APPROXIMATION OF OPERATOR SEMI-GROUPS

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§1. In his book [3], Yosida presents a section (§12 of Ch. IX) entitled 'The Trotter-Kato Theorem' in which is proved a theorem on the convergence of a sequence of C_0 semi-groups acting on a sequentially complete lctvs (locally convex topological vector space) X. This is parallel to, but does not subsume, the theorem presented by Trotter [2] and Kato (e.g., [1]) on the convergence of a sequence of (discrete or continuous) semi-groups acting on a sequence of approximating spaces; for Trotter and Kato all the spaces are Banach spaces. The aim of this paper is to provide a common generalixation of these results.

In § 2 is presented the setting for the theorems: a net of spaces $\{X_\alpha\}$ approximating an lctvs X. In § 3 an approximation theorem is proved for a net of C_0 semi-groups and in § 4 this is used to treat also the case of discrete-parameter approximating semi-groups. Finally, § 5 contains a converse to the main theorems of §3 and §4.

§2. For any lctvs X we call a set Φ of continuous semi-norms on X a determining set if it determines the topology of X; for simplicity we also assume Φ closed under linear combination with positive coefficients so Φ is a determining set for X iff $\{\{x \in X: \varphi(x) < 1\}: \varphi \in \Phi\}$ is a neighborhood base at 0.

Let $\mathfrak{a}$ be a directed set, $\{X_\alpha: \alpha \in \mathfrak{a}\}$ a net of lctvs's, and $\{\pi_\alpha: \alpha \in \mathfrak{a}\}$ a net of continuous linear surjections $(\pi_\alpha: X \twoheadrightarrow X_\alpha)$.

Let Φ be a determining set of semi-norms on X and let $\varphi \to \varphi_\alpha$ be a map (for each $\alpha \in \mathfrak{a}$) of Φ onto a determining set of semi-norms on X_α. We call such a system (incorrectly, but conveniently, identified as $\{X_\alpha\}$) an <u>approximating net (of spaces) to X</u> if, for each $x \in X$ and each $\varphi \in \Phi$,

(1) $$\varphi_\alpha(\pi_\alpha x) \to \varphi(x)$$

(convergence as $\alpha \uparrow$ in the directed set $\mathfrak{a}$).

<u>Example</u>: Let X be any lctvs with dual space X^* and let $\mathfrak{a}$ be the directed set of all finite-dimensional subspaces α of the dual X^* ordered by inclusion. Let X_α be the dual of α for $\alpha \in \mathfrak{a}$ and, define π_α by setting $[\pi_\alpha x](y) = y(x)$ for each $x \in X$ and $y \in \alpha$. Let Φ be any determining set for X and, for $\varphi \in \Phi$, set $\varphi_\alpha(z) = \inf\{\varphi(x) : x \in X, \pi_\alpha x = z\}$ for each $z \in X_\alpha$; since π_α, as defined above, is open as well as continuous, linear and surjective, φ_α will be a continuous semi-norm on X_α. To demonstrate (1), note that for any $\varphi \in \Phi$ and $x_o \in X$ there exists, by the Hahn-Banach Theorem, a $y \in X^*$ such that $|y(x)| \leq \varphi(x)$ for all $x \in X$ while $y(x_o) = \varphi(x_o)$; it follows that we have $\varphi_\alpha(\pi_\alpha x_o) = \varphi(x_o)$ for any α such that $y \in \alpha$. Thus, every lctvs can be approximated in this sense by a net of finite-dimensional spaces.

If Φ is a determining set for the lctvs X and μ is a mapping (μ: Φ Φ), we say the linear operator $L: X \to X$ is <u>μ-continuous</u> if, for every $x \in X$ and every $\varphi \in \Phi$,

$$\varphi(Lx) \leq [\mu\varphi](x) .$$

We observe that every continuous operator on X is μ-continuous for some μ. An operator $L_\alpha: X_\alpha \to X_\alpha$ will also be called μ-continuous if, for every $x_\alpha \in X_\alpha$ and $\varphi \in \Phi$,

$$\Phi_\alpha(L_\alpha x_\alpha) \leq [\mu\varphi]_\alpha(x_\alpha) .$$

A net of operators $\{L_\alpha\}$ $(L_\alpha: X_\alpha \to X_\alpha)$ is called <u>equi-continuous in</u> α if each L_α is μ-continuous with the mapping μ fixed (independent of α).

Let $\{x_\alpha: \alpha \in \mathfrak{a}\}$ be a net of vectors, $x_\alpha \in X_\alpha$. We say x_α <u>converges to</u> x (for some $x \in X$) and write $x_\alpha \to x$ or $x = \lim x_\alpha$ if, for every $\varphi_\alpha \in \Phi$,

$$\varphi_\alpha(x_\alpha - \pi_\alpha x) \to 0.$$

For a net $\{L_\alpha\}$ of operators $(L_\alpha: X_\alpha \to X_\alpha)$ we say $\{L_\alpha\}$ <u>converges (strongly) to</u> L (for some operator L on X) and write $L_\alpha \to L$ or $\lim L_\alpha = L$ if, for every $x \in X$, $L_\alpha \pi_\alpha x \to Lx$ so, for every $\varphi \in \Phi$,

$$\varphi_\alpha(L_\alpha \pi_\alpha x - \pi_\alpha Lx) \to 0 .$$

For future reference, we state the following (obvious) result as a lemma.

<u>Lemma 1</u>: Let $\{A_\alpha\}$, $\{B_\alpha\}$ be nets of operators on the approximating net $\{X\}$ to X. If $A_\alpha \to A$ and $B_\alpha \to B$, then $(A_\alpha + B_\alpha) \to A+B$. If, in addition, $\{A_\alpha\}$ is equi-continuous (i.e., μ-continuous uniformly in α for some μ) then $(A_\alpha B_\alpha) \to AB$.

<u>§3</u>. In this section $\{X_\alpha\}$ will denote an approximating net of spaces to X, with X and each X_α a sequentially complete

lctvs; $\{S_\alpha\} = \{S_\alpha(t): t \geq 0\}$ is a net of C_0 semi-groups $(S_\alpha(t): X_\alpha \to X_\alpha)$ μ-continuous uniformly in $\alpha \in \mathfrak{A}$, $t \geq 0$; that is, for each $x_\alpha \in X_\alpha$ and $\varphi \in \Phi$,

$$\varphi_\alpha(S_\alpha(t)x_\alpha) \leq [\mu\varphi]_\alpha(x_\alpha) \quad . \tag{2}$$

Denote by A_α the infinitesimal generator of the semi-group S_α and by $R_\alpha(\lambda)$ the resolvent of A_α ($R_\alpha(\lambda) = (\lambda - A_\alpha)^{-1}$ where this inverse exists as a continuous operator).

<u>Theorem 1</u>: Let $\{S_\alpha\}$ be a uniformly μ-continuous net of C_0 semi-groups, as above, on the net $\{X_\alpha\}$ approximating X. Suppose that, for some $\lambda_0 > 0$,

$$R(\lambda_0)x = \lim R_\alpha(\lambda_0)\pi_\alpha x$$

exists for each $x \in X$ and that the range of $R(\lambda_0)$ (i.e., the set $\{\lim R_\alpha(\lambda_0)\pi_\alpha x: x \in X\}$) is dense in X. Then, for $\mathrm{Re}\lambda > 0$, $R_\alpha(\lambda)$ converges strongly to an operator $R(\lambda)$ on X which is the resolvent of the infinitesimal generator A of a μ-continuous C_0 semi-group $S = \{S(t): t \geq 0\}$ on X; further, $S_\alpha(t)$ converges to $S(t)$ uniformly in t, for t in any bounded interval.

<u>Proof</u>: That $R_\alpha(\lambda)$ is defined for $\mathrm{Re}\lambda > 0$ is Corollary 1 of [Y: IX, 4]*. Formula (10) of that section gives us, for each $x_\alpha \in X_\alpha$ $\mathrm{Re}\lambda > 0$, $n = 0, 1, \ldots,$

$$[\lambda R_\alpha(\lambda)]^{n+1} x_\alpha = \frac{\lambda^{n+1}}{n!} \int_0^\infty e^{-\lambda t} t^n \, S_\alpha(t) x_\alpha \, dt$$

(that the [Bochner] integral is well-defined follows from (2)

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*References of this form are to chapter and section of Yosida [3]; e.g., this refers to §4 of Chapter IX.

and the sequential completeness of X_α) so, for each $\varphi\in\Phi$,

(3) $$\varphi_\alpha([\lambda R_\alpha(\lambda)]^{n+1} x_\alpha) \leq \frac{\lambda^{n+1}}{n!} \int_0^\infty e^{-\lambda t} t^n [\mu\varphi]_\alpha(x_\alpha)\,dt = [\mu\varphi]_\alpha(x_\alpha)$$

if λ is real and positive; i.e., $\{[\lambda R_\alpha(\lambda)]^n\}$ is μ-continuous uniformly in $\lambda(\lambda > 0)$, $n(n=1,2,\ldots)$ and $\alpha(\alpha\in\mathfrak{A})$.

From the linearity of each $R_\alpha(\lambda_0)$ follows that of $R(\lambda_0)$. Using (3) with $\lambda = \lambda_0$, we may apply Lemma 1 recursively in n to show $[\lambda_0 R_\alpha(\lambda_0)]^n \rightarrow [\lambda_0 R(\lambda_0)]^n$ $(n=1,2,\ldots)$. It then follows that $[\lambda_0 R(\lambda_0)]^n$ is μ-continuous (for $n=1,2,\ldots$) on X since

$$\begin{aligned}\varphi([\lambda_0 R(\lambda_0)]^n x) \leq\ & |\varphi_\alpha(\pi_\alpha[\lambda_0 R(\lambda_0)]^n x) - \varphi([\lambda_0 R(\lambda_0)]^n x)| \\ & + \varphi_\alpha([\lambda_0 R_\alpha(\lambda_0)]^n \pi_\alpha x - \pi_\alpha[\lambda_0 R(\lambda_0)]^n x) \\ & + \varphi_\alpha([\lambda_0 R_\alpha(\lambda_0)]^n \pi_\alpha x) \\ \leq\ & [\mu\varphi]_\alpha(\pi_\alpha x) + \epsilon_\alpha\end{aligned}$$

where $\epsilon_\alpha \rightarrow 0$ by (3) and the convergence of $[\lambda_0 R_\alpha(\lambda_0)]^n$, and $[\mu\varphi]_\alpha(\pi_\alpha x) \rightarrow [\mu\varphi](x)$ by (1).

Now set $\theta = (\lambda_0 - \lambda)/\lambda_0$ and, for λ such that $|\theta| < 1$, set

(4) $$R(\lambda)x = \lambda_0^{-1} \Sigma_0^\infty \theta^n [\lambda_0 R(\lambda_0)]^{n+1} x .$$

The series converges absolutely - - i.e., the related series $|\lambda_0|^{-1} \Sigma_0^\infty |\theta|^n \varphi([\lambda_0 R(\lambda_0)]^{n+1} x)$ converges - - by the uniform μcontinuity of $\{[\lambda_0 R(\lambda_0)]^{n+1}\}$. Hence $R(\lambda)$ is well-defined by (4) since Φ is a determining set and X is sequentially complete . Observe that, similarly, for $|\lambda - \lambda_0| < \lambda_0$

$$\lambda_0^{-1} \Sigma_0^\infty \theta^n [\lambda_0 R_\alpha(\lambda_0)]^{n+1}$$

converges absolutely and may easily be seen to converge to $R_\alpha(\lambda)$. Let R^N and R^N_α be the corresponding partial sums (Σ_0^N) so, for $\epsilon > 0$, there exists $N(\epsilon) = N(x,\varphi,\epsilon)$ independent of α $(x\epsilon X, \varphi\epsilon\Phi)$ such that

$$\varphi_\alpha([R_\alpha - R^N_\alpha]\pi_\alpha x),\ \varphi([R - R^N]x) < \epsilon$$

$(R_\alpha = R_\alpha(\lambda),\ R = R(\lambda))$ for $N > N(\epsilon)$ in which case

$$\begin{aligned}\varphi_\alpha(\pi_\alpha Rx - R_\alpha\pi_\alpha x) &\\ \leq\ & \varphi_\alpha(\pi_\alpha R^N x - R^N_\alpha\pi_\alpha x)\\ &+ |\varphi_\alpha(\pi_\alpha[R - R^N]x) - \varphi([R - R^N]x)|\\ &+ \varphi([R - R^N]x)\\ &+ \varphi_\alpha(R_\alpha - R^N_\alpha]\pi_\alpha x)\\ \longrightarrow\ & 0\ ;\end{aligned}$$

the first term going to 0 as $\theta^n[\lambda_0 R_\alpha(\lambda_0)]^{n+1} \to \theta^n[\lambda_0 R(\lambda_0)]^{n+1}$ for $n = 0,\ldots,N$ and the second by (1). Thus $R_\alpha(\lambda) \to R(\lambda)$ for $|\lambda - \lambda_0| < \lambda_0$ and, as above, this implies that (for λ real and $0 < \lambda < 2\lambda_0$) $[\lambda R(\lambda)]^n$ is μ-continuous (for $n=1,2,\ldots$). Choosing a 'new λ_0' one can now repeat the process to obtain convergence in a larger disc and so, eventually, $R_\alpha(\lambda) \to R(\lambda)$ for $\mathrm{Re}\,\lambda > 0$ and, for λ real, positive and $n=1,2,\ldots$, $[\lambda R(\lambda)]^n$ is μ-continuous.

Since each $R_\alpha(\cdot)$ satisfies the resolvent equation, so does $R(\cdot)$ (at least for positive real λ) by the equi-continuity of $\lambda R_\alpha(\lambda)$ and Lemma 1. The assumed density of the range of $R(\lambda_0)$ now guarantees (c.f., [Y: IX, 7]) the existence of an operator A of which $R(\lambda)$ is the resolvent and the uniform μ-continuity of $\{[\lambda R(\lambda)]^n: \lambda > 0;\ n=1,2,\ldots\}$ guarantees that this A is the infinitesimal generator of a μ-continuous C_0 semi-group $S = \{S(t): t \geq 0\}$ on X. We need only show the convergence of $S_\alpha(t)$ to $S(t)$.

For $N = 1,2,\ldots,$ $n = 1,2,\ldots,$ $t > 0$, set

$$(5)\qquad \begin{aligned} S^N(t;n)x &= e^{-nt} \Sigma_0^N \frac{n^k t^k}{k!} [nR(n)]^k x \\ S_\alpha^N(t;n)x_\alpha &= e^{-nt} \Sigma_0^N \frac{n^k t^k}{k!} [nR_\alpha(n)]^k x_\alpha \end{aligned}$$

for $x \in X$, $x_\alpha \in X_\alpha$; by the uniform μ-continuity of $\{[nR(n)]^k, [nR_\alpha(n)]^k\}$ we obtain the μ-continuity of $S^N(t;n)$, $S_\alpha^N(t;n)$, $S(t;n) = \lim_N S^N(t;n)$, and $S_\alpha(t;n) = \lim_N S_\alpha^N(t;n)$ (convergence of $\{S^N\}$ and $\{S_\alpha^N\}$ follows as before from the absolute convergence of the series). By [Y: IX, 7], $S(t;n)$ and $S_\alpha(t;n)$ are C_0 semi-groups which converge strongly, as $n \to \infty$, to $S(t)$ and $S_\alpha(t)$ respectively.

We may differentiate (5) term-by-term (justifiable by the absolute convergence - locally uniform in t - of the derived series) to obtain

$$\frac{d}{dt} S(t;n) = [n^2 R(n) - n] S(t;n).$$

Observing that (we set $R(1) = R$)

$$[n^2R(n) - n]R = nR(n)[R-1] \quad \text{and}$$
$$[mR(m) - nR(n)]R = \left(\frac{n-m}{nm}\right) nR(n)mR(m)[R-1],$$

we have

$$\begin{aligned}[S(t;n) - S(t;m)]R^2x &= \int_0^t \frac{d}{ds} [S(t-s;n)S(s;m)]Rxds \\ &= \left(\frac{n-m}{nm}\right) nR(n)mR(m)[R-1] \int_0^t S(t-s;n)S(s;m)xds.\end{aligned}$$

It follows that, for $x \in X$ and $n,m=1,2,\ldots,$

$$\varphi([S(t;n) - S(t;m)]Rx) \leq t\left|\frac{1}{n} - \frac{1}{m}\right| [\mu'\varphi](x)$$

where $\mu'\varphi = \mu^5\varphi + \mu^4\varphi$ (exponents denoting iterates). Thus, as $S(t;n) \to S(t)$,

$$\varphi([S(t) - S(t;m)]Rx) \leq \frac{t}{m} [\mu'\varphi](x). \tag{6}$$

Similarly,

$$\varphi_\alpha([S_\alpha(t) - S_\alpha(t;m)]R_\alpha x_\alpha) \leq \frac{t}{m} [\mu'\varphi]_\alpha(x_\alpha). \tag{6'}$$

Notice that the absolute convergence of $\{S_\alpha^N\}$ is uniform in α ; there exists $\epsilon_N = \epsilon_N(t,n)$, such that $\epsilon_N \to 0$ uniformly in t (t bounded) as $N \to \infty$ and

$$\begin{aligned} \varphi_\alpha([S_\alpha^N(t;n) - S_\alpha(t;n)]x_\alpha) &\leq \epsilon_N[\mu\varphi]_\alpha(x_\alpha), \\ \varphi([S^N(t;n) - S(t;n)]x) &\leq \epsilon_N[\mu\varphi](x). \end{aligned} \tag{7}$$

We now have, for $y = Rx$ and all $\alpha \in \mathfrak{A}$, N, n, $t > 0$, $\varphi \in \Phi$,

$$\begin{aligned} &\varphi_\alpha([\pi_\alpha S(t) - S_\alpha(t)\pi_\alpha]y) \\ &\quad\leq \varphi_\alpha(\pi_\alpha[S(t) - S_\alpha^N(t;n)]y) \\ &\qquad + \varphi_\alpha(\pi_\alpha S^N(t;n)y - S_\alpha^N(t;n)\pi_\alpha y) \\ &\qquad + \varphi_\alpha([S_\alpha^N(t;n) - S_\alpha(t)]\pi_\alpha y \\ &\quad\leq |\varphi_\alpha(\pi_\alpha[S(t) - S^N(t;n)]y) - \varphi([S(t) - S^N(t;n)]y)| \\ &\qquad + \varphi([S(t) - S(t;n)]Rx) + \varphi([S(t;n) - S^N(t;n)]y) \\ &\qquad + \varphi_\alpha([S_\alpha(t) - S_\alpha(t;n)]R_\alpha\pi_\alpha x) + \varphi_\alpha([S_\alpha(t;n) - S_\alpha^N(t;n)]\pi_\alpha y) \\ &\qquad + \varphi_\alpha([S_\alpha(t) - S_\alpha(t;n)][R_\alpha\pi_\alpha - \pi_\alpha R]x) \\ &\qquad + e^{-nt}\sum_0^N \frac{n^k t^k}{k!} \varphi_\alpha\left(\pi_\alpha[nR(n)]^k y - [nR_\alpha(n)]^k\pi_\alpha y\right). \end{aligned}$$

Given φ, y and a bounded t-interval, choose n large enough so that $(t/n)[\mu'\varphi](x)$ and $(t/n)[\mu'\varphi]_\alpha(\pi_\alpha x)$ are small (uniformly in α for $\alpha > \alpha_o$ -- so $[\mu'\varphi]_\alpha(\pi_\alpha x) \approx [\mu'\varphi](x)$); next fix N so $\epsilon_N[\mu\varphi](y)$, $\epsilon_N[\mu\varphi]_\alpha(\pi_\alpha y)$ are small (uniformly in α for $\alpha > \alpha_1$); finally, take α large enough that all the remaining terms are small (possible by (1), the equicontinuity of $S_\alpha(t)$ and $S_\alpha(t;n)$ and the convergence of R_α to R).

Thus $S_\alpha(t)y$ converges to $S(t)y$ (uniformly in t for t in a bounded interval) when y is in the range of $R = R(1)$. Since this range is dense in X, the continuity of $S(t)$ (uniformly in t) and the equi-continuity in α of $\{S_\alpha(t)\}$ suffice to ensure the convergence of $S_\alpha(t)y$ to $S(t)y$ for <u>all</u> $y \in X$ and the local uniformity in t of this convergence.

QED

<u>§4.</u> In this section $\{X_\alpha\}$ and X will be as in §3. For $\alpha \in \mathfrak{a}$, let $\delta_\alpha \geq 0$ with $\delta_\alpha \to 0$; set $T_\alpha = \{n\delta_\alpha: n=0,1,\ldots\}$ if $\delta_\alpha > 0$ and $T_\alpha = [0,\infty)$ if $\delta_\alpha = 0$. Let $\{\hat{S}_\alpha\} = \{\hat{S}_\alpha(t): t \in T_\alpha\}$ be a semi-group: $\hat{S}_\alpha(n\delta_\alpha) = (\tilde{S}_\alpha)^n$ $(n=0,1,\ldots)$ if $\delta_\alpha > 0$ and $\hat{S}_\alpha$ a C_o semi-group if $\delta_\alpha = 0$. Set $A_\alpha = [\tilde{S}_\alpha - 1]/\delta_\alpha$ if $\delta_\alpha > 0$ and let A_α be the infinitesimal generator of the C_o semi-group $\hat{S}_\alpha$ if $\delta_\alpha = 0$; in any case let $R_\alpha(\lambda)$ be the resolvent of A_α. We assume the semi-groups $\{\hat{S}_\alpha\}$ are μ-continuous uniformly in $\alpha \in \mathfrak{a}$, $t \in T_\alpha$ so, for each $x_\alpha \in X_\alpha$, $t_\alpha \in T_\alpha$, and $\varphi \in \Phi$,

(8) $$\varphi_\alpha(\hat{S}_\alpha(t_\alpha)x_\alpha) \leq [\mu\varphi]_\alpha(x_\alpha).$$

Theorem 2: Let $\{\hat{S}_\alpha\}$ be a uniformly μ-continuous net of (possibly discrete) semi-groups as above on the net $\{X_\alpha\}$ approximating X. Suppose that for some $\lambda_0 > 0$

$$R(\lambda_0)x = \lim R_\alpha(\lambda_0)\pi_\alpha x$$

exists for each $x \in X$, $R(\lambda_0)$ has dense range and $t_\alpha \to t$ (with $t_\alpha \in T_\alpha$) uniformly on bounded t-intervals. Then $R_\alpha(\lambda)$ exists for Re $\lambda > 0$ and converges there to $R(\lambda)$ which is the resolvent of the infinitesimal generator A of a μ-continuous C_0 semi-group $\{S(t) : t \geq 0\}$; further, if $t_\alpha \in T_\alpha$ and $t_\alpha \to t$, then $\hat{S}_\alpha(t_\alpha)\pi_\alpha x \to S(t)x$.

Proof: When $\delta_\alpha > 0$, define, for $t \geq 0$, $x_\alpha \in X_\alpha$,

(9) $$S_\alpha(t)x_\alpha = e^{-s} \Sigma_0^\infty \frac{s^k}{k!} S_\alpha^k x_\alpha$$

where $s = t/\delta_\alpha$; if $\delta_\alpha = 0$ set $S_\alpha = \hat{S}_\alpha$. Observe that, by (8), (9) converges absolutely and

(10) $$\begin{aligned} \varphi_\alpha(S_\alpha(t)\tilde{S}_\alpha^m x_\alpha) &\leq e^{-s} \Sigma_0^\infty \frac{s^k}{k!} \varphi_\alpha(\tilde{S}_\alpha^{m+k} x_\alpha) \\ &\leq e^{-s} \Sigma_0^\infty \frac{s^k}{k!} [\mu\varphi]_\alpha(x_\alpha) = [\mu\varphi]_\alpha(x_\alpha) . \end{aligned}$$

By the usual series manipulations, it follows that $\{S_\alpha(t)\}$ is a semi-group and, by term-by-term differentiation of (9), that its infinitesimal generator is A_α $(\delta_\alpha > 0)$; that S_α is of type C_0 follows easily from (8) and (9).

Thus, the net $\{S_\alpha\}$ satisfies the conditions of Theorem 1 so $S_\alpha(t) \to S(t)$, where $S(t)$ is the well-defined μ-continuous C_0 semi-group whose infinitesimal generator has resolvent $R(\lambda) = \lim R_\alpha(\lambda)$. We need only prove that $[\hat{S}_\alpha(t_\alpha) - S_\alpha(t)]\pi_\alpha x$

must become small as $t_\alpha \to t(\delta_\alpha \to 0)$.

We set $R_\alpha = R_\alpha(1)$, $R = R(1)$ for convenience and first show that, for $x_\alpha \in X_\alpha$, $t_\alpha \in T_\alpha$, $\varphi \in \Phi$,

$$(11) \qquad \varphi_\alpha\big([\hat{S}_\alpha(t_\alpha) - S_\alpha(t_\alpha)]R_\alpha^2 x_\alpha\big) \le \tfrac{1}{2}\, t_\alpha \delta_\alpha [\mu\varphi]_\alpha\big([R_\alpha - 1]^2 x_\alpha\big);$$

this is trivial for $\delta_\alpha = 0$, in which case $S_\alpha \equiv \hat{S}_\alpha$, and we may assume $\delta_\alpha > 0$. Then, noting that all operators involved commute,

$$\frac{d}{dt} S_\alpha(t) R_\alpha^2 x_\alpha = S_\alpha(t) R_\alpha (R_\alpha - 1) x_\alpha$$

$$\frac{d^2}{dt^2} S_\alpha(t) R_\alpha^2 x_\alpha = S_\alpha(t)(R_\alpha - 1)^2 x_\alpha$$

so

$$S_\alpha(t) R_\alpha^2 x_\alpha = R_\alpha^2 x_\alpha + \int_0^t \Big[R_\alpha(R_\alpha - 1)x_\alpha + \int_0^s S_\alpha(r)(R_\alpha - 1)^2 x_\alpha\, dr\Big] ds$$

whence, as $[R_\alpha^2 + \delta_\alpha R_\alpha(R_\alpha - 1)] = \tilde{S}_\alpha R_\alpha^2$,

$$[S_\alpha(\delta_\alpha) - \tilde{S}_\alpha] R_\alpha^2 x_\alpha = \int_0^{\delta_\alpha} (\delta_\alpha - t) S_\alpha(t)(R_\alpha - 1)^2 x_\alpha\, dt.$$

Now, for $t_\alpha = n\delta_\alpha$,

$$\begin{aligned}[S_\alpha(t_\alpha) - \hat{S}_\alpha(t_\alpha)] R_\alpha^2 x_\alpha &= [S_\alpha(n\delta_\alpha) - S_\alpha^n] R_\alpha^2 x_\alpha \\ &= \Sigma_{k=1}^n S_\alpha([n-k]\delta_\alpha)\tilde{S}_\alpha^{k-1}[S_\alpha(\delta_\alpha) - \tilde{S}_\alpha] R_\alpha^2 x_\alpha \\ &= \Sigma_1^n \int_0^{\delta_\alpha} (\delta_\alpha - t) S_\alpha([n-k]\delta_\alpha + t)\tilde{S}_\alpha^{k-1}(R_\alpha - 1)^2 x_\alpha\, dt\end{aligned}$$

whence, using (10),

$$\begin{aligned}&\varphi_\alpha([S_\alpha(t_\alpha) - \hat{S}_\alpha(t_\alpha)] R_\alpha^2 x_\alpha) \\ &\quad \le \Sigma_1^n \int_0^{\delta_\alpha} (\delta_\alpha - t)[\mu\varphi]_\alpha([R_\alpha - 1]^2 x_\alpha)\, dt \\ &\quad = n\left(\tfrac{1}{2}\delta_\alpha^2\right)[\mu\varphi]_\alpha([R_\alpha - 1]^2 x_\alpha).\end{aligned}$$

which is just (11).

Next we show that for $t,s \geq 0$, $x_\alpha \in X_\alpha$, $\varphi \in \Phi$ we have

$$(12) \quad \varphi_\alpha([S_\alpha(t) - S_\alpha(s)]R_\alpha x_\alpha) \leq |t-s|\,[\mu\varphi]_\alpha([R_\alpha - 1]x_\alpha).$$

We may clearly assume $t > s$ and set $\epsilon = t-s > 0$. Noting that

$$\frac{d}{dt} S_\alpha(t) R_\alpha x_\alpha = S_\alpha(t)[R_\alpha - 1]\, x_\alpha ,$$

we have

$$S_\alpha(\epsilon) R_\alpha x_\alpha = R_\alpha x_\alpha + \int_o^\epsilon S_\alpha(r)[R_\alpha - 1]x_\alpha dr$$

whence

$$\varphi_\alpha([S_\alpha(t) - S_\alpha(s)]R_\alpha x_\alpha) \leq \int_o^\epsilon \varphi_\alpha(S_\alpha(s+r)[R_\alpha - 1]x_\alpha)\,dr$$
$$\leq \epsilon\, [\mu\varphi]_\alpha([R_\alpha - 1]x_\alpha)$$

which is just (12).

Now, for $y = R^2 x \in X$, $\varphi \in \Phi$, we have

$$\begin{aligned} (13) \quad & \varphi_\alpha\big([\hat{S}_\alpha(t_\alpha) - S_\alpha(t)]\pi_\alpha y\big) \\ & \leq \varphi_\alpha\big([\hat{S}_\alpha(t_\alpha) - S_\alpha(t_\alpha)]R_\alpha^2 \pi_\alpha x\big) \\ & \quad + \varphi_\alpha\big([S_\alpha(t_\alpha) - S_\alpha(t)]R_\alpha^2 \pi_\alpha x\big) \\ & \quad + \varphi_\alpha\big([\hat{S}_\alpha(t_\alpha) - S_\alpha(t)][R_\alpha^2 \pi_\alpha - \pi_\alpha R^2]x\big). \end{aligned}$$

By (11), the first term on the right in (13) is bounded by $\frac{1}{2} t_\alpha \delta_\alpha [\mu\varphi]_\alpha([R_\alpha - 1]^2 \pi_\alpha x)$ which, in turn is less than $\frac{1}{2} t_\alpha \delta_\alpha$ times

$$\begin{aligned} & [\mu\varphi]([R-1]^2 x) \\ & + \big|\,[\mu\varphi]([R-1]^2 x) - [\mu\varphi]_\alpha(\pi_\alpha(R-1)^2 x)\,\big| \\ & + [\mu\varphi]_\alpha(\pi_\alpha(R-1)^2 x - (R_\alpha - 1)^2 \pi_\alpha x) \end{aligned}$$

which is bounded -- so the first term goes to 0 as $\delta_\alpha \to 0$ (as α increases in $\mathfrak{a}$) uniformly on bounded t-intervals. By (12), the second term on the right in (13) is bounded by $|t_\alpha - t|[\mu\varphi]_\alpha([R_\alpha - 1]R_\alpha \pi_\alpha x)$ which is less than $|t_\alpha - t|$ times

$$[\mu\varphi]([R-1]Rx)$$
$$+ \big|[\mu\varphi]([R-1]Rx) - [\mu\varphi]_\alpha(\pi_\alpha[R-1]Rx)\big|$$
$$+ [\mu\varphi]_\alpha(\pi_\alpha[R-1]Rx - [R_\alpha - 1]R_\alpha\pi_\alpha x)$$

which is bounded - - so the second term goes to 0 as $t_\alpha \to t$ (as α increases in $\mathfrak{a}$) uniformly on bounded t-intervals. Finally, the last term in (13) is less than $2[\mu\varphi]_\alpha([R_\alpha^2 \pi_\alpha - \pi_\alpha R^2]x)$ which goes to 0 (independently of t) as α increases in $\mathfrak{a}$. Thus, for each y in the range of R^2, $\hat{S}_\alpha(t_\alpha)\pi_\alpha y \to S(t)y$ (uniformly on bounded t-intervals) as α increases in $\mathfrak{a}$. Since the range of R^2 is dense in X, S(t) continuous (uniformly in t on bounded intervals), and $\hat{S}_\alpha(t_\alpha)$ equi-continuous, this implies that $\hat{S}_\alpha(t_\alpha)\pi_\alpha y \to S(t)y$ for <u>all</u> $y \in X$ (whether or not y is in the range of R^2) uniformly on bounded t-intervals.

QED

<u>§5.</u> In this section it is shown that the consistency condition, $R_\alpha(\lambda_o) \to R(\lambda_o)$, is necessary for the approximating semi-groups to converge.

<u>Theorem 3</u>: Let $\{S_\alpha\}$ be a uniformly μ-continuous net of C_o semi-groups on the net X_α approximating X (as in §3). Suppose there is a μ-continuous C_o semi-group $\{S\}$ on X such that $S_\alpha(t) \to S(t)$ strongly, uniformly on bounded t-intervals. Then

$R_\alpha(\lambda) \to R(\lambda)$ for $\operatorname{Re}\lambda > 0$ where $R_\alpha(\lambda)$ is the resolvent $(\lambda - A_\alpha)^{-1}$ of the infinitesimal generator A_α of $\{S_\alpha\}$ and similarly for $R(\lambda) = (\lambda - A)^{-1}$.

Proof: Recall (see, e.g., [Y: IX, 4]) that we have the representation

$$R_\alpha(\lambda)x_\alpha = \int_0^\infty e^{-\lambda t} S_\alpha(t)x_\alpha dt$$

for $\operatorname{Re}\lambda \geq 0$, $x_\alpha \in X_\alpha$, and similarly for $R(\lambda)$. Then, for any $\varphi \in \Phi$ and any $x \in X$,

$$\begin{aligned}
&\varphi_\alpha(\pi_\alpha R(\lambda)x - R_\alpha(\lambda)\pi_\alpha x) \\
&\quad \leq \int_0^\infty e^{-(\operatorname{Re}\lambda)t} \varphi_\alpha(\pi_\alpha S(t)x - S_\alpha(t)\pi_\alpha x)\,dt \\
&\quad \leq \int_0^M e^{-(\operatorname{Re}\lambda)t} \varphi_\alpha(\pi_\alpha S(t)x - S_\alpha(t)\pi_\alpha x)\,dt \\
&\qquad + 2\int_M^\infty e^{-(\operatorname{Re}\lambda)t}\,dt\,[\mu\varphi](x).
\end{aligned}$$

The last term may be made small by taking M large enough. Then, by the strong convergence of $S_\alpha(t)$ to $S(t)$ uniformly on $[0,M]$, the preceding term becomes small as $\alpha\uparrow$ in $\mathfrak{G}$.

QED

Remark: In the setting of §4, the consistency condition is still necessary as $\hat{S}_\alpha(t_\alpha) \to S(t)$ and $\varphi_\alpha([S_\alpha(t_\alpha) - S_\alpha(t)]x_\alpha) \to 0$ (uniformly on bounded t-intervals) implies the convergence of $S_\alpha(t)$ to $S(t)$ so the above theorem can be applied.

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