nn equations, differentiation, contour integration, llorera's theorem, and harmonic polynomials,, The theory is more than an analogy because for a common class of boundary value problems it *-proves* possible to obtain upper and lo^er bounds for the classical Dirichlet integral in terms of discrete harmonic functions.

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POTENTIAL THEORY OK A iffiOIJBIC LATTICE

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!• Introduction. There is an.extensive literature treating a discrete potential theory 021 the square lattice in the plane* The central problem of concern in those studies is the properties of functions whose value at a lattice point is the mean of the values at the four neighboring lattice points. This gives a discrete analog* of harmonic functions. In this paper a similar problem is studied, in which the lattice of squares of side h is replaced by a lattice of rhombs of side h. Such a rhombic lattice can be quite irregular and consequently the methods of difference equations can not be used.

The rhombic lattice is embedded in the -complex plane* A discrete analytic function is defined as a complex valued function on the lattice points whose difference quotient across one diagonal of a rhomb is equal to its difference quotient across the other diagonal. This leads directly to analogs of the Cauohy—Riemann. equations a,nd the Laplace equation*

Functions defined on the lattice points can be extened to the lattice lines by linear interpolation* This device permits defining the complex line integral along an arbitrary path in the lattice. Then given, a discrete analytic function the integral defines a lattice function P. Moreover- F is discrete analytic. Repeated integration leads to functions analogous to polynomials in the complex variable.

In the classical function theory the inverse of the integral operation is differentiation. IIowever;in the discrete function theory on the square lattice it was found that the inverse of the integral is another type of integral termed "dual integration" £l]. In this paper it is shown that this integral ojjeration extends to the rhombic lattice.

The finaJL section of this paper is, perhaps, the most interesting part because it is shown that the discrete function theory is more than a mere analogy. Actually the discrete *tYieory* bears a quantitative relation to the classical theory after which it was modeled. This is brought out by analysis of the classical Dirichlet Integra,! I)(u) of a harmonic function U. The integral is to be evaluated over a region of the plane bounded by diagonals of rhombs and the assigned "boundary values are assumed to vary linearly over the diagonals* Let u be a discrete harmonic function- taking on the came boundary'values. Then a certain quadratic form Q'(u) is shown to give an. .upper...bound for l)(l))» A similar quadratic form gives a lover- bound •

2. The rhombic lattice. It is supposed that the complex plane is paved with rhombic blocks each of side lenth h. The sides of the blocks make up, the lattice lines and the vertices of the blocks forirr the lattice points. Thus each rhomb has four[1] lattice lines and four lattice points. The most common lattice of this type is, of course, the square lattice,

A rhomb will be regarded as a closed two-dimensional point set* A jLaJ^J--,cS-,re.ff-LSS ^s defined as the union of rhombs. A simple lattice ^ggion is a connected and simply connected set of a finite number of rhombs. Such-a region is shown in Figure 1.

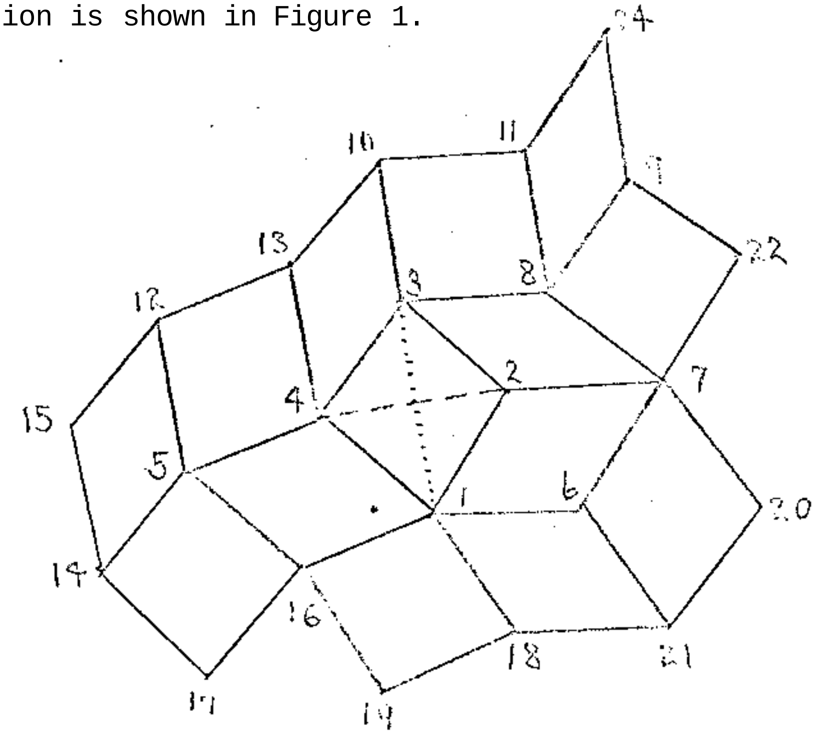

Figure 1« A simple lattice region

It is clear that a closed chain of rhomb: edges consists of an even number of edges. Thus it follows that the lattice- points fall into two disjoint classes: the even class and the odd class. These classes are defined as follows: the neighboring lattice points of an odd point are even points e.nd the neighboring lattice points of an even point are odd points* Of course neighboring points are defined to be *end.* points of an edge.

The lattice points are enumerated with the positive integers so that the even integers name the even lattice points and the odd integers name the odd lattice points. Thus one of the rhombs in Figure 1 has lattice points designated 1,2,3,4. One diagonal of this rhomb is depicted as a dotted line the other diagonal is depicted as a dashed line. The diagonals connecting odd points are termed odd diagonals and the diagonals connecting even points are termed even diagonals.

The diagonals corresponding to the simple lattice region of Figure 1 is shown in Figure 2 but the lattice lines are deleted. The even diagonals of the lattice define the even network and the the odd diagonals define the odd network. Thus the dashed lines in Figure 2 correspond to the even network and the dotted lines correspond to the odd network.

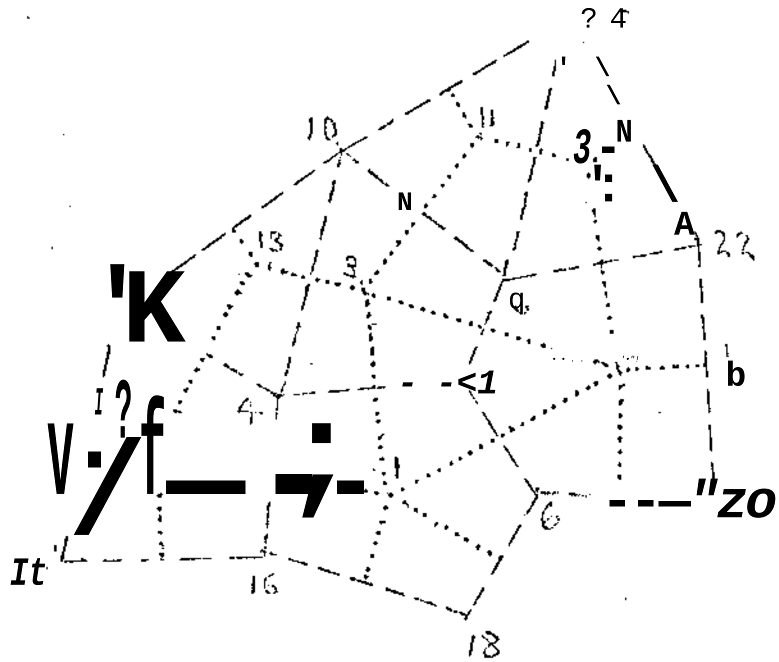

Figure 2. Polygonal cells of the even network

It is seen that there is precisely one node of the odd network in each polygonal cell of the even network and vice versa. Clearly the even and the odd networks are what are termed dual graphs in graph theory. Since the diagonals of a rhomb bisect each other perpendicularly it follows that the edges where the networks cross are perpendicular bisectors.

Since all the rhombs have the same length side it follows that a polygonal cell of the even network is inscribed in a circle of of radius h. Moreover the center of the circle must lie inside the polygonal cell.

The above statements are geometrically obvious and are *given* whith out proof*. It shall be assumed that a rhomb can not have arbitrarily small area in any finite portion of the plane* The lattice and the networks are all graphs. The terminology serves to to distinguish the roles the lattice and the'networks will play.

The reader will observe that some of-the theorems to follow hold for a general lattice of quadrila-fcei*als. Some of the proofs require only .that the quadrilaterals be parallelograms. Other proofs are based on the diagonals being- perpendicular.

3. Analytic fil^^ions and integrals.^^ concern; in this paper[1] are functions f(z) defined on the lattice points of a rhombic lattice If ?v is the complex nuinter giving the position* of the lattice point k then let f_k be a short notation for $f(z_k)$. *i^J^-^^^.S^^ilSR^^.* said to be discrete analytic at a rho^ib if the difference Quotient across one diagonal is equal to the difference quotient across the other. For example sup-pose $1_S 2_9 3_5 4$ s're lattice points of a rhomb. Then f is analytic at this rhomb if

(i) $$\frac{f_3 - f_1}{z_3 - z_1} = \frac{f_4 - f_2}{z_4 - z_2} .$$

This is the discrete, analog of the property of the derivative of a poorer series being independent of direction.

Let f(z) "be a function defined on the lattice points. Let G be a chain of lattice points having comple3: values $z_1, z_2, \ldots, z_n$. Successive points in the sequence are neighbors. Let $z_1 = a$ and $z_n = b$ then the J^^^^^l^i^^JlJ^SE^ °^ ^(z) on' '^e chain G is defined as

(2) $$\int_a^b f(z)\, dz = \sum_1^{n-1} (f_{k+1} + f_k)(z_{k+1} - z_k)/2.$$

It is clear that this is equivalent to the ordinary complex line integral along the lattice edges of the chain G if f(z) is defined along an edge by linear interpolation.

The following statement is an analog of Korera's theorem.

Theorem 1. A fun[illegible] lattice region if and only if the discrete line integral around every simple closed chain vanishes.

Proof. The line integral around the rhomb. (l,2,3>4) is "by definition

$$2 \int f(z)\, dz = (f_2 + f_1)(z_2 - z_1) + (f_3 + f_2)(z_3 - z_2)$$
$$+ (f_4 + f_3)(z_4 - z_3) + (f_1 + f_4)(z_1 - z_4).$$

Reforming the right side gives

(3) $$2 \int f(z)\, dz = (f_1 - f_3)(z_2 - z_4) - (f_2 - f_4)(z_1 - z_3) .$$

It is obvious that the vanishing of this integral is equivalent to f(z) being analytic on the rhombs

It is clear that the lino integral of a function around a simple closed chain is the sum of the line integrals around the rhombs contained inside the chain, provided all the integrals are taken in the same sense* This observation together "with the formula (3) completes the proof• The following corollary is now obvious•

Corollary 1* Let f(z) be discrete anaJytic jU\j\j?\iwr)le ^attice region R# ^i2i^a il^(L^a point ^of R the^n: a^ i^.oti^oii F'(^)J i£F^m9^LJ^lVl, integral of f(z),-

$$(4) \qquad P(z) - J_a^{z_0} f(a)\, dz$$

^and P(z) ig^^nde;nondent of ^tho^athjof ^inte^ratio'tTi. ^if^^the JP^WI is in iU

4. Conjugate harmonic functions* Lot z_1, z_2, z_3, z_4 to complex numbers defining the lattice points of a rhomb in counter clockwise order. Then z_1 and z_3 are endpoints of a line of the odd network. The conductance of this diagonal line is defined to be.

(5) V _ i %/ - %o - „ .

Since the diagonals of a rhomb are perpendicular the conductance is a positive number*. The conductance of the even diagonals are defined analogously and so

$$(6) \qquad g_{24} g_{13} = 1 \text{ .}$$

In other nords crossing diagonals have reciprocal values of conductance.

A function defined on the points of the even network is said to be discrete harmonic at a given point if its value at that point is the weighted mean of its values at the neighboring points of the even network* The weights are the corresponding conductances, A similar relation defines discrete harmonic functions on the odd network.

Theorem 2. Let a lattice function

region R ... nterior to R.

Proof. First consider a special case depicted in Figures 1 and 2* Thus the point 2 is an interior point of the region shown in Figure I. In the rhomb (1,2,3*4) the condition for discrete analyticity is

f3 „ f1 = ^-~; i (f4 - f2) .

This can be written in terms of the conductance $g_{0/}$« This relation: and the corresponding relations for the other rhombs surrounding the point 2 become:

f3 "f1 " iS?A (f4 "f2>

f1 -f7 - i e26 (f6 "f2}

f7 " f3 s i ff28 ^f8 ~ f2^ *

Adding these equations gives

(7} (s24+ "?:6+G23} f2 " S?A h+S26 %+g23 f8 *

This is precisely the condition that f bo discrete harmonic at the point 2*

The pattern of the proof when more than three rhombs meet at a point is now clear and the proof is complete. An equation of the form (7) will be termed the *discrete Laplace equation*.

Theorem 3 The real part* u *and the imaginary part* v *of a discrete analytic function* f *satisfy the discrete Cauchy-Riemann. equations*:

(8a) $$\frac{u_3 - u_1}{|z_3 - z_1|} = \frac{v_4 - v_2}{|z_4 - z_2|}$$

(8b) $$\frac{v_3 - v_1}{|z_3 - z_1|} = -\frac{u_4 - u_2}{|z_4 - z_2|}$$

where (1,2,3*4) *denotes a rhomb in counterclockwise order. Moreover- the functions* u *and* v *are discrete harmonic*.

Proof. Equations (8) follow from the substitution f•« u + iv in: equation, (l) and separation! of real and imaginary parts* The last statement of the theorem results from the fact that the coefficients of the discrete Laplace equationi are real.

The pair of functions u and v are termed *conjugate functions* because they satisfy the analog of the Cauchy-Riemann equations.

5* ^r'ouf l2rP ov?or jT^^^i£: ir:<' ^ *G ^PP^rervt that linear functions of the complex variable z ai*o discrete analytic. This raises the question about higher powers of &•

*Tlieorein 4»Q*i ar l;^Vk %xC I.^£H5Q."!*\Qj 1: L_.P? z iH* c LJ^^l!P lu J^^ analytic.*

Proof• It is sufficient to consider the case $f(z) = z^2$. Then the condition for discrete analyticity is

$$\frac{z_3^2 - z_1^2}{z_3 - z_1} = \frac{z_4^2 - z_2^2}{z_4 - z_2}$$

Thus $z_1 - z_2 = z_4 - z_3$ and this is true because a rhomb is a parallelogram.

Theorem 5. The integrals of discrete analytic functions are discrete analytic.

Proof. At first let f be an arbitrarily defined lattice function. Let $\oint f\,dz$ denote the integral of f completely around the rhomb. (1,2,3,4)«

$$2\oint f\,dz = (f_1 + f_2)(z_2 - z_1) + (f_2 + f_3)(z_3 - z_2) + (f_3 + f_4)(z_4 - z_3) + (f_4 + f_1)(z_1 - z_4)\ .$$

Rearranging the terms gives

$$2\oint f\,dz = (f_1 - f_3)(z_2 - z_4) - (f_2 - f_4)(z_1 - z_3).$$

It is convenient to let

$$2\oint f\,dz = (z_2 - z_4)\, Lf$$

so

(9) $$Lf = (f_1 - f_3) - (f_2 - f_4)\, G$$ where

$$G = \frac{z_1 - z_3}{z_2 - z_4}\ .$$

Let the function F be defined by integrating f around the rhomb. Thus

$$2\int_1^3 f\,dz = 2(F_3 - F_1) = (f_1 + f_2)(z_2 - z_1) + (f_2 + f_3)(z_3 - z_2)$$

Cancelling and rearranging terms gives

(10a) $$2(F_3 - F_1) = (f_1 - f_3)(z_2 - z_1) + (f_2 + f_3)(z_3 - z_1)\ .$$

Clearly $2\int_2^4 f\,dz$ is given by the analogous relation

(10b) $$2(F_4 - F_2) = (f_2 - f_4)(z_3 - z_2) + (f_3 + f_4)(z_4 - z_2)\ .$$

Multiplying (10b) by G gives

(11) $$2G(F_4 - F_2) = (f_1 - f_3 - Lf)(z_3 - z_2) + (f_3 + f_4)(z_3 - z_1)\ .$$

Subtracting (10a) from (11) yields

$2\,LP = Lf\,(z_2 - z_3) + (f_1 - f_3)(z_1 - 2z_2 + z_3) + (f_4 - f_2)(z_3 - z_x)$

But $(z_1 - z_3)(f_2 - f_4) - (z_2 - z_4)(f_1 - f_3 - Lf)$ so

(12) $2LP - Lf\,(z_4 - z_3) + (f_1 - f_3)(z_3 - z_2 + z_3 - z_4)$ •

This is a general identity expressing a repeated line integral around a block in terms of the single integral. But since a rhomb is a parallelogram $z_1 - z_2 + z_3 - z_4 = 0$ and so

(13) $2\int P\,dz - \int f\,dz\,(z_4 - z_3)$.

It is then a direct consequence of Theorpml that-if f is dicrete analytic at this rhomb so also is F.

/The higher powers of *z* are not disc>*ete analytic but analogous functions termed pscudo-poorers may be defined which are discrete analytic. The pseudo-power of degree zero is defined to be unity. Then the pseudo-power of degree n relative to the "origin, point" b is defined to be

(14) $(z - b)^{(n)} = n \int (z - b)^{(n-1)}\,dz$.

Theorem 6^ The k pseudo-powers $(z - b)^{(0)}, (z - b)^{(1)}, \ldots, (z - b)^{(k-1)}$ generate a complex vector^ s9^,..,o^f ^i^CTe^^^ analytic functions. This space does not depend *on* the choice of the origini point b.

The proof is straightforward and so is omitted.

6. Dual functions and dual integration. So far the analysis has exhibited a close analogy between classical potential theory and discrete potential theory. However* the next two theorems have no direct analogy im the classical theory.

If f is an arbitrary lattice function let f^- be a function termed the due,! and defined as

(15) $f^- = f^*$ at even lattice points,

$f^- = - f^*$ at odd lattice points.

Here f* denotes the complex conjugate. Clearly the dual of the dual is the original function.

Theorem J. A lattice function f is .discrete analytic if and only if f^- is discrete analytic.

Proof. Consider the rhomb (l,2,3*4) z-ⁿ& let L be the operation, defined in (9) so

$$Lf^1 \gg f_1 \sim G f_2 - f_3 + G f_4 ,$$

$$- Lf^- = f_1 - + G t_2^* - f_3^* - G f_4^* .$$

But G is pure imaginary so

(16) $$Lf = (- Lf^-)^* .$$

This completes the proof of the theorem because f is discrete analytic if and only if Lf = 0.

A "biconstant is a function which has a constant value c OHL the even lattice and a constant value - c on the odd lattice. It is a corollary of Theorem 7 that a biconstant is a discrete analytic function.

Theorem 8. Let f(z) be a discrete analytic function and let

(17) $$F(z): = [I f(z) \& z + c.$$

Here a is an arbitrary lattice point^and c is an. arbitrary constant. The the solution of the integral eriuation (l7) ^is

(18) $$f(z) \ll (4 h^{""2} J^ P^(z) dz + k)""$$

where b is an arbitrary lattice point and *k* is the constant k = f*"(b).

Proof. It follows from relation! (17) that if p and q are neighboring points then

(19) $$\frac{F(P) - F(n)}{} \; a \; \frac{f(p) + f(a)}{} .$$

The complex conjugate of relation! (19) is now formed. Since $(p - q)^* = \bar{h}\ (p - q)^-$ and since p and q have opposite parity it follows that

(20) $$\frac{4}{h^2}\ \frac{F^-(p) + F^-(q)}{2} = \frac{f^-(p) - f^-(q)}{p - q}\ .$$

This defines $f^-(z)$ as an integral of the discrete analytic function. $4\ h^{-2}\ F^-(z)$. Thus

$$f^-(z) = 4\ h^{-2} \int_b^z F^-(z)\ dz + f^-(b)\ .$$

This is equivalent to (18) and the proof is complete.

The operation defined by formula (18) is inverse to integration, and may be termed <u>dual integration</u>. It is analogous, but not closely analogous, to the derivative operation of the calculus.

6# Relating the continuous and the discrete* This section concerns the problem, of relating the discrete potential theory just formulated to the classical continuous theory* Obviously the two theories are qualitatively analogous but it is desired to make this analogy quantitative in some sense* One approach would be to show that the continuous theory is a limit of the discrete theory as cell size vanishes. However what is needed in applied mathematics is an estimate of the error for finite cells* This is the phase of the problem, to be treated here*

The classical potential theory may be regarded as describing the steady flow- of electric current in a plane conductor having unit specific conductance. The discrete potential theory may be regarded as describing the steady flow of electric current in the even network* The number g_{ij} is interpreted as the electrical conductance of the line of the network connecting points i and j. Then a discrete harmonic function is the electric potential of the junction points of the network* The discrete Laplace equation is a simple consequence of the laws of Ohm and Kirchhoff.

The following classical boundary value problem is to be related to a corresponding problem of the even network.

Problem I. Find the value of the Birichlet integral

(21) D(U) * 5 L (<! + U*) dx dy

where R is a region of the (x,y) plane composed of a finite number of polygonal cells of the even network and U is harmonic in R* U is continuous and takes on a prescribed linear variation on the boundary edges*

An example of such a region is depicted by the dashed.lines of Figure 2. To relate the boundary values to a network problem the discrete harmonic functions are interpolated linearly along the network lines* The electrical interpretation of D is the power input to the region *R.* The network analog of D is a quadratic form $Q^?$ giving the power input of the network in R.

Theorem % The Dirichlet integral of Problem I has an upper bound given by the relation

(22) D(u) * Q'(u)

vhere u is discrete harmonic at the -points of the even network interior to R and u = U on the boundary *till*. The quadratic form Q[1] is defined as

(23) $Q'(u) - \sum g'_{ij} (u_i - u_j)^2$

where the summation is over the lines of the even network. Here:

$g'_{ij} = g_{..}$ if the line (i,3) is inside R,

g'_{ij} =i·g•, iC the line (i,j) is on ^R,

g'_{ij}.. » θj if the line (i,j) is outside R.

Proof. This theorem will be deduced from a similar theorem given, in [2, p802]. That theorem concerned a region triangulated in an arbitrary manner by triangular cells. To reduce the present problem to the previous one let the polyhedral cells be triangulated by connecting one of the vertices to the others by auxiliary lines. For example Figure 3 shows a pentagon of the even network With auxiliary lines drawn from vertex 2 to vertices 6 and 8. To obtain an upper bound network each triangle is regarded as a conducting loop of wire. The conductance of a side of a triangle is given by the formula

(24) $g^f = \frac{1}{2}$ cot (angle opposite)

The total network is obtained by connecting all the triangular networks at the vertices. This network was called an upper network because it gave an upper bound to the Dirichlet integral as stated in Theorem 9»

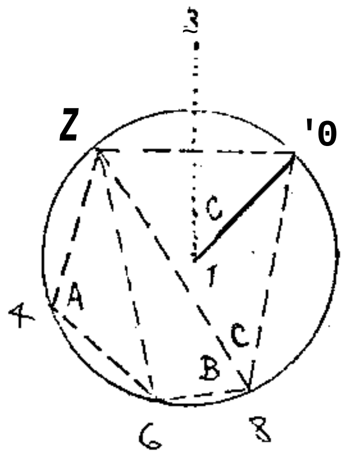

Figure 3* Triangulated polygon

Actually the upper bound network reduces to an even network defined above. To see this note that the total conductance of auxiliary line (2,6) is given by the formula

$$2 g^f = \cot A + \cot 3 .$$

But A + B =180° because of *tao* inscribed angle theorem of the circle. It follows that $gg_6 = 0$ • Hence the auxiliary lines contribute nothing to the quadratic form Q. Let L be the length of the side (2,10) of the polygon. Let W be the length of the line (},3) of the odd network. This line crosses the line (2,10) of the even network.

The conductance of the side (2,10) of the triangle (2,8,10) is given as $g'_{2,10} = \frac{1}{2}\cot\theta$. But a central angle of a circle is twice the corresponding inscribed angle so $\cot C = W/L$. Thus if (2,10) is a line on the boundary of R

(25a) $\qquad g' = W/2L$.

If (2,10) is a line in the interior of it it follows by symmetry that

(25b) $\qquad g' = H/L$.

This shows that the upper network and the even network are equivalent and Theorem 9 is proved.

Theorem 10. The Dirichlet integral of Problem I has a lower bound given by the relation

(26) $\qquad D(U) \geq Q''(w)$

where w is discrete harmonic at the points of the odd network inside R and w = U at the points where the odd network crosses the boundary of R. The quadratic form Q'' is defined as

(27) $\qquad Q''(w) = \sum g''_{ij}(w_i - w_j)^2$

where the summation is over the lines of the odd network. Here:

$g''_{ij} = g_{ij}$ if the line (i,j) is inside R,

$g''_{ij} = 2g_{ij}$ if the line (i,j) crosses ∂R,

$g''_{ij} = 0$ if the line (i,j) is outside R.

Proof. This theorem will be deduced from a similar theorem given in [2, p804] concerning a lower network. A lower network was defined as a network whose power input gives a lower bound to the Dirichlet integral of Problem I for corresponding boundary conditions. To apply this theorem the region R is triangulated as in the proof of the previous theorem. For example Figure 4 shows a polygonal cell triangulated by auxiliary lines. Then the lower network is obtained by constructing the dual network to the network of triangles. The dual network is shown as dotted lines in Figure 4. The resistance of the dotted line is is taken equal to conductance of the line of the upper network which it crosses provided it crosses an interior line. In particular the

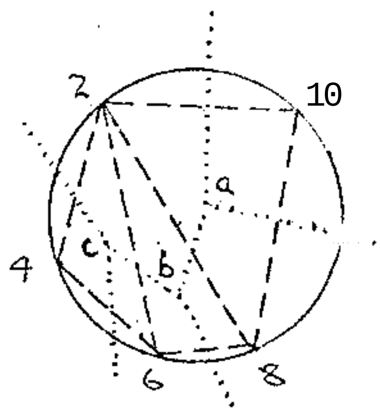

Figure 4. The dual network

line (b,c) is ascribed zero resitance because the auxiliary line (2,6) has zero conductance. Likewise the line (a,b) is ascribed zero resistance. Thus an equivalent network is obtained by shrinking the points a,b,c into a single point placed at the center of the circle of the polygon. If a line such as (2,4) is a boundary line of the region *R* then the corresponding dotted line is terminated at the midpoint of the boundary line and only half the resistance is ascribed. It is seen that the geometry of the shrunken network is the same as that of the portion of the odd network in R. Since resistance is the reciprocal of conductance it follows that Q^f|i is the quadratic form giving the power input to the lower network.

These last two theorems permit various generalizations. Im particular some of the boundary lines of the region *R* can be given the Neumann boundary condition"^ u/Sn = 0 where n denotes the normal. This generalization includes the standard problem of the total conductance of a polygonal plate between two edges. Thus an upper- bound to the conductance of the plate is furnished by the joint conductance of an associated upper network. A lower bound for the conductance is obtained by employing a lower network in the same way. For details reference is made to [2] •

Recent developments in the theory of a complex variable have concerned the geometrical concept of "extremal length". The electrical interpretation, of extremal length is resistance. The concept of extremal length has been extended to the geometry of networks in [3] • These concepts can be carried over directly to the present problem.

As an example of these ideas let "5 be the total conductance between edge (12,14) and *edge* (20,22) of the polygonal plate shown in Figure 2; other edges being insula.ted. Then

(28) ? » £ ^" £ ^ •

Here If^f is the joint conductance of the even network between, points (12,14) and (20,22) When line conductances g_{ij} are as in Theorem 9. Here$^n is the joint conductance of the odd network between points a and b when line conductances g_{ij} are as in Theorem 10.

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