BOUNDED GENERATORS OF LINEAR SPACES*

T. Ito
Hokkaido University, Sapporo, Japan

and

T. Seidman**
Carnegie-Mellon University
Pittsburgh, Pennsylvania

Report 67-32

October, 1967

*This work was initiated while both authors were at Wayne State University, Detroit, Michigan.

** This work was partially supported by contract NONR 760(27).

BOUNDED GENERATORS OF LINEAR SPACES*

T. Ito
Hokkaido University, Sapporo,, Japan

and

T. Seidman **
Carnegie-Mellon University
Pittsburgh, Pennsylvania

Summary

Let S^ = {x *e* X: sup^ <P_a(x) < °°} where <p = {<P_a) is a family of semi-norms determining the topology of X. It is shown that <p may be chosen so S_φ is dense iff X has a bounded generating set iff there is a continuous norm on X*. It is shown that these conditions hold for separable Fréchet spaces and for quotients of products of Banach spaces. An example is given of a Fréchet space containing no bounded generating set thus contradicting an assertion of L. Maté that S_φ is dense for Fréchet spaces.

This work was initiated while both authors were at Wayne State University, Detroit, Michigan.

This work was partially supported by contract NONR 760(27).

All spaces, X, Y, ..., in this paper are linear spaces with locally convex Haussdorf linear topologies. Given X, let $ = $\Phi(X)$ be the set of all families <p = Op^J of continuous semi-norms on X determining the topology; for <p *e* $, let S_φ = (x *e* X: N_φ(x) = $\sup_\alpha$ ^ (x) < °°}. For subsets *T* c: x, let *[T]* be the closed linear hull of *T;* if X has a bounded generator, i.e., if there is a bounded set r c x with [r] = X, we call X a 'BG space[1] . Again, for a subset *T* <= x, let *T*~ be its annihiiator, *T*~ = (4 e X*: £(x) = 0 for x *e* T}, let 1° be its polar set, r° = {£ *e* X^: |4(x)| < 1 for x.€ T}^ and let *v-p* be the Minkowski gauge of r° so, for 4 € X*, $i>_r$(£) = inf {t: ^ € t 1°} = sup { |£ (x) | : x *e* *T].* By the definition of the strong topology on X*, ^p is (strongly) continuous (and everywhere finite) if and only if *T* is bounded in X; it is a norm if and only if *[T]* = X.

It is asserted in Maté [l] that, given a Fréchet space X and <p *e* $(X), <p = {<p, *<Po>* • • • 3^ the set S_φ is dense in X. That this need not hold in general may be seen from the following elementary example.

Example 1: Let X = s be the space of all sequences x = [x_n, x_o, . . .] with the topology of coordinate-wise convergence. We may take <p *e* *$* to be given by ($p_n$ (x) = n max{ | x_k | : k < n} (n = 1, 2, . . .) or ^ *e* <& with θ_n(x) = l^xnl- Then S^ is the set of all bounded sequences (which is dense in X) but S^φ = {0} since, for k > n and any x *e* X, $|x_k|$ < P_n(x)/n < N^φ (x)/n.

This makes it clear that the density of S_φ in X depends on the choice of <p e $ and raises the question: Can one (in particular, if X is a Fréchet space) choose <p e <£(X) so that S_φ is dense in X? or, somewhat more generally: Can one choose <p e $(X) so that the closure of S_φ is a specified closed subspace Y? We begin by showing the equivalence of certain conditions on X, Y.

Theorem 1: Given a space X and a closed subspace Y c: x, the following conditions are equivalent:

(A) There exists <p e $(X) such that $\bar{S}_\varphi$ = Y,

(B) There exists a bounded set r c x with [r] = Y.

(C) Y is a BG space.

(D) There exists a (strongly) continuous semi-norm $\hat{v}$ on X*
such that $\hat{v}$"1^) = Y^x,

(E) There exists a (strongly) continuous norm 1/ on Y*.

Proof: The equivalence of (B) and (C) is immediate. The equivalence of (D) and (E) follows from the observation that we may take $\hat{v}$ = v • r (where T: X^ —> Y^ is the 'restriction of domain' map) and noting that T is continuous, open, surjective, and linear with kernel Y^. We now show the equivalence of (A) and (B) and of (C) and (E).

Given (A), set T = (x € X: N (x) < l}. Clearly T is bounded; as r is balanced and convex, [F] = U kl which is just $\bar{S}_\varphi$ = Y. Conversely, suppose we are given (B). Let 0 be a set of continuous semi-norms on X such that fl {^^(O): $a e $} = Y; such a set $ may be obtained from any element of $(X/Y). Let A be the index set of

$ = i$_α }» which we may assume infinite and let B be the set of all finite subsets of A; for j8 e B, x € X, set 0g(x) = n^ max{ ⌊_α (x): a € j8} (where n^ is the cardinality of jS) and let

0 = {#*}. Next choose any <p e $(X) and define p € * as follows: for <p e $ there is, as T is bounded, c > 0 such that T <= c {x e X: £y(x) < 1}; set Py(x) = <py(x)/cy and £ = {^y}. Finally, let <p = 0 U $•; since each 0^ € ^ is continuous and (p e $, we have <p € ^. For x € F and <p* € (p, we have <P*(x) = 0 if ^ G 0 (as T <= y) and <P^(x) ≤ 1 if % e $ so N_φ(x) ≤ 1. Thus T c: S_φ whence Y = [r] 5 S̄_φ. On the other hand, the construction of 0 guarantees that supgtygCx) = °° for x^Y so S_φ 5 Y and S̄^ = Y.

Given (C), there is a bounded generating set T for Y and v→v*_Γ is the required continuous norm on Y*• Conversely, given (E), let B = {77 G Y^: v(77) ≤ l}; the strong continuity of v means B is a neighborhood of 0 in Y* and so there exists a bounded set F e y such that I°5^B* Then v^ is an everywhere finite continuous semi-norm on Y* and, as u-p ^> v and 1/ is a norm, v-p is a norm on Y* and so [T] = Y. QED

For convenience we state separately the results above in the case X = Y.

<u>Corollary</u>: The following conditions on a space X are equivalent:

(A) There exists <p e $(X) such that S_φ is dense.

(B) There is a bounded generating set for X (i.e., X is a BG space).

(C) There exists a continuous norm v on X*.

Remark: Clearly, if X is a Fréchet space the <*p* of condition (A) may be taken to be countable.

Lemma: A Fréchet space X has a pre-compact generating set K if and only if it is separable.

Proof: Suppose X is separable and $\{x_1, x_2, ...\}$ is a countable dense subset. Letting *f* be any metric giving the topology on X, set y_n = ^ / f 1 + n ? ^ x n ^ and K ^ ^ y n ^ * Since y_n —> 0,, K is pre-compact. Since each x_n is in the linear hull of K and [x_{JL} x_o ^ *dt* . . . } is dense, K is a generating set. Conversely, if K is a pre-compact generating set then $\bar{K}$ (being a compact metric space) contains a countable dense subset $\{y_1, y_2, . . .\}$; let S be the set of all finite linear combinations of the $\{y_n\}$ with (complex) rational coefficients. Then S is countable and, as $\{y_1, y_2, \cdots\}$ is dense in the generating set $\bar{K}$, S is dense in X which is thus separable. QED

Since a pre-compact set must be bounded, any separable Fréchet space is a BG space and a positive partial answer to the question raised above is that, for a separable Fréchet space X, one can always choose <p *e* $(X) so that S_φ is dense; from the construction one can clearly arrange that S^ contain any specified countable set. We now collect some conditions under which a positive answer may be given to the question.

Theorem 2: Any of the following is sufficient to ensure that X is a BG space:

(A) X is a Banach space.

(B) X is a separable Fréchet space.

(C) X is a product of BG spaces.

(D) X is the image., under a continuous linear map., of a BG space.

(E) X is a quotient of a product of Banach spaces.

Proof: (A) is trivial, (B) follows from the Lemma above. (C) holds because the product of bounded subsets of the factors is bounded in a product space and the product of generating sets is a generating set. (D) holds because the image, under a continuous map, of a bounded set is bounded, of a dense set is dense. (E) follows immediately from (A), (C) and (D). QED

We now note that the final answer to the question: Does there always exist <p e $ such that S_φ is dense? is negative even when restricted to Fréchet spaces.

Example 2: Let A = {A =[A-.$_1$ A$_2$, . . .]: 1 = A_1. < A$_g$ < ...; A_n integers} and let H be a Hilbert space big enough to contain an orthonormal set [a_λ: A 6 A). Now let X = $\prod_1^\infty H_n$ with each H_n = H; the topology on X is determined by the sequence of semi-norms <p = {<p, ,<p$_o$, . . . }: $<P_n$ (x) = || π_n x|| where π_n: X —> H_n = H is the canonical projection taking x = [x, ,x$_o$, ...] e X into $x_\#$ e H, the norm being that of H. Then X is a Fréchet space; we observe that, by conditions (A) and (C) of Theorem 2, it is a BG space. For each A e A, set b_λ = [a_λ, A$_2$ a^, Ag a_λ, ...] e X and let Y be the closed linear hull of {b_λ: A € A}$_\#$ We assert that the Fréchet space Y (we give Y the induced topology,, determined by <p) contains no bounded generating sets, i.e., Y is not a BG space.

Proof: We suppose, to the contrary, that *T* is a bounded generating set for Y (we may, and do, assume that *T* is also closed, balanced, and convex) and proceed to show a contradiction. Boundedness of F implies that each $<p_n$ is bounded on *T* so $c_n = \sup\{<p_n(x) : x \in T\} <$ ∞ $(n = 1,2,...)$. Choose M *e* A such that $\sup_n\{/i_n/(1+c_n)\} = \infty$; we show that b_{1f} £ [F] so *T* cannot be a generating set. Suppose, now, that be [T]; then, as *[T]* = ILkT', there would exist, for some k = k*, an $x^{(0)}$ € k* T such that $9_n(x^{(0)} - b_{)(})$ < 1/4. By the definition of Y, there is, for each n, a linear combination $x^{(n)}$ = S, ./S^b, (a A€A A A finite sum: |3 = i^^(n) vanishes for A not in some finite set, in general depending on n) such that both $<P_1(x^{(0)} - x^{(n)}) < 1/4$ and $<P_n(x^{(0)} - x^{(n)}) < 1$. Then $<P_1(x^{(n)} - b_n) < 1/2$ so 1/4 > $\|\pi_1(x^{(n)}$ " V " 2= L A ^ ^ A I 2+ I*M ~ 1 ! 2wnence (n)| > 1/2. Now $<P_n(x^{(0)} - x^{(n)}) < 1$ implies that 1 > H l ^ A ^ - $7r_n x^{(0)}$|| $\geq$ ‖V A V A‖ - $\|x_n^{(0)}\| \geq$ | ^ (n)| μ_n - ' M $x^{(0)}$) $\geq$ V 2 - k*η' This, however, would imply that $/i_n/2 \leq 1 + k^c_n$ so $/i_n/(1+c_n) \leq$ 2k^- for n = 2,3,... which contradicts the choice of the sequence /i- QED

It is known that there may exist in Banach spaces, closed subspaces which are not the range of any continuous projection; considering X and Y of the example above in the light of condition (D) of Theorem 2 gives the following analogue.

Corollary: There exist Fréchet spaces (indeed, countable products of Banach spaces) containing closed subspaces which are not the range of any continuous linear operator on the space.

A number of open questions may be mentioned here.

(1) Ordering 0 by inclusion, let $\$_{o}$ be the set of all minimal families of semi-norms determining the topology on X. In Example 1 we have 0 *e* $\$_{o}$ and S^ dense while <p *i* $*_{o}$ and S_{φ} not dense. If X is a BG space and <p *e* $\$_{o}$, must S_{φ} be dense?

(2) Call a space X a ᵀhereditary BG space¹ (HBG space) if every closed subspace is a BG space. Every Banach space and every separable Fréchet space is HBG. Are there any other HBG spaces? Is the product of two HBG spaces necessarily HBG? In particular, is the product of the spaces X of Example 1 and H of Example 2 an HBG space?

(3) Is every BG space a quotient of a product of Banach spaces? In particular, is every Fréchet BG space a quotient of a countable product of Banach spaces?

REFERENCE

[l] Maté, *JL.,* On a semi-group of operators in Fréchet space, Doklady Akad. Nauk SSSR, 142 (1962) pp. 1247-1250. (English translation in: Sov. Math. Doklady 3^ (1962), pp. 288-292.)