REALIZATION OF TEMPERATURE DISTRIBUTIONS

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Abstract: It is shown that an arbitrary temperature distribution may be approximated in the $\mathcal{L}_2$ sense by controlling the boundary temperatures over a preceding time interval.

Let & be a bounded open subset of R^n with a piecewise smooth boundary SB having finite (n-1)-dimensional volume. Only a rather restricted class of temperature distributions u(= u(x), xeft) can "arise",, that is, can have a Tpast history[1] j e.g.,, such a u must be analytic in x. We ask, however, whether one could start with a given distribution - say, u = 0 - and,, by manipulating the boundary temperatures over some finite time interval, arrange to approximate arbitrarily well a specified distribution.

Thus, if tr(= TT(t,x) , t€(-l,O), xedft) is given one may obtain the solution $\tilde{u}$ of the heat equation with 0 initial data and $\bar{u}$ as boundary data and then define the operator T from a space of admissible boundary data to the space of possible temperature distributions by setting $T\bar{u} = u$ where $u(x) = \tilde{u}(0,x)$ for X6&. We ask whether the range of T is dense in £~(R).

It is easily seen that the answer to this question is independent of any reasonable choice of the space of admissible boundary data.

The function $\tilde{u}$, then, satisfies the conditions

$$\tilde{u}_t = A\tilde{u} \qquad te(-l,O), xe££$$

(1) $$\tilde{u}(-l,x) = 0 \qquad xeft$$

$$\tilde{u}(t,x) = \bar{u}(t,x) \qquad te(-l,O), x\in\partial\Omega$$

and, with $u(x) = \tilde{u}(0,x)$, we have the integral representation

(2) $$u(x) = \int_{-1}^{0}\int_{9R} G_\nu(x,y,-t)\,\bar{u}(t,y)\,dy\,dt$$

where G^ν is the normal derivative (with respect to its second variable) of the Green's function for the heat equation in a. As is well-known, G may be- expressed, for x,yeft and 0 < -t, as

(3) G(x,y,-t) = ££ ^ W a ^y) exp[? ^t]

where C-A, } are the eigenvalues of the Laplacian operator in ℛ and {su$_k$) are ^xe corresponding normalized eigenfunctions; thus

$$\Delta a_k = -\lambda_k a_k, \quad a_k(\partial \mathcal{R}) = 0, \quad \|a_k\|_2 = 1. \tag{4}$$

We recall that the i³^) are an orthogonal basis of £«₂W J the {A$_k$} are positive and that, if we let {jJL$_j$-} be the distinct .

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eigenvalues, ordered by increasing magnitude, then {/i$_j$/j } is bounded away from 0 and °° . The kernel G$_\nu$ of (2) is now given by G$_\nu$(x^y^-t) = v$_y$• V$_y$Gfx^y^-t) and, for xeR,yed&, 0 < -t,

(5) G^(x,y,-t) = S^ a$_k$(x)b$_k$(y) expf^t]

where v$_y$ is the unit outward normal at y (undefined on the subset - assumed negligible - at which BR is not smooth) and ^(y) = i>$_y$ • Va$_k$(y) ^ por fixed ^t > 0 and XGR^ we have G^ continuous (in fact, C$^{co *}$ where this is meaningful on oft) in y and, for fixed -t < 0 and yed&j G$_\nu$ is analytic in x.

We assert that, as (y, t) ranges over dft X (-.1,0) , the 'cross-sections' G$_\nu$(x,y,~t) generate <£$_2$ (&) when considered as functions of x. It then follows that, any u(x) in £$_9$(R) can be approximated arbitrarily well in £$_?$(&) by finite sums of the form

3 V⁵, <*•**•-V-

Now we may find, for e > 0/ C$^\delta$ functions 0$_m$(t,y) , for (t,y) e (-1,0) x 5ft, which vanish for jy-y$_{m}$| > e or |t-t$_m$| > e and satisfy $\int_{-1}^{0}\int_{\partial\mathcal{R}}$ 0$_m$ dy dt = 1; hence, using (2) and the continuity of G$^\nu$ in (i;-t) , u(x) can be approximated arbitrarily well by

T[2V c θ] so T has dense range. It remains to prove the assertion.

Suppose, to the contrary, that the linear span of ш = {G (•jYj-t): (Y* t) €dftx(-l,o)) were not dense in £9(&). We could then find a non-trivial (pelft^ i.e., a function cpe^2(&) such that <p / 0 but

(6) 0 = <<P,G^(-,y,-t)> (y,t)€dftX(-l,O).

Expanding <p in terms of the orthonormal basis t3^} gives <p = £k j8k ak with j8k = < <P,ak> and (6) becomes

(7) 0 = \P^(Y) exp[Akt] yeBft, 0 < -.t < 1.

We observe that this series is absolutely convergent for each (y,t) since {bk (y) exp [Ak. t] : k = 1,2,. . .} is square-summable, being the expansion coefficients of Gν which is in SI 2(fo) when, as here, considered as a function of x for yedft, t < 0.

If we collect the terms in (7) associated with multiple eigenvalues we obtain in terms of the distinct eigenvalues /zj

(8) 0 = Ej>'jz en < z < 1

where $z = e^t$ (-1 < t < 0) and, putting Kj = {k;Ak = /i3} (note that each Kj is finite),

(9) $\gamma_j = \gamma_j(y) = \Sigma_{k \in K_j} \beta_k b_k(y)$.

The absolute convergence of the series in (8) for, say, z = 3/4 and the sign of the exponents {jμj} guarantee the absolute convergence of the series uniformly in the disk |z| ≤ 3/4. Since each of the functions $z^{\mu_j} = \exp[\mu_j \log z]$ is analytic in the half-disk

$\{z: |z| \leq 3/4, \text{Re } z > 0\}$ the series converges in the half-disk to an analytic function which, by (8), must be 0. It follows that

$$o = \lim_{z \to 0} z^{-1} \sum_j \gamma_j z^{M_j} = \gamma_1$$

and, recursively, we obtain $0 = \gamma_2 = \gamma_3 = \dots$ so

(10) $0 = \wedge \wedge \wedge (y)_j \qquad y \in B, \ j = 1,2,\dots$

This implies that $\sum_{\kappa \in K_j} \wedge v \wedge (x)$ has $b^0 \wedge^1$ ° Dirichlet data, by (4), and 0 Neumann data and so vanishes whence, by the independence of the a_κ, we have $\xi_1 = \xi_2 = \dots = 0$ contradicting the assumption that $\varphi \neq 0$.

It follows that tU and, therefore, the range of T is dense in $\mathcal{L}_9 W$ • We remark that the same proof works for any strictly parabolic equation if the coefficients are independent of t. Presumably the same result would obtain for temporally inhomogeneous processes, under mild conditions on the time dependence, but a more delicate argument would be required.

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