ON LIFTING CHARACTERS IN FINITE GROUPS

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September 16, 1966

66-2

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1. INTRODUCTION

The theory of exceptional characters was first developed by R. Brauer (see, for instance, [1]) and M. Suzuki [11], and later was generalized by W. Feit [6 and 7]. It has been a powerful tool in its many applications, in that, under certain conditions, it yields information about the characters of a finite group $\mathcal{G}$ from information about the characters of a certain subgroup $\mathcal{L}$. More recently Feit and Thompson [9] obtained the theory of coherent sets of characters, which applies under more general conditions on $\mathcal{G}$ and $\mathcal{L}$. The earlier theories had required the existence of a trivial intersection set $\hat{\mathcal{L}}$ in $\mathcal{L}$, whereas they introduced the weaker condition that $\hat{\mathcal{L}}$ be tamely imbedded in $\mathcal{G}$. The theory of coherent sets of characters played a crucial role in their proof of the solvability of groups of odd order.

All these theories concern a linear isometry σ mapping certain generalized characters of $\mathcal{L}$ into those of $\mathcal{G}$ and the extension of σ to an isometry with a larger domain. When an extension is possible, the larger domain is called coherent. In the theory of exceptional characters, σ is the induction operator. Recently Dade [5] proved the existence of a suitable operator σ under more general conditions on $\mathcal{G}$, $\mathcal{L}$, and $\hat{\mathcal{L}}$ than those of Feit and Thompson. At the same time, related operators were

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*The first author was supported by National Science Foundation Grant GP-4240. An earlier version of much of the paper was contained in the second author's dissertation, which was submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the Carnegie Institute of Technology. Part of her work was done while she was on the faculty of Mount Mercy College, Pittsburgh.

employed by Brauer [2] and Suzuki [12] in their studies of the existence of normal complements of subgroups.

Our purpose here is to make a new study of σ and of coherence, further generalizing the existing theories, and to indicate some applications. In §2 we study the existence and properties of σ, and we obtain a result on the existence of normal complements. In §§3 and 4 the existence of σ is assumed and results concerning the existence and uniqueness of its extensions σ^* are obtained. Assuming an extension σ^* of σ exists, we study the multiplicities of the constituents of $\theta|\mathfrak{L}$ for certain characters θ of $\mathfrak{G}$ in §5. Finally, as an application involving many of the earlier results of the paper, we state in §5 a theorem giving a lower bound on the degrees of certain characters of a class of groups having a Frobenius section.

Most of the notation and terminology are standard. All groups discussed are assumed to be finite, and all characters are over the field of complex numbers. By a <u>generalized</u> <u>character</u> we mean a linear combination of irreducible characters with coefficients in the ring $\mathbb{Z}$ of integers. We reserve the term 'character' for linear combinations of irreducible characters with positive integer coefficients. The inner product of class functions has the usual meaning. By the <u>weight</u> of a generalized character α we mean (α,α), that is, $\|\alpha\|^2$. By the kernel of a class function α of $\mathfrak{G}$ we mean the intersection of the kernels of all irreducible representations for whose characters χ we have $(\alpha,\chi) \neq 0$.

If $\mathfrak{G}$ is a group then $Z(\mathfrak{G})$ denotes the center of $\mathfrak{G}$. If $\mathfrak{T}$ is a set then $|\mathfrak{T}|$ denotes the number of elements in $\mathfrak{T}$. If $\mathfrak{T}$ is a subset of $\mathfrak{G}$ then $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{T})$ and $C_{\mathfrak{G}}(\mathfrak{T})$ denote the normalizer and centralizer of $\underset{\sim}{T}$ in $\mathfrak{G}$, respectively, and $\underset{\sim}{T}^{\#}$ denotes $\underset{\sim}{T} - \{1\}$. We call $\underset{\sim}{T}$ a <u>trivial intersection set</u> in $\mathfrak{G}$ if every pair of conjugates of $\underset{\sim}{T}$ has its intersection contained in $\langle 1 \rangle$.

Let π denote a set of primes and π' the set of all primes not in π. Any integer divisible only by primes in π is a π-<u>number</u>. $\mathfrak{L}$ is a π-<u>group</u> if $|\mathfrak{L}|$ is a π-number. G is a π-<u>element</u> of a group $\mathfrak{G}$ if the group $\langle G \rangle$ generated by G is a π-group. Any element G can be represented uniquely as a product $G = G_{\pi} \cdot G_{\pi'} = G_{\pi'} \cdot G_{\pi}$ of commuting π and π'-elements. G_{π} is the π-<u>part</u> of G. If $\mathfrak{R}$ is a subset of a group $\mathfrak{G}$ then $\mathfrak{R}_{\pi}$ denotes the set of all π-elements in $\mathfrak{R}$.

A subgroup $\mathfrak{L}$ of $\mathfrak{G}$ is a <u>Hall subgroup</u> if $(\mathfrak{G}:\mathfrak{L})$ and $|\mathfrak{L}|$ are relatively prime. $\mathfrak{L}$ is a π-<u>Hall subgroup</u> if it is a π-group and $(\mathfrak{G}:\mathfrak{L})$ is a π'-number.

Let $\mathfrak{G}$ be a finite group, $\mathfrak{L}$ a subgroup and $\mathfrak{L}_o$ a normal subgroup of $\mathfrak{L}$. A normal subgroup $\mathfrak{G}_o$ of $\mathfrak{G}$ is a <u>normal complement of</u> $\mathfrak{L}$ <u>over</u> $\mathfrak{L}_o$, if

$$\mathfrak{G} = \mathfrak{G}_o\mathfrak{L}, \quad \mathfrak{L}_o = \mathfrak{G}_o \cap \mathfrak{L}$$

and hence $\mathfrak{G}/\mathfrak{G}_o \cong \mathfrak{L}/\mathfrak{L}_o$.

2. THE OPERATOR σ AND THE EXISTENCE OF NORMAL COMPLEMENTS

Here we study the existence and properties of a lifting operator σ. Hypothesis 2.1 is a generalization of Dade's assumptions [5]. Theorem 2.1 gives conditions under which σ preserves inner products and gives analogues of the Frobenius Reciprocity Theorem. Theorem 2.2 gives conditions under which α^{σ} is a generalized character. These results are generalizations

$$= \frac{(\mathcal{G}:C_{\mathcal{G}}(L))(\mathcal{J}(L):\mathcal{K}(L))}{|\mathcal{G}|} \Sigma_{K\in\mathcal{K}(L)}\Theta(LK) = \frac{|\mathcal{J}(L)|}{|C_{\mathfrak{L}}(L)||\mathcal{J}(L)|} \Sigma_{K\in\mathcal{K}(L)}\Theta(LK)$$

$$= \frac{(\mathfrak{L}:C_{\mathfrak{L}}(L))}{|\mathfrak{L}|} \Sigma_{K\in\mathcal{K}(L)}\Theta(LK) = \frac{1}{|\mathfrak{L}|} \Sigma_{\delta(L)}\Theta(G).$$

Equation (2.4) can be obtained by letting $\Theta = 1$, and (2.4a) follows from (2.1), (2.4) and (2.2).

If α is a complex-valued class function on $\mathfrak{L}$ such that $\alpha(L) = \alpha(L_\pi)$ for $L\in\hat{\mathfrak{L}}$ then define α^σ by

$$(2.6) \qquad \alpha^\sigma(G) = \begin{cases} 0 & G\notin\varphi(\hat{\mathfrak{L}}) \\ \alpha(L) & G\in\varphi(L), L\in\hat{\mathfrak{L}}_\pi. \end{cases}$$

Equation (2.1) implies that α^σ is a well-defined class function of $\mathcal{G}$ and (2.2) implies that $\alpha^\sigma|\mathfrak{L} = \alpha|\hat{\mathfrak{L}}$. Clearly the domain of σ is a complex vector space and σ is a linear transformation. It is clear from (2.6) that (2.5) can be applied to α^σ for all $L\in\hat{\mathfrak{L}}_\pi$.

In particular, for any class function α of $\mathfrak{L}$ having $\mathcal{K}(L)$ in its kernel for every $L\in\hat{\mathfrak{L}}_\pi$, α^σ is defined.

THEOREM 2.1. *Assume* $\mathfrak{L}$ *and* $\hat{\mathfrak{L}}$ *satisfy Hypothesis* 2.1. *Let* Θ *be a class function of* $\mathcal{G}$ *such that*

$$(2.7) \qquad \Sigma_{J\in\mathcal{J}(L)}\Theta(LJ) = (\mathcal{J}(L):\mathcal{K}(L))\Sigma_{K\in\mathcal{K}(L)}\Theta(LK)$$

for all $L\in\hat{\mathfrak{L}}_\pi$. *Let* α_1 *be a class function of* $\mathfrak{L}$ *which vanishes outside* $\hat{\mathfrak{L}}$ *such that* $\alpha_1(L) = \alpha_1(L_\pi)$ *for* $L\in\hat{\mathfrak{L}}$. *Then*

$$(2.8) \qquad (\Theta,\alpha_1^\sigma)_{\mathcal{G}} = (\Theta|\mathfrak{L},\alpha_1)_{\mathfrak{L}}.$$

If α_2 *is a class function of* $\mathfrak{L}$ *such that* $\alpha_2(L) = \alpha_2(L_\pi)$ *for* $L\in\hat{\mathfrak{L}}$ *then*

$$(2.9) \qquad (\alpha_2^\sigma,\alpha_1^\sigma)_{\mathcal{G}} = (\alpha_2,\alpha_1)_{\mathfrak{L}}.$$

If α_3 *is a class function of* $\mathfrak{L}$ *which vanishes on* $\varphi(\hat{\mathfrak{L}})\cap(\mathfrak{L}-\hat{\mathfrak{L}})$ *such that* $\alpha_3(L) = \alpha_3(L_\pi)$ *for* $L\in\mathfrak{L}$ *then*

$$(2.10) \qquad (\alpha_3^\sigma,\alpha_1^\sigma)_{\mathcal{G}} = (\alpha_3,\alpha_1^\sigma|\mathfrak{L})_{\mathfrak{L}}.$$

Proof. Since α_1^σ vanishes outside $\varphi(\hat{\mathfrak{L}})$, (2.1) implies that

$$(\theta,\alpha_1^\sigma)_{\mathfrak{G}} = \frac{1}{|\mathfrak{G}|}\Sigma_{\varphi(\hat{\mathfrak{L}})}\theta\overline{\alpha_1^\sigma}(G) = \frac{1}{|\mathfrak{G}|}\Sigma_{R\in\mathfrak{R},G\in\varphi(R)}\theta\overline{\alpha_1^\sigma}(G)$$

where $\mathfrak{R}$ is a system of representatives of the classes of $\hat{\mathfrak{L}}_\pi$.
Applying (2.5) to θ and using (2.2),

$$(\theta,\alpha_1^\sigma)_{\mathfrak{G}} = \frac{1}{|\mathfrak{L}|}\Sigma_{R\in\mathfrak{R},G\in\delta(R)}\theta\overline{\alpha_1^\sigma}(G) = \frac{1}{|\mathfrak{L}|}\Sigma_{\hat{\mathfrak{L}}}\theta\overline{\alpha_1^\sigma}(G).$$

Since $\theta\overline{\alpha_1^\sigma}|\hat{\mathfrak{L}} = \theta\overline{\alpha_1}|\hat{\mathfrak{L}}$ and since α_1 vanishes outside $\hat{\mathfrak{L}}$,

$$(\theta,\alpha_1^\sigma)_{\mathfrak{G}} = \frac{1}{|\mathfrak{L}|}\Sigma_{\hat{\mathfrak{L}}}\theta\overline{\alpha_1}(G) = (\theta|\mathfrak{L},\alpha_1)_{\mathfrak{L}}.$$

The proofs of the other equations of the theorem are similar.

Let $\mathfrak{R} = \{R_1,\ldots,R_k\}$ denote any non-empty subset of $\hat{\mathfrak{L}}_\pi$. Let

$$\mathfrak{J}(\mathfrak{R}) = \mathfrak{J}(R_1)\cap\cdots\cap\mathfrak{J}(R_k).$$

and let

$$\mathfrak{K}(\mathfrak{R}) = \mathfrak{K}(R_1)\cap\ldots\cap\mathfrak{K}(R_k).$$

Then $\mathfrak{J}(\mathfrak{R})$ is the normal π'-Hall subgroup of $C_{\mathfrak{G}}(\mathfrak{R})$ and $C_{\mathfrak{L}}(\mathfrak{R})\cap\mathfrak{J}(\mathfrak{R}) = \mathfrak{K}(\mathfrak{R})$. Furthermore $\mathfrak{N}_{\mathfrak{L}}(\mathfrak{R})$ normalizes $\mathfrak{J}(\mathfrak{R})$ and

$$\mathfrak{N}_{\mathfrak{L}}(\mathfrak{R})\cap\mathfrak{J}(\mathfrak{R}) = C_{\mathfrak{L}}(\mathfrak{R})\cap\mathfrak{J}(\mathfrak{R}) = \mathfrak{K}(\mathfrak{R}) \subseteq \mathfrak{L}_0.$$

Thus any class function α of $\mathfrak{L}$ which has $\mathfrak{K}(\mathfrak{R})$ in its kernel defines a class function $\alpha_{\mathfrak{R}}$ on $\mathfrak{N}_{\mathfrak{L}}(\mathfrak{R})\cdot\mathfrak{J}(\mathfrak{R})$ by

$$\alpha_{\mathfrak{R}}(NJ) = \alpha(N) \text{ for all } N\in\mathfrak{N}_{\mathfrak{L}}(\mathfrak{R}), J\in\mathfrak{J}(\mathfrak{R}).$$

Assume α has $\mathfrak{K}(L)$ in its kernel for every $L\in\hat{\mathfrak{L}}_\pi$. Then $\alpha(M) = \alpha(M_\pi)$ for all $M\in\hat{\mathfrak{L}}$ and we can define α^σ and $\alpha_{\mathfrak{R}}$. In particular any class function α of $\mathfrak{L}/\mathfrak{L}_0$ satisfies the conditions necessary for defining α^σ and $\alpha_{\mathfrak{R}}$.

THEOREM 2.2. <u>Assume $\mathfrak{L}$ and $\hat{\mathfrak{L}}$ satisfy Hypothesis 2.1. If α is a generalized character of $\mathfrak{L}$ which has $\mathfrak{K}(L)$ in its kernel for all $L\in\hat{\mathfrak{L}}_\pi$ and which vanishes outside $\hat{\mathfrak{L}}$ then α^σ is a generalized character of $\mathfrak{G}$.</u>

Proof. According to a theorem of Brauer (See [3] and the references there or [4, (40.8)].) it is sufficient to show that $\alpha^\sigma|\mathfrak{E}$ is a generalized character of $\mathfrak{E}$ for all elementary subgroups $\mathfrak{E}$ of $\mathfrak{G}$. Let $\mathfrak{E}$ be an elementary group. Then $\mathfrak{E} = \mathfrak{E}_0 \times \mathfrak{E}_1$ is the direct product of $\mathfrak{E}_0$, a π-group, and $\mathfrak{E}_1$, a π'-group. Let ρ be the projection of $\mathfrak{E}$ onto $\mathfrak{E}_0$. Then $E\in\mathfrak{E}$ has the π-factor $E_\pi = \rho(E)$ and it is clear from the definition of α^σ that $\alpha^\sigma(E) = \alpha^\sigma(\rho(E))$, that is, $\alpha^\sigma|\mathfrak{E} = \alpha^\sigma\rho$. It will, therefore, suffice to prove that $\alpha^\sigma|\mathfrak{E}_0$ is a generalized character of $\mathfrak{E}_0$.

Suppose $\mathfrak{E}$ is an elementary π-group. We will show that $\alpha^\sigma|\mathfrak{E} = \beta|\mathfrak{E}$ where

$$\beta = -\Sigma_{\mathfrak{R}\neq\emptyset,\mathfrak{R}\subseteq\hat{\mathfrak{L}}_\pi} \frac{(-1)^{|\mathfrak{R}|}\alpha^*_{\mathfrak{R}}}{(\mathfrak{L}:\mathfrak{N}_{\mathfrak{L}}(\mathfrak{R}))}.$$

If $G\in\mathfrak{E}$ then clearly

$$\beta(G) = -\Sigma_{\mathfrak{R},Y} \frac{(-1)^{|\mathfrak{R}|}\alpha_{\mathfrak{R}}(G^Y)}{(\mathfrak{L}:\mathfrak{N}_{\mathfrak{L}}(\mathfrak{R}))\,|\mathfrak{N}_{\mathfrak{L}}(\mathfrak{R})\mathfrak{J}(\mathfrak{R})|}$$

summed over all $\mathfrak{R}$, Y such that $\mathfrak{R}$ is a non-empty subset of $\hat{\mathfrak{L}}_\pi$, $Y\in\mathfrak{G}$ and $G^Y\in\mathfrak{N}_{\mathfrak{L}}(\mathfrak{R})\cdot\mathfrak{J}(\mathfrak{R})$. The denominators are just $|\mathfrak{L}|\cdot|\mathfrak{J}(\mathfrak{R})|/|\mathfrak{K}(\mathfrak{R})|$. If $N\in\mathfrak{N}_{\mathfrak{L}}(\mathfrak{R})$ is one of the $|\mathfrak{K}(\mathfrak{R})|$ elements satisfying $G^Y\in N\mathfrak{J}(\mathfrak{R})$ then $\alpha_{\mathfrak{R}}(G^Y) = \alpha(N)$. This vanishes unless $N\in\hat{\mathfrak{L}}$. Therefore

$$\beta(G) = -\frac{1}{|\mathfrak{L}|}\,\Sigma_{\mathfrak{R},Y,N} \frac{(-1)^{|\mathfrak{R}|}\alpha(N)}{|\mathfrak{J}(\mathfrak{R})|}\;\frac{|\mathfrak{K}(\mathfrak{R})|}{|\mathfrak{K}(\mathfrak{R})|}$$

where $\mathfrak{R}\neq\emptyset$, $\mathfrak{R}\subseteq\hat{\mathfrak{L}}_\pi$, $Y\in\mathfrak{G}$, $N\in\mathfrak{N}_{\mathfrak{L}}(\mathfrak{R})\cap\hat{\mathfrak{L}}$, $G^Y\in N\mathfrak{J}(\mathfrak{R})$.

By a lemma of Dade [5, p. 595]

$$\begin{aligned} N_\pi\cdot N_{\pi'}\mathfrak{J}(\mathfrak{R}) &\subseteq N_\pi\langle N_{\pi'}\rangle\mathfrak{J}(\mathfrak{R}) \\ &= [N_\pi C_{\langle N_{\pi'}\rangle\mathfrak{J}(\mathfrak{R})}(N_\pi)]^{\langle N_{\pi'}\rangle\mathfrak{J}(\mathfrak{R})} \\ &= [N_\pi\langle N_{\pi'}\rangle C_{\mathfrak{J}(\mathfrak{R})}(N_\pi)]^{\langle N_{\pi'}\rangle\mathfrak{J}(\mathfrak{R})} \\ &= [N_\pi\langle N_{\pi'}\rangle C_{\mathfrak{J}(\mathfrak{R})}(N_\pi)]^{\mathfrak{J}(\mathfrak{R})}\;. \end{aligned}$$

Thus if $G^Y\in N\mathfrak{J}(\mathfrak{R})$ then

$$G^Y \in [N_\pi < N_{\pi'} > C_{\mathfrak{I}(\mathfrak{R})}(N_\pi)]^{\mathfrak{I}(\mathfrak{R})}$$

or

$$G^Y \in [N_\pi]^{\mathfrak{I}(\mathfrak{R})}$$

since G is a π-element. But then

$$G^Y \in [N_\pi C_{\mathfrak{I}(\mathfrak{R})}(N_\pi)]^{\mathfrak{I}(\mathfrak{R})} = N_\pi \mathfrak{I}(\mathfrak{R})$$

and

$$N\mathfrak{I}(\mathfrak{R}) = N_\pi \mathfrak{I}(\mathfrak{R}).$$

Therefore $G^Y \in N\mathfrak{I}(\mathfrak{R})$ if and only if $G^Y \in N_\pi^{\mathfrak{I}(\mathfrak{R})}$ and $N_{\pi'} \in \mathfrak{I}(\mathfrak{R})$.

There are $|\mathfrak{I}(\mathfrak{R})|/|C_{\mathfrak{I}(\mathfrak{R})}(N_\pi)|$ elements in $N_\pi^{\mathfrak{I}(\mathfrak{R})}$ and $|C_{\mathfrak{G}}(G)|$ Y's give the same conjugate of G. Therefore, there are $|C_{\mathfrak{G}}(G)|\,|\mathfrak{I}(\mathfrak{R})|/|C_{\mathfrak{I}(\mathfrak{R})}(N_\pi)|$ Y's such that $G^Y \in N\mathfrak{I}(\mathfrak{R})$. Furthermore since $N_\pi \in \hat{\mathfrak{L}}_\pi$

$$C_{\mathfrak{I}(\mathfrak{R})}(N_\pi) = \mathfrak{I}(\mathfrak{R}) \cap C_{\mathfrak{G}}(N_\pi) = \mathfrak{I}(\mathfrak{R} \cup \{N_\pi\}).$$

Thus

$$\beta(G) = -\frac{1}{|\mathfrak{L}|}\Sigma_{\mathfrak{R},N}\frac{(-1)^{|\mathfrak{R}|}\alpha(N)\,|C_{\mathfrak{G}}(G)|\cdot|\mathfrak{I}(\mathfrak{R})|}{|\mathfrak{I}(\mathfrak{R} \cup \{N_\pi\})|\cdot|\mathfrak{I}(\mathfrak{R})|}$$

summed over all $\mathfrak{R} \neq \emptyset$, $\mathfrak{R} \subseteq \hat{\mathfrak{L}}_\pi$, $N \in \mathcal{N}_{\mathfrak{L}}(\mathfrak{R}) \cap \hat{\mathfrak{L}}$ such that G is conjugate to N_π in $\mathfrak{G}$ and $N_{\pi'} \in \mathfrak{I}(\mathfrak{R})$.

Clearly if $G \notin \varphi(\hat{\mathfrak{L}})$ then $\beta(G) = 0$. If $G \in \varphi(\hat{\mathfrak{L}})$ then since G is a π-element we may assume $G \in \hat{\mathfrak{L}}$. Then $\alpha(N) = \alpha(G)$ and $|C_{\mathfrak{G}}(G)| = |C_{\mathfrak{G}}(N_\pi)|$. Now hold N fixed and vary $\mathfrak{R}$. If $N_\pi \notin \mathfrak{R}$ and $\mathfrak{R}$ occurs in the sum then $\mathfrak{R} \cup \{N_\pi\}$ also occurs in the sum and it is clear that the two terms cancel. Therefore

$$\beta(G) = \frac{-\alpha(G)}{|\mathfrak{L}|}\Sigma_N(-1)\frac{|C_{\mathfrak{G}}(N_\pi)|}{|\mathfrak{I}(N_\pi)|} = \Sigma_N \frac{\alpha(G)}{(\mathfrak{L}:C_{\mathfrak{L}}(N_\pi))\,|\mathfrak{K}(N_\pi)|}$$

where the sum is over all N such that $N \in \hat{\mathfrak{L}}$ and G is conjugate to N_π in $\mathfrak{G}$. There are $|\mathfrak{K}(N_\pi)|\cdot(\mathfrak{L}:C_{\mathfrak{L}}(N_\pi))$ such elements N. Thus

$$\beta(G) = \alpha(G).$$

To complete the proof of the theorem it is sufficient to note that

$$\beta = -\Sigma_{\mathfrak{R}}(-1)^{|\mathfrak{R}|}\alpha^*_{\mathfrak{R}}$$

where $\mathfrak{R}$ ranges over a system of representatives of the collection of conjugage classes of non-empty subsets of $\hat{\mathfrak{L}}_\pi$ in $\mathfrak{L}$.

THEOREM 2.3. Let $\mathfrak{G}$ be a finite group, $\mathfrak{L}$ a subgroup of $\mathfrak{G}$ and $\mathfrak{L}_o$ a normal subgroup of $\mathfrak{L}$. Suppose there is a set π of primes such that $(\mathfrak{L}:\mathfrak{L}_o)$ is a π-number and such that:

1. Whenever two π-elements of $\mathfrak{L} - \mathfrak{L}_o$ are conjugate in $\mathfrak{G}$, they are already conjugate in $\mathfrak{L}$.

2. If $L \in \mathfrak{L} - \mathfrak{L}_o$ is a π-element then $C_{\mathfrak{G}}(L) = \mathfrak{J}(L)C_{\mathfrak{L}}(L)$ where $\mathfrak{J}(L)$ is a normal π'-Hall subgroup of $C_{\mathfrak{G}}(L)$ and $C_{\mathfrak{L}}(L) \cap \mathfrak{J}(L) = \mathfrak{K}(L) \subseteq \mathfrak{L}_o$.

Then there exists a unique normal complement $\mathfrak{G}_o$ of $\mathfrak{L}$ over $\mathfrak{L}_o$, and $\mathfrak{G}_o = \mathfrak{G} - \varphi(\mathfrak{L} - \mathfrak{L}_o)$.

Proof. Clearly $\mathfrak{L}$ and $\hat{\mathfrak{L}} = \mathfrak{L} - \mathfrak{L}_o$ satisfy Hypothesis 2.1. Let θ be a non-principal irreducible character of $\mathfrak{L}/\mathfrak{L}_o$ with degree d. Then by Theorem 2.2 and (2.8) $(\theta - d)^\sigma$ is a generalized character of $\mathfrak{G}$ and $\|(\theta - d)^\sigma\|^2 = 1 + d^2$. (2.7) implies that $((\theta - d)^\sigma, 1) = -d$ and hence $(\theta - d)^\sigma = \chi - d$ where χ is an irreducible character of $\mathfrak{G}$. The intersection of the kernels of all the characters χ as θ ranges over all non-principal irreducible characters of $\mathfrak{L}/\mathfrak{L}_o$ is $\mathfrak{G} - \varphi(\hat{\mathfrak{L}})$. Thus $\mathfrak{G} - \varphi(\hat{\mathfrak{L}})$ is a normal subgroup of $\mathfrak{G}$ and $[\mathfrak{G} - \varphi(\hat{\mathfrak{L}})] \cap \mathfrak{L} = \mathfrak{L}_o$. By (2.4a)

$$|\mathfrak{G} - \varphi(\hat{\mathfrak{L}})| = (\mathfrak{G}:\mathfrak{L})|\mathfrak{L}| - (\mathfrak{G}:\mathfrak{L})|\hat{\mathfrak{L}}| = (\mathfrak{G}:\mathfrak{L})|\mathfrak{L}_o|.$$

Therefore $\mathfrak{G} = (\mathfrak{G} - \varphi(\hat{\mathfrak{L}}))\mathfrak{L}$, and $\mathfrak{G} - \varphi(\hat{\mathfrak{L}})$ is a normal complement of $\mathfrak{L}$ over $\mathfrak{L}_o$ in $\mathfrak{G}$.

Let $\mathfrak{G}_o$ be an arbitrary normal complement of $\mathfrak{L}$ over $\mathfrak{L}_o$ in $\mathfrak{G}$. Then $\mathfrak{G}/\mathfrak{G}_o \cong \mathfrak{L}/\mathfrak{L}_o$ is a π-group and all π'-elements of

$\mathfrak{G}$ belong to $\mathfrak{G}_o$. Suppose $G \in \mathfrak{G}_o \cap \varphi(\hat{\mathfrak{L}})$. Then $G_\pi \in \mathfrak{G}_o$ and $G_\pi{}^Y \in \mathfrak{L} - \mathfrak{L}_o$ for some $Y \in \mathfrak{G}$. But $G_\pi{}^Y \in \mathfrak{L} \cap \mathfrak{G}_o = \mathfrak{L}_o$ since $\mathfrak{G}_o$ is normal in $\mathfrak{G}$. This is a contradiction. Therefore $\mathfrak{G}_o \subseteq \mathfrak{G} - \varphi(\hat{\mathfrak{L}})$. Since $|\mathfrak{G}_o| = |\mathfrak{G}|/(\mathfrak{L}:\mathfrak{L}_o) = |\mathfrak{G} - \varphi(\hat{\mathfrak{L}})|$, $\mathfrak{L}$ has a unique normal complement over $\mathfrak{L}_o$ in $\mathfrak{G}$.

REMARKS. Notice we have not assumed $(\mathfrak{L}:\mathfrak{L}_o)$ and $(\mathfrak{G}:\mathfrak{L})$ are relatively prime. Theorem 2.3 is related to a theorem of Brauer [2, Theorem 1]. It generalizes the sufficiency part of a theorem of Suzuki [12, Theorem 1]. (The last paragraph of his proof provides a deduction of his theorem from ours.) And two theorems of Wielandt [13, Satze 1 and 2] can be deduced from it. These deductions can be made in the same way that Brauer makes them [2, p. 79]. The proof of theorem 2.3 is similar to Suzuki's proof.

3. COHERENT SETS OF CHARACTERS

Let $\mathfrak{G}$ be a finite group, let $\mathfrak{L}$ be a subgroup of $\mathfrak{G}$, let $\mathfrak{S}$ be a set of characters of $\mathfrak{L}$, and let $\mathfrak{I} = \mathfrak{I}(\mathfrak{S})$ be the $\mathfrak{Z}$-module of generalized characters of $\mathfrak{L}$ generated by $\mathfrak{S}$. Let $\mathfrak{I}_o = \mathfrak{I}_o(\mathfrak{S})$ be the submodule of $\mathfrak{I}$ consisting of the members of degree zero, and suppose there is a linear isometry σ of $\mathfrak{I}_o$ into the $\mathfrak{Z}$-module $\mathfrak{M}_\mathfrak{G}$ of all generalized characters of $\mathfrak{G}$. We call $\mathfrak{S}$ σ-<u>coherent</u> if $\mathfrak{I}_o \neq \{0\}$ and there exists an extension σ^* of σ which is a linear isometry of $\mathfrak{I}$ into $\mathfrak{M}_\mathfrak{G}$. This definition of coherence is the same as that of Feit [8, p. 181]. We let $\mathfrak{M}_{\mathfrak{G}o}$ denote the submodule of $\mathfrak{M}_\mathfrak{G}$ consisting of the members of degree zero.

In this section we give conditions which imply the coherence of certain sets of characters or which imply the uniqueness of σ^*. Proposition 3.1 is a simple well-known fact. Proposition 3.2 is much the same as a result of Feit [8, p. 182, ¶¶. 7-10] and its proof is also the same as the proof of the existence part of [9, Lemma 10.1], of which it is a generalization. Proposition 3.3 is a generalization of the uniqueness part of [8, (31.2)] and its proof is much the same. For the sake of unity we give the proof here. Lemma 3.1 and Proposition 3.4 are related, for example, to [10, Corollary 2.2.].

Theorem 3.1, the main result in this section, together with Propositions 3.3 and 3.5, generalize a major theorem of Feit and Thompson [9, Theorem 10.1] and a similar theorem of Feit [8, (31.2)]. Our definition of subcoherence is essentially the same as that of Feit and Thompson [9] except that we have separated it from the concept of a tamely embedded subset of a group and from our corresponding Hypothesis 2.1, and we have weakened it in one other respect.

PROPOSITION 3.1. *Let* $\mathfrak{L}$ *be a subgroup of* $\mathfrak{G}$, *and let* $\underset{\sim}{S} = \{\lambda_i | 1 \leq i \leq n\}$ *be a set of characters of* $\mathfrak{L}$. *If there exist integers* χ_i *for* $i = 1,\ldots,n$ *such that* $\chi_1 = 1$ *and* $\lambda_i(1) = \chi_i\lambda_i(1)$ *for all* i, *then* $\underset{\sim}{I}_o(\underset{\sim}{S})$ *is generated by* $\{\lambda_i - \chi_i\lambda_1 | 1 \leq i \leq n\}$.

Proof. If $\Sigma a_i\lambda_i \in \underset{\sim}{I}_o(\underset{\sim}{S})$ then $\Sigma a_i\chi_i = 0$. Therefore $\Sigma a_i\lambda_i = \Sigma a_i(\lambda_i - \chi_i\lambda_1)$.

PROPOSITION 3.2. *Suppose* $\mathfrak{L}$ *is a subgroup of a group* $\mathfrak{G}$ *and* $\underset{\sim}{S}$ *is a set of irreducible characters of* $\mathfrak{L}$ *all of the same degree. Suppose there is a linear isometry* σ *of* $\underset{\sim}{I}_o(\underset{\sim}{S})$ *into* $\mathfrak{M}_{\mathfrak{G}o}$. *Then* $\underset{\sim}{S}$ *is* σ-*coherent*.

REMARK. If $\mathfrak{L}$ and $\hat{\mathfrak{L}}$ satisfy Hypothesis 2.1 and $1 \notin \hat{\mathfrak{L}}$, and if $\underset{\sim}{S} = \{\lambda_i | 1 \leq i \leq n\}$ with $n \geq 2$ is a set of irreducible characters of $\mathfrak{L}/\mathfrak{L}_o$ such that $\lambda_i(L) = \lambda_1(L)$ for all $L \in \mathfrak{L} - \hat{\mathfrak{L}}$ and for $1 < i \leq n$, then by (2.8) and Theorem 2.2 the mapping σ

of §2 is a linear isometry of $I_o(S)$ into $M_{\mathcal{G}}$, and by (2.6) $\mu^\sigma(1) = 0$ for all $\mu \in I_o$. Hence by Proposition 3.2 S is σ-coherent.

PROPOSITION 3.3. *Suppose $\mathfrak{L}$ is a subgroup of $\mathcal{G}$ and $S = \{\lambda_i \mid 1 \le i \le n\}$ with $n > 1$ is a set of orthogonal characters of $\mathfrak{L}$. Suppose there is an isometry σ from $I_o(S)$ into $M_{\mathcal{G}}$. Suppose λ_1 and λ_2 are irreducible and $(\lambda_1(1)\lambda_2 - \lambda_2(1)\lambda_1)^\sigma(1) = 0$. Then either there is at most one extension σ^* of σ to $I(S)$ or $n = 2$, $\lambda_1(1) = \lambda_2(1)$, and there are exactly two extensions σ^* and $\sigma^\#$ of σ to $I(S)$; namely $\lambda_i^{\sigma^\#} = -\lambda_{3-i}^{\sigma^*}$ for $i = 1,2$.*

Proof. Let σ^* and $\sigma^\#$ be extensions of σ to $I(S)$. Denote $\lambda_i(1)$ by χ_i. Then for all i

$$(3.1)\quad (\chi_1\lambda_i - \chi_i\lambda_1)^\sigma = \chi_1\lambda_i^{\sigma^*} - \chi_i\lambda_1^{\sigma^*} = \chi_1\lambda_i^{\sigma^\#} - \chi_i\lambda_1^{\sigma^\#}.$$

Clearly our conclusion follows if $n = 2$, since (3.1) must be the difference of two characters. Suppose $n > 2$. Then either $\lambda_i^{\sigma^\#} = \lambda_i^{\sigma^*}$ for $i = 1,2$ or $\lambda_i^{\sigma^\#} = -\lambda_{3-i}^{\sigma^*}$ for $i = 1,2$. In the former case our conclusion follows. In the latter case we have

$$\chi_1^2\, \|\lambda_3^{\sigma^\#}\|^2 = \|\chi_1\lambda_3^{\sigma^*} - \chi_3\lambda_1^{\sigma^*} - \chi_3\lambda_2^{\sigma^*}\|^2 = \chi_1^2\|\lambda_3\|^2 + \chi_3^2\|\lambda_1\|^2 + \chi_3^2\|\lambda_2\|^2,$$

a contradiction. It follows that $\sigma^* = \sigma^\#$.

LEMMA 3.1. *Suppose $\mathfrak{L}$ is a subgroup of $\mathcal{G}$ and $S = \{\kappa_i \mid 1 \le i \le m\}$ and $T = \{\lambda_j \mid 1 \le j \le n\}$ are coherent sets of orthogonal characters of $\mathfrak{L}$ with respect to isometries σ and τ, respectively. Assume $\mu^\sigma(1) = 0$ and $\nu^\tau(1) = 0$ for all $\mu \in I_o(S)$ and $\nu \in I_o(T)$. If $I_o(S)^\sigma$ and $I_o(T)^\tau$ are orthogonal and $(\kappa_i^{\sigma^*}, \lambda_j^{\tau^*}) = 0$ whenever either κ_i or λ_j is reducible, then S^{σ^*} and T^{τ^*} are orthogonal.*

Proof. Denote $\kappa_i(1)$ and $\lambda_j(1)$ by k_i and ℓ_j, respectively, for all i and j. Suppose for some i and j that κ_i and λ_j are irreducible and $(\kappa_i^{\sigma*}, \lambda_j^{\tau*}) = \varepsilon = \pm 1$. Choose $\kappa_u \in \mathfrak{S}$ and $\lambda_v \in \mathfrak{T}$ with $u \neq i$ and $v \neq j$. Then

$$0 = ((k_i \kappa_u - k_u \kappa_i)^{\sigma}, (\ell_j \lambda_v - \ell_v \lambda_j)^{\tau})$$
$$= k_i \ell_j (\kappa_u^{\sigma*}, \lambda_v^{\tau*}) + \varepsilon\, k_u \ell_v .$$

Then κ_u and λ_v are irreducible, $(\kappa_u^{\sigma*}, \lambda_v^{\tau*}) = -\varepsilon$, and $k_i \ell_j = k_u \ell_v$. Since $\mu^{\sigma}(1) = 0$ for all $\mu \in I_o(S)$,

$$0 = k_i \kappa_u^{\sigma*}(1) - k_u \kappa_i^{\sigma*}(1) = -\varepsilon k_i \lambda_v^{\tau*}(1) - \varepsilon\, k_u \lambda_j^{\tau*}(1).$$

But

$$0 = \ell_j \lambda_v^{\tau*}(1) - \ell_v \lambda_j^{\tau*}(1).$$

These two equations imply $\lambda_j^{\tau*}(1) = 0$ because the k_i and ℓ_j are positive integers. This contradicts the fact that $\|\lambda_j^{\tau*}\| = 1$.

As an easy corollary of this lemma we have:

PROPOSITION 3.4. _Suppose_ $\mathfrak{L}$ _is a subgroup of_ $\mathfrak{G}$ _and_ S_i _for_ $1 \leq i \leq k$ _are disjoint_ σ_i_-coherent sets of irreducible characters of_ $\mathfrak{L}$. _Suppose_ $\mu_i^{\sigma_i}(1) = 0$ _for all_ $\mu_i \in I_o(S_i)$. _Assume that_ $I_o(S_h)^{\sigma_h}$ _and_ $I_o(S_i)^{\sigma_i}$ _are orthogonal whenever_ $h \neq i$. _Let_ $S = \bigcup_{i=1}^{k} S_i$. _Then there is an isometry_ σ^* _from_ $I(S)$ _into_ $M_{\mathfrak{G}}$ _such that_ $\sigma^* | I_o(S_i) = \sigma_i^*$ _for each_ i.

REMARKS. If in Proposition 3.4 S_i consists of characters of the same degree for some i, then the σ_i-coherence of S_i need not be assumed since it is implied by Proposition 3.2. Proposition 3.4 does not conclude exactly that S is coherent, because $I_o(S)$ was not considered.

The following hypotheses and Theorem 3.1 provide sufficient conditions for the coherence of the union S of certain orthogonal sets of characters, but assuming an isometry σ on $\underset{\sim}{I}_o(\underset{\sim}{S})$ is given. However, it is not explicitly assumed that the sets are coherent or that they consist of irreducible characters, and therefore Lemma 3.1 is not applicable in all cases. But the assumption that certain subsets which don't consist of irreducible characters are subcoherent in the union gives a substitute for this lemma.

If $\underset{\sim}{T}$ is a set of characters of $\mathfrak{G}$, let $\underset{\sim}{R}(\underset{\sim}{T})$ denote the module of class functions of $\mathfrak{G}$ generated by members of $\underset{\sim}{T}$ over the rational numbers. If $\mathfrak{L}$ is a subgroup of the group $\mathfrak{G}$, let $\underset{\sim}{S} = \{\lambda_i | 1 \leq i \leq n\}$ be a set of pairwise orthogonal characters of $\mathfrak{L}$. Suppose there is an isometry σ from $\underset{\sim}{I}_o$ into $\underset{\sim}{M}_{\mathfrak{G}}$. If $\underset{\sim}{S}_1 \subseteq \underset{\sim}{S}$, let $x(\underset{\sim}{S}_1)$ denote the smallest weight of any character in $\underset{\sim}{S}_1$ with minimum degree. If $\underset{\sim}{S}_1$ and $\underset{\sim}{T}$ are σ-coherent subsets of $\underset{\sim}{S}$ with extensions σ_1 and σ_2 respectively, define

$$\underset{\sim}{A}(\underset{\sim}{S}_1,\sigma_1;\underset{\sim}{T},\sigma_2) = \{\alpha|\ \text{i)}\ \alpha|\underset{\sim}{I}_o \ \text{ and }\ \text{ii)}\ \alpha^\sigma = \Delta_1 + \Delta_2,$$

where

a) $\Delta_2 \epsilon \underset{\sim}{R}(\underset{\sim}{T}^{\sigma_2})$,

b) Δ_1 is orthogonal to $\underset{\sim}{R}(\underset{\sim}{T}^{\sigma_2})$,

c) Δ_1 is not orthogonal to $\underset{\sim}{I}_o(\underset{\sim}{S}_1)^\sigma$,

d) $\|\Delta_1\|^2 \leq x(\underset{\sim}{S}_1)\}$.

DEFINITION. <u>Let</u> $\underset{\sim}{S}_1$ <u>be a</u> σ-<u>coherent subset of</u> $\underset{\sim}{S}$ <u>and let</u> σ^* <u>be an extension of</u> σ <u>to</u> $\underset{\sim}{S}_1$. <u>The pair</u> $(\underset{\sim}{S}_1,\sigma^*)$ <u>is subcoherent in</u> $\underset{\sim}{S}$ <u>if the following conditions are satisfied</u>: <u>If</u> $\underset{\sim}{T}$ <u>is a</u> σ-<u>coherent subset of</u> $\underset{\sim}{S}$ <u>which is orthogonal to</u> $\underset{\sim}{S}_1$ <u>and if</u> σ_1 <u>and</u> σ_2 <u>are extensions of</u> σ <u>to</u> $\underset{\sim}{S}_1$ <u>and</u> $\underset{\sim}{T}$ <u>respectively</u>, <u>then</u>:

i) $S_1^{\sigma_1}$ *is orthogonal to* T^{σ_2}.

ii) *If* $\alpha \in A(S_1, \sigma_1; T, \sigma_2)$ *then* α^{σ} *is a sum of two generalized characters, one of which is orthogonal to* $S_1^{\sigma^*}$ *and the other is in* $\pm S_1^{\sigma^*}$.

We first obtain a uniqueness statement in terms of subcoherence.

PROPOSITION 3.5. *Suppose* $\mathfrak{L}$ *is a subgroup of* $\mathfrak{G}$ *and* S *is a set of orthogonal characters of* $\mathfrak{L}$. *Suppose there is an isometry* σ *from* $I_o(S)$ *into* $M_{\mathfrak{G}}$ *and that* S *is* σ-*coherent with fixed extension* σ^* *of* σ *to* $I(S)$. *Assume there is a subset* S_1 *of* S *such that* (S_1, σ_1) *is subcoherent in* S *for some extension* σ_1 *of* σ *to* $I(S_1)$ *and that* $|S - S_1| \geq 2$. *If* $S_2 = S_1 \cup \{\lambda\}$ *for some* $\lambda \in S - S_1$ *then* $\sigma^*|S_2$ *is the only extension of* σ *to* $I(S_2)$. *If there exists* $\lambda_1 \in S - S_1$ *such that* $\lambda_1(1)$ *divides the minimum degree of characters in* S_1, *then either* $\sigma^*|S_1 = \sigma_1$ *or* $S_1 = \{\mu_1, \mu_2\}$ *for some* μ_1, μ_2, *and* $\mu_1(1) = \mu_2(1)$, $\mu_i^{\sigma^*} = -\mu_{3-i}^{\sigma_1}$ *for* $i = 1,2$.

Proof. Let σ_2 be an extension of σ to $I(S_2)$. Then for $\mu_j \in S_1$, $(\mu_j^{\sigma_2}, \lambda^{\sigma_2}) = 0$, and by the definition of subcoherence $(u_j^{\sigma_2}, \lambda^{\sigma^*}) = 0$. But if $m_j = \mu_j(1)$ and $\chi = \lambda(1)$ then $m_j\lambda^{\sigma_2} - \chi\mu_j^{\sigma_2} = m_j\lambda^{\sigma^*} - \chi\mu_j^{\sigma^*}$. Therefore $\|u_j\|^2 = (\mu_j^{\sigma_2}, \mu_j^{\sigma_2}) = (\mu_j^{\sigma_2}, \mu_j^{\sigma^*})$. Hence $\mu_j^{\sigma_2} - \mu_j^{\sigma^*} = 0$ for all $\mu_j \in S_1$, and consequently $\lambda^{\sigma_2} = \lambda^{\sigma^*}$. This proves the first statement.

To prove the second statement, let $S_1 = \{\mu_1, \ldots, \mu_k\}$. Choose the notation so that μ_1 has smallest weight of any character in S_1 of minimum degree and so that $\lambda_1(1) | \mu_1(1)$. Denote $\lambda_1(1)$ and $\mu_j(1)$ by χ_1 and m_j, respectively, and m_1/χ_1 by $\bar{m}_1$. Then

$$(\overline{m}_1\lambda_1 - \mu_1)^{\sigma} = \overline{m}_1\lambda_1^{\sigma*} - \mu_1^{\sigma*}.$$

Also, by the first part of the definition of subcoherence,

$$(-\mu_1^{\sigma*}, (m_1\mu_2 - m_2\mu_1)^{\sigma})$$

$$= (\overline{m}_1\lambda_1^{\sigma*} - \mu_1^{\sigma*}, (m_1\mu_2 - m_2\mu_1)^{\sigma})$$

$$= m_2\|\mu_1\|^2 \neq 0.$$

Therefore $\overline{m}_1\lambda_1 - \mu_1 \in \underset{\sim}{A}(\underset{\sim}{S}_1, \sigma_1; \underset{\sim}{S} - \underset{\sim}{S}_1, \sigma^*)$, so $(\overline{m}_1\lambda_1 - \mu_1)^{\sigma} = \Lambda_1 + \Lambda_2$, where $\Lambda_1 \in \pm\underset{\sim}{S}_1^{\sigma_1}$ and Λ_2 is orthogonal to $\underset{\sim}{S}_1^{\sigma_1}$. Since $\lambda_1^{\sigma*}$ is orthogonal to $\underset{\sim}{S}_1^{\sigma_1}$, it follows that $\Lambda_1 + \mu_1^{\sigma*}$ is orthogonal to $\underset{\sim}{S}_1^{\sigma_1}$, and in particular to Λ_1. Hence

$$(3.2) \qquad (\Lambda_1, \Lambda_1) = -(\Lambda_1, \mu_1^{\sigma*}).$$

Suppose $\|\Lambda_1\| = \|\mu_1\|$. Then it follows from (3.2) that $\Lambda_1 = -\mu_1^{\sigma*}$. But $\Lambda_1 \in \pm\underset{\sim}{S}_1^{\sigma_1}$ and

$$(3.3) \qquad m_1\mu_j^{\sigma_1} - m_j\mu_1^{\sigma_1} = m_1\mu_j^{\sigma*} - m_j\mu_1^{\sigma*}.$$

Hence if $\Lambda_1 = \pm\mu_1^{\sigma_1} = -\mu_1^{\sigma*}$ then $\Lambda_1 = -\mu_1^{\sigma_1}$ and then the conclusion holds. If $\Lambda_1 = \pm\mu_2^{\sigma_1}$ then (3.3) implies $\Lambda_1 = \mu_2^{\sigma_1}$ and $m_1 = m_2$. Also $k = 2$ because otherwise (3.3) implies

$$m_1^2\|\mu_3^{\sigma*}\|^2 = \|m_1\mu_3^{\sigma_1} - m_3\mu_1^{\sigma_1} - m_3\mu_2^{\sigma_1}\|^2$$

$$= m_1^2\|\mu_3\|^2 + m_3^2\|\mu_1\|^2 + m_3^2\|\mu_2\|^2,$$

a contradiction. Again the conclusion holds.

Thus we may assume $\|\Lambda_1\| \neq \|\mu_1\|$. Then we can assume $\Lambda_1 = \pm\mu_2^{\sigma_1}$ and $\|\mu_2\| \neq \|\mu_1\|$. Equation (3.2) implies

$$\|\Lambda_1\|^2 = |(\Lambda_1, \mu_1^{\sigma*})| \leq \|\Lambda_1\|\|\mu_1^{\sigma*}\|.$$

Hence $\|\mu_2\| < \|\mu_1\|$. The original choice of μ_1 implies $m_2 > m_1$. We know that $\mu_1^{\sigma_1}$ is orthogonal to Λ_1, Λ_2, and $\lambda_1^{\sigma*}$. Hence the equations for $(\bar{m}_1, \lambda_1 - \mu_1)^\sigma$ imply $(\mu_1^{\sigma_1}, \mu_1^{\sigma^*}) = 0$. Therefore

$$(m_1\mu_2^{\sigma_1} + m_2\mu_1^{\sigma^*},\ \mu_1^{\sigma_1}) = 0.$$

Equation (3.3) implies

$$(m_1\mu_2^{\sigma^*} + m_2\mu_1^{\sigma_1}, \mu_1^{\sigma_1}) = 0.$$

Therefore

$$\|m_1\mu_2^{\sigma^*} + m_2\mu_1^{\sigma_1}\|^2 = m_1^2\|\mu_2\|^2 - m_2\|\mu_1\|^2,$$

which, because of our earlier inequalities, is negative. This is a contradiction, and the proof is complete.

HYPOTHESIS 3.1.

1. $\mathfrak{L}$ is a subgroup of the finite group $\mathfrak{G}$.

2. For $1 \leq i \leq k$, $\underset{\sim}{S}_i = \{\lambda_{is} | 1 \leq s \leq n_i\}$. The sets $\underset{\sim}{S}_i$ are pairwise disjoint and $\underset{\sim}{S} = \bigcup_{i=1}^{k} \underset{\sim}{S}_i$ consists of pairwise orthogonal characters of $\mathfrak{L}$.

3. For $1 \leq i \leq k$, $1 \leq s \leq n_i$, there exist positive integers χ_{is} such that $\chi_{11} = 1$ and $\lambda_{is}(1) = \chi_{is}\lambda_{11}(1)$ for all λ_{is}. There is a linear isometry σ from $\underset{\sim}{I}_o$ to $\underset{\sim}{M}_{\mathfrak{G}o}$.

4. λ_{11} is irreducible.

5. $\underset{\sim}{S}_1$ is σ-coherent with extension σ_1 of σ to $\underset{\sim}{I}(\underset{\sim}{S}_1)$.

6. For each $\underset{\sim}{S}_m$ either a or b is true.

a. $\underset{\sim}{S}_m$ consists of irreducible characters. For each t with $1 \leq t \leq n_m$

$$\sum_{i=1}^{m-1} \sum_{s=1}^{n_i} \frac{\chi_{is}^2}{\|\lambda_{is}\|^2} \geq 2\chi_{mt} + 1 \quad \text{if} \quad m > 1. \tag{3.4}$$

b. $\underset{\sim}{S}_m$ is σ-coherent with extension σ_m of σ. $\underset{\sim}{S}_m$ is partitioned into sets $\underset{\sim}{S}_{mj}$ such that each $\underset{\sim}{S}_{mj}$ either consists of irreducible characters or $(\underset{\sim}{S}_{mj}, \sigma_{mj})$ is subcoherent in $\underset{\sim}{S}$ for some σ_{mj}. Furthermore

(3.5) $$\Sigma_{i=1}^{m-1} \Sigma_{s=1}^{n_i} \frac{\chi_{is}^2}{\|\lambda_{is}\|^2} > 2\chi_{m1} \quad \text{if} \quad m > 1.$$

THEOREM 3.1. <u>Suppose Hypothesis</u> 3.1 <u>is satisfied</u>. <u>Then</u> $\underset{\sim}{S}$ <u>is</u> σ-<u>coherent, and there is an extension</u> σ^* <u>of</u> σ <u>to</u> $\underset{\sim}{I}(\underset{\sim}{S})$ <u>such that</u> $\sigma^*|\underset{\sim}{S}_1 = \sigma_1$.

<u>Proof</u>. We use induction on k. If $k = 1$ the theorem is true by hypothesis. Assume the theorem is true for $k - 1$. Then there is an extension of σ to a linear isometry σ^* on $\underset{\sim}{I}(\cup_{i=1}^{k-1} \underset{\sim}{S}_i)$ such that $\sigma^*|\underset{\sim}{S}_1 = \sigma_1$.

We may assume $\chi_{k1} \leq \chi_{ks}$ for all s. Choose the notation so that λ_{k1} has minimum weight among the members λ_{ks} of $\underset{\sim}{S}_k$ for which $\chi_{ks} = \chi_{k1}$. If $\underset{\sim}{S}_k$ satisfies 6b let $\underset{\sim}{S}_{k1}$ denote the set $\underset{\sim}{S}_{kj}$ containing λ_{k1}. For $1 \leq s \leq n_k$ define

$$\beta_s = \chi_{ks}\lambda_{11} - \lambda_{ks}.$$

Define the integers a_s by

(3.6) $$(\lambda_{11}{}^{\sigma*}, \beta_s{}^{\sigma}) = \chi_{ks} - a_s \quad \text{for} \quad 1 \leq s \leq n_k.$$

Since $\beta_s \epsilon \underset{\sim}{I}_o$

$$(\lambda_{it}{}^{\sigma*}, \beta_s{}^{\sigma}) = (\chi_{it}\lambda_{11}{}^{\sigma*}, \beta_s{}^{\sigma})$$

(3.7) $$- (\gamma_{it}\lambda_{11}{}^{\sigma*} - \lambda_{it}{}^{\sigma*}, \beta_s{}^{\sigma})$$

$$= -a_s\chi_{it} \text{ for } 1 \leq i \leq k - 1,\ 1 \leq t \leq n_i, (i,t) \neq (1,1).$$

Since λ_{11} is irreducible and σ is an isometry on $\underset{\sim}{I}_o$,

(3.8) $$\|\beta_s{}^{\sigma}\|^2 = \chi_{ks}{}^2 + \|\lambda_{ks}\|^2 \quad \text{for} \quad 1 \leq s \leq n_k.$$

We have

(3.9) $$\beta_s{}^{\sigma} = \chi_{ks}\lambda_{11}{}^{\sigma*} - a_s \Sigma_{i=1}^{k-1} \Sigma_{t=1}^{n_i} \frac{\chi_{it}}{\|\lambda_{it}\|^2} \lambda_{it}{}^{\sigma*} + \Delta_s$$

for some class function Δ_s of $\mathfrak{G}$. It is clear that $(\Delta_s, \lambda_{11}{}^{\sigma*}) = 0$.

If $(i,t) \neq (1,1)$ and $1 \leq i \leq k-1$, $1 \leq t \leq n_i$ then, since the $\lambda_{it}^{\sigma*}$ are pairwise orthogonal, (3.7) implies $(\Delta_s, \lambda_{it}^{\sigma*}) = 0$. Therefore (3.8) and (3.9) yield that

$$(3.10)\quad \chi_{ks}^2 - 2\chi_{ks}a_s + a_s^2 \Sigma_{i=1}^{k-1}, \sum_{t=1}^{n_i} \frac{\chi_{it}^2}{\|\lambda_{it}\|^2} + \|\Delta_s\|^2 = \chi_{ks}^2 + \|\lambda_{ks}\|^2.$$

Now we shall complete the proof in the case that $\underset{\sim}{S}_k$ satisfies 6a. If $a_s \neq 0$ then (3.4) and (3.10) imply

$$\|\lambda_{ks}\|^2 - \|\Delta_s\|^2 \geq 2\chi_{ks}(a_s^2 - a_s) + a_s^2 \geq a_s^2$$

since a_s is an integer. Since λ_{ks} is irreducible this implies $\Delta_s = 0$ if $a_s \neq 0$. Hypothesis 3.1.3 and (3.9) imply

$$0 = \beta_s^{\sigma}(1) = \chi_{ks}\lambda_{11}^{\sigma*}(1) - a_s \Sigma \frac{\chi_{it}}{\|\lambda_{it}\|^2} \lambda_{it}^{\sigma*}(1)$$

$$= [\chi_{ks} - a_s \Sigma \frac{\chi_{it}^2}{\|\lambda_{it}\|^2}]\lambda_{11}^{\sigma*}(1)$$

if $\Delta_s = 0$. Since λ_{11} is irreducible, $\lambda_{11}^{\sigma*}(1) \neq 0$. Hence the preceding equation implies that if $\Delta_s = 0$ then $a_s > 0$ and together with (3.4) it implies $\chi_{ks} > 2a_s\chi_{ks}$, a contradiction. Thus $\Delta_s \neq 0$ and hence $a_s = 0$. Define $\lambda_{ks}^{\sigma*}$ to be $-\Delta_s$. Then $\lambda_{ks}^{\sigma*}$ is orthogonal to $\cup_{i=1}^{k-1} \underset{\sim}{S}_i^{\sigma*}$, (3.8) and (3.9) imply $\|\lambda_{ks}^{\sigma*}\| = 1$, and, since $(\beta_s^{\sigma}, \beta_t^{\sigma}) = \chi_{ks}\chi_{kt}$ for $s \neq t$, the generalized characters $\lambda_{ks}^{\sigma*}$ are orthogonal. Thus the proof is complete in this case.

From now on we assume $\underset{\sim}{S}_k$ satisfies 6b. If $a_1 \neq 0$ then since a_1 is an integer (3.5) and (3.10) imply

$$0 \leq 2\chi_{k1}(a_1^2 - a_1) < \|\lambda_{k1}\|^2 - \|\Delta_1\|^2.$$

Therefore

$$(3.11)\qquad \|\Delta_1\|^2 < \|\lambda_{k1}\|^2 \quad \text{if} \quad a_1 \neq 0.$$

We will show that $a_1 = 0$. First we note that if $\lambda \in \cup_{i=1}^{k-1} S_i$ and $\mu \in S_k$ then $(\lambda^{\sigma^*}, \mu^{\sigma_k}) = 0$. For if either λ or μ is reducible this follows from the definition of subcoherence, and hence by Lemma 3.1 it follows for irreducible λ and μ. Thus $S_k^{\sigma_k}$ is orthogonal to $\cup_{i=1}^{k-1} S_i$. Therefore for $2 \le s \le n_k$,

$$(3.12) \quad (\Delta_1, \chi_{ks}\lambda_{k1}^{\sigma_k} - \chi_{k1}\lambda_{ks}^{\sigma_k}) = (\beta_1^{\sigma}, \chi_{ks}\lambda_{k1}^{\sigma_k} - \chi_{k1}\lambda_{ks}^{\sigma_k}) = -\chi_{ks}\|\lambda_{k1}\|^2.$$

In particular Δ_1 is not orthogonal to $S_{k1}^{\sigma_k}$, and its inner product with each member of $S_{k1}^{\sigma_k}$ is an integer. Hence if S_{k1} consists of irreducible characters then (3.11) implies that $a_1 = 0$.

Suppose (S_{k1}, σ_{k1}) is subcoherent in S. If $a_1 \ne 0$ then by (3.11) $\|\Delta_1\|^2 < \|\lambda_{k1}\|^2$ where $\|\lambda_{k1}\|^2 = x(S_{k1})$. Since, by (3.12), Δ_1 is not orthogonal to $(I_o(S_{k1}))^{\sigma}$, (3.9) and the definition of subcoherent imply

$$(3.13) \qquad \beta_1^{\sigma} = \Lambda_1 + \Lambda_2$$

where $\Lambda_2 \in \pm S_{k1}^{\sigma_{k1}}$ and Λ_1 is orthogonal to $S_{k1}^{\sigma_{k1}}$. Since $S_{k1}^{\sigma_{k1}}$ is orthogonal to $\cup_{i=1}^{k-1} S_i^{\sigma^*}$, (3.9) and (3.13) imply Λ_2 is the perpendicular projection of Δ_1 onto $(S_{k1})^{\sigma_{k1}}$, and hence $\|\Lambda_2\|^2 \le \|\Delta_1\|^2$. Therefore if $a_1 \ne 0$ (3.11) implies $\Lambda_2 = \pm \lambda_{ks}^{\sigma_{k1}}$ for some $s \ne 1$. Hence (3.13) and the analogue for σ_{k1} of (3.12) imply

$$-\chi_{k1}\|\lambda_{ks}\|^2 = -\chi_{ks}\|\lambda_{k1}\|^2,$$

so $\|\lambda_{k1}\|^2 \le \|\Lambda_2\|^2 \le \|\Delta_1\|^2$. Hence (3.11) implies that $a_1 = 0$ in all cases.

Since $a_1 = 0$ (3.6) implies

$$(3.14) \qquad (\lambda_{11}^{\sigma^*}, \beta_1^{\sigma}) = \chi_{k1}.$$

For $1 \le s \le n_k$,

$$\beta_s = \frac{\chi_{ks}}{\chi_{k1}} \beta_1 + (\frac{\chi_{ks}}{\chi_{k1}} \lambda_{k1} - \lambda_{ks}).$$

Therefore (3.14) implies that $(\lambda_{11}^{\sigma*}, \beta_s^{\sigma}) = \chi_{ks}$ for all s, so by (3.6) $a_s = 0$ for all s. Almost exactly as in the case 6a we can verify that σ^* can be extended to $\underset{\sim}{I}(\underset{\sim}{S})$. This completes the proof.

4. COHERENCE IN GROUPS HAVING A NILPOTENT SECTION

Here we apply Theorem 3.1 to groups satisfying Hypothesis 4.2 stated below. This hypothesis generalizes assumptions of Feit and Thompson [9, Hypothesis 11.1] in several ways. We have separated the hypothesis from the concept of a tamely imbedded subset and from our corresponding Hypothesis 2.1. We have replaced their assumption that $|\mathfrak{L}|$ is odd by the weaker assumption 4.2.4. And we have introduced the subgroup $\mathfrak{N}$ into the Hypotheses (which is {1} under Feit and Thompson's Hypotheses). This allows, for instance, for a non-trivial center in $\mathfrak{G}$, and makes the theory available to the study of collineation groups. (See Theorem 5.1.)

Theorem 4.1 is a generalization of Feit and Thompson's Theorem 11.1 [9] and of [8, (31.3)].

HYPOTHESIS 4.1.

1. $\mathfrak{L}$ _is a subgroup of the finite group_ $\mathfrak{G}$, $\mathfrak{L}_o$ _is a normal subgroup of_ $\mathfrak{L}$, _and_ $\hat{\mathfrak{L}}$ _is a union of cosets in_ $\mathfrak{L}$ _of_ $\mathfrak{L}_o$, _with_ $1 \notin \hat{\mathfrak{L}}$.

2. $\mathfrak{H}$, $\mathfrak{M}$ _and_ $\mathfrak{N}$ _are normal subgroups of_ $\mathfrak{L}/\mathfrak{L}_o$ _such that_ $\mathfrak{H}$ _is nilpotent_, $\mathfrak{H} \subseteq \mathfrak{M}$, _and_

$$[\cup_{H \in \mathfrak{H}^{\#}} C(H) \cap \mathfrak{M}] \times \mathfrak{N} - \mathfrak{N} \subseteq \hat{\mathfrak{L}}/\mathfrak{L}_o \subseteq \mathfrak{M} \times \mathfrak{N} = \mathfrak{T} \subseteq \mathfrak{L}/\mathfrak{L}_o.$$

Denote $\mathfrak{L}/\mathfrak{L}_o$ and $\hat{\mathfrak{L}}/\mathfrak{L}_o$ by $\mathfrak{K}$ and $\hat{\mathfrak{K}}$, respectively. Often we regard characters of $\mathfrak{K}$ as characters of $\mathfrak{L}$. If $\underset{\sim}{S}$ is a set of characters of $\mathfrak{L}$ we denote by $\underset{\sim}{I}(\underset{\sim}{S},\hat{\mathfrak{L}})$ the set of members of $\underset{\sim}{I}(\underset{\sim}{S})$ which vanish on $\mathfrak{L} - \hat{\mathfrak{L}}$.

Throughout this section we let η denote any fixed irreducible character of $\mathfrak{N}$, and let ξ_i denote any non-principal irreducible character of $\mathfrak{M}$. Let $\tilde{\xi}_{i\eta}$ denote the character of $\mathfrak{K}$ induced by $\xi_i\eta$, and let $\mathfrak{J}$ denote the inertia group of η in $\mathfrak{K}$. Let

$$\underset{\sim}{S} = \{\tilde{\xi}_{i\eta} \mid \xi_i \neq 1 \text{ and } \tilde{\xi}_{i\eta} \text{ vanishes on } \mathfrak{K} - (\hat{\mathfrak{K}} \cup \mathfrak{N})\}.$$

LEMMA 4.1. _Suppose Hypothesis_ 4.1 _is satisfied_. _Then_ $\underset{\sim}{S}$ _is a set of orthogonal characters_. _Suppose that for some_ $\tilde{\xi}_{1\eta} \in \underset{\sim}{S}$ _we have_ $\xi_1(1) \mid \xi_i(1)$ _for all_ $\tilde{\xi}_{i\eta} \in \underset{\sim}{S}$. _If_ $\lambda_i = \xi_i(1)/\xi_1(1)$ _then_

$$\tilde{\xi}_{i\eta}(L) = \lambda_i \tilde{\xi}_1(L) \text{ for } L \in \mathfrak{L} - \hat{\mathfrak{L}}.$$

Furthermore $\underset{\sim}{I}(\underset{\sim}{S},\hat{\mathfrak{L}}) = \underset{\sim}{I}_o(\underset{\sim}{S})$.

Proof. Clearly $\underset{\sim}{S}$ consists of orthogonal characters. The equation for $\tilde{\xi}_{i\eta}$ holds by definition of $\underset{\sim}{S}$ and because it holds for $L = N \in \mathfrak{N}$. Since $1 \notin \hat{\mathfrak{L}}$, $\underset{\sim}{I}(\underset{\sim}{S},\hat{\mathfrak{L}}) \subseteq \underset{\sim}{I}_o(\underset{\sim}{S})$, and now Proposition 3.1 implies that $\underset{\sim}{I}(\underset{\sim}{S},\hat{\mathfrak{L}}) = \underset{\sim}{I}_o(\underset{\sim}{S})$.

REMARK. If $\mathfrak{G}$, $\mathfrak{L}$, and $\hat{\mathfrak{L}}$ satisfy Hypothesis 2.1 as well as Hypothesis 4.1 then, since $\underset{\sim}{I}(\underset{\sim}{S},\hat{\mathfrak{L}}) = \underset{\sim}{I}_o(\underset{\sim}{S})$, Theorems 2.1 and 2.2 and (2.6) imply that the mapping σ in §2 is an isometry from $\underset{\sim}{I}_o(\underset{\sim}{S})$ into $\underset{\sim}{M}_{\mathfrak{G}o}$. In Hypothesis 4.2 below we shall assume the existence of such a function.

LEMMA 4.2. _Assume Hypothesis_ 4.1 _is satisfied_. _If the kernel of_ ξ_i _does not contain_ $\mathfrak{H}$ _then_ $\tilde{\xi}_{i\eta} \in \underset{\sim}{S}$.

Proof. We must show $\tilde{\xi}_{i\eta}$ vanishes on $\mathfrak{K} - (\hat{\mathfrak{K}} \cup \mathfrak{N})$. Clearly $\tilde{\xi}_{i\eta}$ vanishes on $\mathfrak{K} - \mathfrak{J}$. By [9, Lemma 4.3] ξ_i vanishes on

$\mathfrak{M} - \cup_{H \in \mathfrak{H}^{\#}} C(H) \cap \mathfrak{M}$. If $J \in \mathfrak{J} - (\hat{\mathfrak{K}} \cup \mathfrak{N})$ then $J = MN$ where $M \in \mathfrak{M} - \cup_{H \in \mathfrak{H}^{\#}} C(H) \cap \mathfrak{M}$ and $N \in \mathfrak{N}$. Then $\xi_i(M) = 0$ so $\tilde{\xi}_{i\eta}(MN) = 0$.

Two characters $\theta_1, \theta_2 \in \underset{\sim}{S}$ will be called <u>equivalent</u> if $\theta_1(1) = \theta_2(1)$ and $\|\theta_1\| = \|\theta_2\|$. If $\mathfrak{a}$ is a normal subgroup of $\mathfrak{L}$ let $\underset{\sim}{S}(\mathfrak{a})$ denote $\{\theta \in \underset{\sim}{S} \mid \theta$ is equivalent to some character in $\underset{\sim}{S}$ which has $\mathfrak{a}$ in its kernel$\}$.

HYPOTHESIS 4.2.

1. *Hypothesis 4.1 is satisfied. There is a linear isometry σ from $\underset{\sim}{I}_o(\underset{\sim}{S})$ into $\underset{\sim}{M}_{GO}$.*

2. *There exists a positive integer e such that $e \mid \xi_i(1)$ for all $\tilde{\xi}_{i\eta} \in \underset{\sim}{S}$ and $\underset{\sim}{S}$ contains some irreducible character of degree $e\eta(1)(\mathfrak{K}:\mathfrak{J})$.*

3. *Each equivalence class of $\underset{\sim}{S}$ is either subcoherent in $\underset{\sim}{S}$ or consists of irreducible characters.*

4. *$\mathfrak{H}_1$ is a normal subgroup of $\mathfrak{K}$ with $\mathfrak{H}_1 \subseteq \mathfrak{H}$ and one of the following is true:*

   a. *$\underset{\sim}{S}$ consists of irreducible characters.*

   b. *Each equivalence class in $\underset{\sim}{S} - \underset{\sim}{S}(\mathfrak{H}_1)$ has at least two members.*

   c. *$e^2 \mid (\mathfrak{M}:\mathfrak{H})$ and $(\mathfrak{F}:\mathfrak{J}) \mid [(\mathfrak{H}:\mathfrak{H}_2) - 1]$ for each subgroup $\mathfrak{H}_2$ of $\mathfrak{H}_1$ which is normal in $\mathfrak{K}$.*

THEOREM 4.1. *Suppose Hypothesis 4.2 is satisfied. Let a be the square free part of $(\mathfrak{H}:\mathfrak{H}_1)$. Suppose that either $a = (\mathfrak{H}:\mathfrak{H}_1)$ and*

$$((\mathfrak{H}:\mathfrak{H}_1) - 1)^2 > 4e^2(\mathfrak{F}:\mathfrak{J})^2 \tag{4.1}$$

or $a \neq (\mathfrak{H}:\mathfrak{H}_1)$ and

$$a(\mathfrak{H}:\mathfrak{H}_1) > 4e^2(\mathfrak{F}:\mathfrak{J})^2. \tag{4.2}$$

<u>If $\underset{\sim}{S}(\mathfrak{H}_1)$ is σ-coherent and contains an irreducible character of degree</u> $e\eta(1)(\mathfrak{K}:\mathfrak{J})$ <u>then $\underset{\sim}{S}$ is coherent</u>.

<u>Proof</u>. Let $\mathfrak{H}_2$ be a normal subgroup of $\mathfrak{K}$ which is contained in $\mathfrak{H}_1$ and is minimal with respect to the property that $\underset{\sim}{S}(\mathfrak{H}_2)$ is σ-coherent with respect to $\underset{\sim}{I}_0(\underset{\sim}{S}(\mathfrak{H}_2))$. Suppose that $\mathfrak{H}_2 \neq \langle 1 \rangle$. Choose $\mathfrak{H}_3 \subset \mathfrak{H}_2$ such that $\mathfrak{H}_2/\mathfrak{H}_3$ is a chief factor of $\mathfrak{K}$. Let $\underset{\sim}{S}(\mathfrak{H}_2) = \underset{\sim}{S}_1 = \{\lambda_{1s} | 1 \leq s \leq n_1\}$, where λ_{11} is irreducible and $\lambda_{11}(1) = e\eta(1)(\mathfrak{K}:\mathfrak{J})$. Let $\underset{\sim}{S}_2, \ldots, \underset{\sim}{S}_k$ be the equivalence classes of $\underset{\sim}{S}(\mathfrak{H}_3) - \underset{\sim}{S}(\mathfrak{H}_2)$. For $2 \leq i \leq k$ let $\chi_{i1}\lambda_{11}(1)$ be the common degree of the characters in $\underset{\sim}{S}_i$. Clearly our assumptions and Lemma 4.1 imply $\underset{\sim}{S}(\mathfrak{H}_3)$ satisfies Hypothesis 3.1 except possibly (3.4) or (3.5). If $\underset{\sim}{S}_i$ with $i > 1$ consists of irreducible characters and has more than one member, then by Proposition 3.2 S_i is σ-coherent.

We will now verify that either (3.4) or (3.5) is satisfied. Consider only the characters ξ_j of $\mathfrak{M}$ whose kernel does not contain $\mathfrak{H}$. With η fixed let $\xi_j\eta$ range over a set of representaitives of the classes of characters of $\mathfrak{J}/\mathfrak{H}_2$ which are conjugate under $\mathfrak{K}$. Then ξ_j ranges once over a set of representatives of classes of characters of $\mathfrak{M}/\mathfrak{H}_2$ with respect to conjugacy in $\mathfrak{J}$. Let a_j denote the number of characters of $\mathfrak{J}$ which are conjugate in $\mathfrak{K}$ to $\xi_j\eta$. Then $a_j/(\mathfrak{K}:\mathfrak{J})$ is the number of conjugates of ξ_j relative to $\mathfrak{J}$. By lemma 4.2 all the characters $\tilde{\xi}_{j\eta}$ are in $\underset{\sim}{S}_1$ and

$$\Sigma_s \frac{\lambda_{1s}(1)^2}{\|\lambda_{1s}\|^2} \geq \Sigma_{\xi_j\eta} \frac{\tilde{\xi}_{j\eta}(1)^2}{\|\tilde{\xi}_{j\eta}\|^2}$$

$$= \Sigma_{\xi_j\eta} \frac{(\xi_j\eta)(1)^2(\mathfrak{K}:\mathfrak{J})}{\|\tilde{\xi}_{j\eta}\|^2} = \eta(1)^2 \, \Sigma_{\xi_j} \xi_j(1)^2 a_j(\mathfrak{K}:\mathfrak{J})$$

$$= \eta(1)^2(\mathfrak{K}:\mathfrak{J})(\mathfrak{K}:\mathfrak{J})[(\mathfrak{M}:\mathfrak{H}_2) - (\mathfrak{M}:\mathfrak{H})].$$

Equivalently
$$\Sigma_s \frac{\chi_{1s}^2}{\|\lambda_{1s}\|^2} \geq \frac{(\mathcal{K}:\mathcal{J})(\mathfrak{M}:\mathcal{H})[(\mathcal{H}:\mathcal{H}_2) - 1]}{e^2(\mathcal{K}:\mathcal{J})}.$$

Since $\mathcal{H}/\mathcal{H}_3$ is nilpotent, $\mathcal{H}_2/\mathcal{H}_3 \cap Z(\mathcal{H}/\mathcal{H}_3) \neq \langle 1\rangle$. Therefore, since $\mathcal{H}_2/\mathcal{H}_3$ is a chief factor of $\mathcal{K}$, $\mathcal{H}_2/\mathcal{H}_3 \subseteq Z(\mathcal{H}/\mathcal{H}_3)$. If φ is an irreducible character of $\mathcal{H}/\mathcal{H}_3$ then [9, Lemma 4.1] implies that $\varphi(1)^2 | (\mathcal{H}:\mathcal{H}_2)$, since $\mathcal{H}$ is nilpotent. Let b be the square free part of $(\mathcal{H}_1:\mathcal{H}_2)$ and let $c = (a,b)$. Then the square free part of $(\mathcal{H}:\mathcal{H}_2)$ is ab/c^2. Thus $\varphi(1)^2 | (\mathcal{H}:\mathcal{H}_2)c^2/ab$. Every irreducible character of $\mathfrak{M}$ is a constituent of some character induced by an irredubible character of $\mathcal{H}$. Hence

(4.4) $\chi_{m1} e \leq [(\mathcal{H}:\mathcal{H}_2)c^2/ab]^{1/2}(\mathfrak{M}:\mathcal{H})$, $(2 \leq m \leq k)$.

Suppose now that (3.4) or (3.5) is violated for some value of m. Under Hypothesis 4.2.4a the summation in (3.4) is an integer. In case 4.2.4b we may assume (3.5) is violated. And in case 4.2.4c the right hand side of (4.3) is an integer. Hence in all cases (4.3) implies

$$\frac{(\mathfrak{M}:\mathcal{H})[(\mathcal{H}:\mathcal{H}_2) - 1]}{e^2(\mathcal{J}:\mathcal{J})} \leq 2\chi_{m1}.$$

This and (4.4) yield that

$$(\mathcal{H}:\mathcal{H}_2) - 1 \leq 2e(\mathcal{J}:\mathcal{J})[(\mathcal{H}:\mathcal{H}_2)c^2/ab]^{1/2}$$

Let $ab/c^2 = s$ and $2e(\mathcal{J}:\mathcal{J}) = t$. Then $(\mathcal{H}:\mathcal{H}_2) = r^2 s$ for some positive integer r. We have $r^2 s - 1 \leq tr$. This implies $rs - 1 \leq rs - 1/r \leq t$.

If $r = 1$ then $(\mathcal{H}:\mathcal{H}_2)$ is square-free, and hence $(\mathcal{H}:\mathcal{H}_1)$ is square-free. Then

$$(\mathcal{H}:\mathcal{H}_1) - 1 \leq s - 1 \leq 2e(\mathcal{J}:\mathcal{J})$$

contrary to (4.1).

Therefore $r \neq 1$, $rs - 1 < rs - 1/r$ and $rs \leq t$, so

$$(\mathfrak{H}:\mathfrak{H}_2)\,ab/c^2 \leq 4e^2(\mathfrak{F}:\mathfrak{T})^2.$$

Then

$$(\mathfrak{H}:\mathfrak{H}_1) \leq \frac{1}{b}(\mathfrak{H}:\mathfrak{H}_2) \leq \frac{c^2}{ab^2}\, 4e^2(\mathfrak{F}:\mathfrak{T})^2 \leq \frac{1}{a}\, 4e^2(\mathfrak{F}:\mathfrak{T})^2.$$

But this is incompatible with both (4.1) and (4.2). Therefore either (3.4) or (3.5) is true and all assumptions of Theorem 3.1 are satisfied. Hence $\underset{\sim}{S}(\mathfrak{H}_3)$ is coherent contrary to the minimal nature of $\mathfrak{H}_2$. This finally implies that $\mathfrak{H}_2 = \langle 1 \rangle$. Therefore $\underset{\sim}{S} = \underset{\sim}{S}(\mathfrak{H}_2)$ is coherent. This completes the proof of the theorem.

As applications of Theorem 4.1 we mention that Lemmas 11.1 and 11.2 in [9] can be generalized. In particular the inequalities in those lemmas can be replaced by $a(\mathfrak{H}:\mathfrak{H}') \geq 4(\mathfrak{L}:\mathfrak{H})^2$ and $a(\mathfrak{H}:\mathfrak{H}') < 4(\mathfrak{L}:\mathfrak{H})^2$, respectively, where a is the square free part of $(\mathfrak{H}:\mathfrak{H}')$. As another application we state without proof the following theorem, which is a generalization of [8, (31.5)] and [10, Theorem 3.1]. The proof is very much like Feit's proof [8, (31.5)], or it can be based on the generalizations of the lemmas just mentioned. Only the case that $\mathfrak{H}$ is a non-abelian 2-group with $(\mathfrak{H}:\mathfrak{H}') = 4$ requires a separate argument, and this is easily given by the method of proof of [9, Lemma 11.3].

THEOREM 4.2. Suppose that Hypothesis 4.2.1 is satisfied and that $\mathfrak{H} = \mathfrak{M}$, $\hat{\mathfrak{K}} = \mathfrak{H} \times \mathfrak{N} - \mathfrak{N}$, and $\mathfrak{K}/\mathfrak{N}$ is a Frobenius group with Frobenius kernel $(\mathfrak{H} \times \mathfrak{N})/\mathfrak{N}$. Then

$\underset{\sim}{S} = \{\tilde{\xi}_{i\eta} \mid \xi_i$ is a non-principal irreducible character of $\mathfrak{H}\}$.

Then one of the following must occur:

(i) $|\underset{\sim}{S}| = 1$ and hence $\mathfrak{H}$ is an elementary abelian p-group with $|\mathfrak{H}| - 1 = (\mathfrak{F}:\mathfrak{T}) = (\mathfrak{K}:\mathfrak{T})$.

(ii) $\mathfrak{H}$ *is a non-abelian p-group for some prime* p *with* $a(\mathfrak{H}:\mathfrak{H}') < 4(\mathfrak{J}:\mathfrak{J})^2$ *where* a *is the square-free part of* $(\mathfrak{H}:\mathfrak{H}')$.

(iii) $\mathfrak{S}$ *is* σ-*coherent*.

5. THE RESTRICTION OF CERTAIN CHARACTERS OF $\mathfrak{G}$ TO $\mathfrak{L}$

In this section we obtain results concerning the multiplicities of the constituents of $\Theta|\mathfrak{L}$ for characters Θ of $\mathfrak{G}$ assuming $\mathfrak{G}$, $\mathfrak{L}$, and $\hat{\mathfrak{L}}$ satisfy Hypothesis 2.1 and assuming a coherent set of irreducible characters is given. To do this we apply Theorem 2.1, and for this it is necessary that Θ satisfy (2.7), i.e., that the average values of Θ over $L\mathfrak{J}(L)$ and $L\mathfrak{K}(L)$ be equal for each $L\in\hat{\mathfrak{L}}_\pi$. As an application of these results we state without proof Theorem 5.1, which gives a lower bound on the degrees of certain characters of a class of groups having a Frobenius section. First we obtain conditions under which (2.7) holds. Throughout this section we assume Hypothesis 2.1 is satisfied, and σ denotes the operator discusses in §2.

PROPOSITION 5.1. *Suppose* $\mathfrak{L}$ *and* $\hat{\mathfrak{L}}$ *satisfy Hypothesis* 2.1 *and* $L\in\hat{\mathfrak{L}}_\pi$. *If* Θ *is a character of* $\mathfrak{G}$ *let* $\Theta|\langle L\rangle\mathfrak{J}(L) = \psi_1 + \psi_2$ *where the kernel of* ψ_1 *contains* $\mathfrak{J}(L)$ *and no constituent of* ψ_2 *contains* $\mathfrak{J}(L)$ *in its kernel*. *Then*

(5.1) $$\Sigma_{J\in\mathfrak{J}(L)}\Theta(LJ) = (\mathfrak{J}(L):\mathfrak{K}(L))\,\Sigma_{K\in\mathfrak{K}(L)}\Theta(LK)$$

if and only if

(5.2) $$\Sigma_{K\in\mathfrak{K}(L)}\ \psi_2(LK) = 0.$$

If $(\Theta|\mathfrak{J}(L),\ 1_{\mathfrak{J}(L)}) = (\Theta|\mathfrak{K}(L),\ 1_{\mathfrak{K}(L)})$ *then* Θ *satisfies* (5.1).

Proof. Let $\xi\gamma$ denote any irreducible constituent of ψ_2 where ξ is an irreducible character of $\langle L\rangle$ and γ is an irreducible character of $\mathfrak{J}(L)$. Then

$$0 = (\gamma,1) = \frac{1}{|\mathfrak{J}(L)|}\,\Sigma_{J\in\mathfrak{J}(L)}\gamma(J).$$

Therefore

$$0 = \Sigma_{J\in\mathfrak{J}(L)}\,\xi\nu(LJ) = \Sigma_{J\in\mathfrak{J}(L)}\,\psi_2(LJ).$$

Clearly (5.1) holds for ψ_1; hence (5.1) holds for Θ if and only if (5.2) holds for ψ_2. If $(\Theta|\mathfrak{J}(L),\ 1_{\mathfrak{J}(L)}) = (\Theta|\mathfrak{K}(L),\ 1_{\mathfrak{K}(L)})$ then it is easily seen that $0 = (\psi_2|\mathfrak{J}(L),\ 1_{\mathfrak{J}(L)}) = (\psi_2|\mathfrak{K}(L),\ 1_{\mathfrak{K}(L)})$. This implies that ψ_2 satisfies (5.2) and completes the proof.

Clearly if $\Theta = \lambda_i^{\sigma *}$ is constant on the cosets of $\mathfrak{J}(L)$ which lie in $\langle L\rangle\mathfrak{J}(L) - \mathfrak{J}(L)$ then (2.7) is satisfied and (2.8) can be applied to Θ. Lemma 5.1 and Proposition 5.2 given conditions under which this is the case. Lemma 5.1 is related to Feit and Thompsons Lemma 10.2 [9], Lemma 5.2 to their Lemma 9.2, and Proposition 5.2 to their Lemma 10.3.

If all irreducible characters of $\mathfrak{G}$ are constant on $\varphi(L)$ for all $L\in\hat{\mathfrak{L}}_\pi$ then, since character tables are non-singular matrices, $\varphi(L)$ must be a single class of conjugate elements and hence $\mathfrak{J}(L) = \mathfrak{K}(L) = \langle 1\rangle$ for every $L\in\hat{\mathfrak{L}}_\pi$. Then $\hat{\mathfrak{L}}_\pi = \hat{\mathfrak{L}}$ and $C_{\mathfrak{G}}(L) \subseteq \mathfrak{L}$ for every $L\in\hat{\mathfrak{L}}$, and Hypothesis 2.1 implies $\hat{\mathfrak{L}}$ is a trivial intersection set.

Throughout this section $\underset{\sim}{I}'(\varphi,\mathfrak{L})$ will denote the set of all generalized characters of $\mathfrak{L}$ which vanish on $\varphi(\hat{\mathfrak{L}}) \cap (\mathfrak{L} - \hat{\mathfrak{L}})$ and contain $\mathfrak{K}(L)$ in their kernels for all $L\in\hat{\mathfrak{L}}_\pi$. $\underset{\sim}{I}'(\hat{\mathfrak{L}})$ will denote the submodule of $\underset{\sim}{I}'(\varphi,\hat{\mathfrak{L}})$ consisting of those generalized characters which vanish outside $\hat{\mathfrak{L}}$.

If β is a generalized character of a subgroup of $\mathfrak{G}$, denote by β^* the generalized character of $\mathfrak{G}$ induced by β.

LEMMA 5.1. <u>Suppose</u> $\mathfrak{L}$ <u>and</u> $\hat{\mathfrak{L}}$ <u>satisfy</u> <u>Hypothesis</u> 2.1 <u>and</u> <u>that</u> $L\in\hat{\mathfrak{L}}_\pi$. <u>Let</u> α <u>be</u> <u>any</u> <u>irreducible</u> <u>character</u> <u>of</u> $\mathfrak{J}(L)$. <u>Let</u> $\xi_0, \xi_1, \ldots, \xi_n$ <u>be</u> <u>the</u> <u>irreducible</u> <u>characters</u> <u>of</u> $\langle L\rangle$, <u>and</u> <u>let</u>

$\underset{\sim}{S}_\alpha = \{\xi_s\alpha\}$. If θ is a generalized character of $\mathfrak{G}$ such that $(\theta, (\xi_s\alpha - \xi_t\alpha)^*) = 0$ for all $\xi_s\alpha, \xi_t\alpha \in \underset{\sim}{S}_\alpha$ and for all $\alpha \neq 1$ then θ is constant on the cosets of $\mathfrak{J}(L)$ which lie in $\langle L\rangle\mathfrak{J}(L) - \mathfrak{J}(L)$.

Proof. If $\alpha \neq 1$ then

$$(\theta, (\xi_s\alpha - \xi_t\alpha)^*) = (\theta|\langle L\rangle\mathfrak{J}(L), \xi_s\alpha - \xi_t\alpha) = 0.$$

Thus

$$\theta|\langle L\rangle\mathfrak{J}(L) = \Sigma_{s=0}^{n} \xi_s\eta + \beta$$

where η is some character of $\mathfrak{J}(L)$ and β is a character of $\langle L\rangle\mathfrak{J}(L)/\mathfrak{J}(L)$. Thus clearly $\theta(N) = \beta(N)$ for $N \in \langle L\rangle\mathfrak{J}(L) - \mathfrak{J}(L)$.

LEMMA 5.2. Assume $\mathfrak{L}$ and $\hat{\mathfrak{L}}$ satisfy Hypothesis 2.1 and that $\alpha \in \underset{\sim}{I}'(\hat{\mathfrak{L}})$. If $L \in \hat{\mathfrak{L}}_\pi$ then $\alpha^\sigma|C_{\mathfrak{G}}(L)$ is a generalized character of $C_{\mathfrak{G}}(L)/\mathfrak{J}(L)$.

Proof. By a Lemma of Dade [5, p. 595] if $C \in C_{\mathfrak{L}}(L)$ then

$$C\mathfrak{J}(L) = C_\pi\mathfrak{J}(L) = [C_\pi C_{\mathfrak{J}(L)}(C_\pi)]^{\mathfrak{J}(L)}.$$

Thus if $N \in C\mathfrak{J}(L)$ then $N \in \varphi(L)$ if and only if $C \in \varphi(L)$. If $C \in \varphi(L)$ then $\alpha^\sigma(N) = \alpha(L)$ for all $N \in C\mathfrak{J}(L)$. If $C \notin \varphi(\hat{\mathfrak{L}})$ then $\alpha^\sigma(N) = 0$ for all $N \in C\mathfrak{J}(L)$. Let β be the generalized character of $C_{\mathfrak{G}}(L)/\mathfrak{J}(L)$ which satisfies

$$\beta|C_{\mathfrak{L}}(L) = \alpha^\sigma|C_{\mathfrak{L}}(L)$$

By the above remarks

$$\beta = \alpha^\sigma|C_{\mathfrak{G}}(L).$$

Thus since β is a combination of characters of $C_{\mathfrak{G}}(L)/\mathfrak{J}(L)$ the same is true of $\alpha^\sigma|C_{\mathfrak{G}}(L)$.

HYPOTHESIS 5.1. Assume $\mathfrak{L}$ and $\hat{\mathfrak{L}}$ satisfy Hypothesis 2.1. Let $\mathfrak{N}_L = \mathfrak{N}_{\mathfrak{G}}(\mathfrak{J}(L))$ for $L \in \hat{\mathfrak{L}}_\pi$.

1. With the notation of Lemma 5.1, if $\tilde{\xi}_{s\alpha}$ and $\tilde{\xi}_{t\alpha}$ are the characters of $\mathfrak{N}_L$ induced by $\xi_s\alpha$ and $\xi_t\alpha$ with $\alpha \neq 1$, then

$\tilde{\xi}_{s\alpha} = \Sigma_{j=1}^{n} \eta_j$ and $\tilde{\xi}_{t\alpha} = \Sigma_{j=1}^{n} \theta_j$, where the η_j and θ_j are irreducible characters of $\mathcal{N}_L$ such that $\theta_j(1) = \eta_j(1)$ for $j = 1,2,\ldots,n$.

2. Let

$$\hat{\mathcal{H}}_L = [\cup_{J\in\mathfrak{J}(L)^{\#}} C_{\mathcal{N}_L}(J)] - \mathfrak{J}(L).$$

Then $\hat{\mathcal{H}}_L$ is a trivial intersection set in $\mathcal{G}$ with $\mathcal{N}_{\mathcal{G}}(\hat{\mathcal{H}}_L) = \mathcal{N}_L$.

Note that if Hypothesis 10.2 of Feit and Thompson [9] is assumed with $\mathcal{H}_i$ replaced by $\mathfrak{J}(L)$ then condition 1 above is satisfied.

PROPOSITION 5.2. Assume that $\mathfrak{L}$ and $\hat{\mathfrak{L}}$ satisfy Hypothesis 2.1 and that $1\notin\hat{\mathfrak{L}}$. Suppose Hypothesis 5.1 is satisfied for some element $L\in\hat{\mathfrak{L}}_\pi$. Let $\underset{\sim}{S}$ be a set of characters of $\mathfrak{L}$ such that

$$\underset{\sim}{I}_o(\underset{\sim}{S}) \subseteq \underset{\sim}{I}'(\hat{\mathfrak{L}}).$$

Assume that $\underset{\sim}{S}$ is σ-coherent. Assume further that $\underset{\sim}{S}$ contains at least two irreducible characters. If $\lambda\in\underset{\sim}{S}$ then $\lambda^{\sigma*}$ is constant on the cosets of $\mathfrak{J}(L)$ which lie in $\langle L\rangle\mathfrak{J}(L) - \mathfrak{J}(L)$.

Proposition 5.2 is related to Feit and Thompsons Lemma 10.3 [9], and its proof, based on our Lemmas 5.1 and 5.2, is nearly the same as their proof with our η_j, θ_j in the role of their θ_1, θ_2.

Propositions 5.3 and 5.4 are generalizations of [10, Corollary 2.1] and Proposition 5.5 generalizes Feit and Thompsons Lemma 10.5 [9].

If $\underset{\sim}{S} = \{\lambda_i\}$ is a coherent set of irreducible characters and $\{\mathcal{X}_i\}$ is a set of irreducible characters of $\mathcal{G}$ such that $\lambda_i^{\sigma*} = \varepsilon_i \mathcal{X}_i$ where $\varepsilon_i = \pm 1$ for each i, then we call $\mathcal{X}_i$ the exceptional character of $\mathcal{G}$ associated with λ_i. All other irreducible characters of $\mathcal{G}$ are called non-exceptional. If $1\notin\mathfrak{L}$ and $\underset{\sim}{I}_o(\underset{\sim}{S}) \subseteq \underset{\sim}{I}'(\hat{\mathfrak{L}})$ then ε_i is independent of i since $\mu^{\sigma}(1) = 0$

for all $\mu \in \underset{\sim}{I}'(\hat{\mathcal{L}})$. In the remainder of the section we use this notation.

PROPOSITION 5.3. <u>Assume</u> $\mathcal{G}$, $\mathcal{L}$, <u>and</u> $\hat{\mathcal{L}}$ <u>satisfy Hypothesis 2.1 and</u> $1 \notin \hat{\mathcal{L}}$. <u>Suppose</u> $\underset{\sim}{S} = \{\lambda_i \mid 1 \leq i \leq n\}$ <u>is a</u> σ-<u>coherent set of irreducible characters of</u> $\mathcal{L}$ <u>such that</u> $\underset{\sim}{I}_0(\underset{\sim}{S}) \subseteq \underset{\sim}{I}'(\hat{\mathcal{L}})$. <u>Let</u> $\ell_i = \lambda_i(1)$. <u>Then for</u> $G \in \mathcal{G}$ <u>and for all</u> j

$$(5.3) \qquad \ell_1 \chi_j(G) = \ell_j \chi_1(G) + \begin{cases} \varepsilon(\ell_1\lambda_j - \ell_j\lambda_1)(L) \\ 0 \end{cases}$$

<u>where the first case occurs if</u> $G \in \varphi(L)$, $L \in \hat{\mathcal{L}}_\pi$ <u>and the second if</u> $G \notin \varphi(\hat{\mathcal{L}}_\pi)$.

<u>Let</u> Θ <u>be a generalized character of</u> $\mathcal{G}$ <u>satisfying</u> (2.7) <u>which is orthogonal to</u> $\underset{\sim}{S}^{\sigma *}$. <u>Then there is an integer</u> d <u>and a generalized character</u> μ <u>of</u> $\mathcal{L}$ <u>orthogonal to</u> $\underset{\sim}{S}$ <u>such that</u>

$$(5.4) \qquad \Theta|\mathcal{L} = \frac{d}{\ell_1} \Sigma_i \ell_i \lambda_i + \mu.$$

<u>If</u> $\lambda_1^{\sigma *}$ <u>satisfies</u> (2.7) <u>then each</u> $\lambda_j^{\sigma *}$ <u>satisfies</u> (2.7) <u>and then there exist integers</u> d_j <u>and characters</u> μ_j <u>of</u> $\mathcal{L}$ <u>orthogonal to</u> $\underset{\sim}{S}$ <u>such that</u>

$$(5.5) \qquad \chi_j|\mathcal{L} = \varepsilon\lambda_j + \frac{d_j}{\ell_1} \Sigma_i \ell_i \lambda_i + \mu_j.$$

<u>Proof</u>. Equation (5.3) follows from (2.6). If Θ satisfies (2.7) then (2.8) implies that for $j > 1$

$$(\Theta|\mathcal{L}, \ell_1\lambda_j - \ell_j\lambda_1) = \begin{cases} \varepsilon\ell_1 & \text{if } \Theta = \chi_j, \\ -\varepsilon\ell_j & \text{if } \Theta = \chi_1, \\ 0 & \text{if } \Theta \text{ is orthogonal to } \{\chi_1, \chi_j\}. \end{cases}$$

By (2.6), $(\ell_1\lambda_j - \ell_j\lambda_1)^\sigma$ satisfies (2.7). Therefore if $\lambda_1^{\sigma *}$ satisfies (2.7) then so does $\lambda_j^{\sigma *}$ for every j. This yields (5.4) and (5.5).

PROPOSITION 5.4. <u>Suppose all the assumptions of Proposition 5.3 are satisfied. If</u> $\alpha \in \underset{\sim}{I}'(\varphi, \hat{\mathcal{L}})$ <u>and</u> $(\alpha, \ell_1\lambda_j - \ell_j\lambda_1) = 0$ <u>then</u>

(5.6) $\chi_1(\mathcal{X}_j|\mathfrak{L},\alpha) = \chi_j(\mathcal{X}_1|\mathfrak{L},\alpha)$.

<u>Assume</u> $\underset{\sim}{S} \subseteq \underset{\sim}{I}'(\varphi,\hat{\mathfrak{L}})$. <u>If</u> $\lambda_1^{\sigma*}$ <u>satisfies</u> (2.7) <u>then in</u> (5.5) <u>we have</u> $d_j = \chi_j d_1/\chi_1$. <u>If also each constituent of</u> μ_j <u>and</u> μ_1 <u>is in</u> $\underset{\sim}{I}'(\varphi,\hat{\mathfrak{L}})$ <u>then</u> $\mu_j = \frac{\chi_j}{\chi_1}\mu_1$.

<u>Proof</u>. If $\alpha\in\underset{\sim}{I}'(\varphi,\hat{\mathfrak{L}})$ then (2.10) and (2.8) imply that

$$\varepsilon((\chi_1\mathcal{X}_j - \chi_j\mathcal{X}_1)|\mathfrak{L},\alpha) = ((\chi_1\lambda_j - \chi_j\lambda_1)^{\sigma}|\mathfrak{L},\alpha)$$
(5.7)
$$= ((\chi_1\lambda_j - \chi_j\lambda_1)^{\sigma},\alpha^{\sigma}) = ((\chi_1\lambda_j - \chi_j\lambda_1,\alpha).$$

This yields (5.6). If $\underset{\sim}{S} \subseteq \underset{\sim}{I}'(\varphi,\hat{\mathfrak{L}})$ then putting $\alpha = \lambda_1$ in (5.7) we have for $j > 1$

$$\varepsilon(d_j\chi_1 - \chi_j(\varepsilon + d_1)) = -\chi_j.$$

Hence $d_j = \chi_j d_1/\chi_1$. Equation (5.6) yields the last statement.

REMARK. Often it occurs that $\varphi(\hat{\mathfrak{L}}) \cap (\mathfrak{L} - \hat{\mathfrak{L}})$ is empty. If also $\mathfrak{L}_0$ has no π'-elements then every character of $\mathfrak{L}$ is in $\underset{\sim}{I}'(\varphi,\hat{\mathfrak{L}})$ and Proposition 5.4 is more readily applicable.

PROPOSITION 5.5. <u>Suppose all the assumptions of Proposition 5.3 are satisfied</u>. <u>Let</u> Θ <u>be a generalized character of</u> $\mathfrak{G}$ <u>satisfying</u> (2.7). <u>Then there exist rational numbers</u> b <u>and</u> c <u>and generalized characters</u> β <u>and</u> γ <u>of</u> $\mathfrak{L}$ <u>orthogonal to</u> $\underset{\sim}{S}$ <u>such that if</u> $L\in\mathfrak{L}^{\#}$ <u>then</u> $\Theta(L) = b\beta(L)$ <u>if</u> Θ <u>is orthogonal to</u> $\underset{\sim}{S}^{\sigma*}$, <u>and</u> $\mathcal{X}_j(L) = \varepsilon\lambda_j(L) + c\gamma(L)$ <u>if</u> $\Theta = \mathcal{X}_j$.

<u>Proof</u>. Let $\xi = \Sigma_i\chi_i\lambda_i$ where λ_i ranges over $\underset{\sim}{S}$. There exists a character ξ' of $\mathfrak{L}$ which is orthogonal to $\underset{\sim}{S}$ such that $\xi + \xi' = \rho_{\mathfrak{L}}$, the character of the regular representation of $\mathfrak{L}$. By (5.4) if Θ is orthogonal to $\underset{\sim}{S}^{\sigma*}$ then

$$\Theta|\mathfrak{L} = b_1\xi + \mu$$

where μ is orthogonal to ξ. Similarly

$$\mathcal{X}_j|\mathfrak{L} = \varepsilon\lambda_j + c_1\xi + \mu_j$$

if $\theta = \mathcal{X}_j$ by (5.5). Since $\rho_{\mathfrak{L}}(L) = 0$ for $L \in \mathfrak{L}^{\#}$

$$\theta(L) = -b_1 \xi'(L) + \mu,$$

$$\mathcal{X}_j(L) = \varepsilon \lambda_j(L) - c_1 \xi'(L) + \mu_j(L),$$

and the Lemma results by a suitable change in notation.

THEOREM 5.1. Suppose $\mathfrak{G}$, $\mathfrak{L}$, and $\hat{\mathfrak{L}}$ satisfy Hypotheses 2.1 and 4.1 and that $\mathfrak{H}=\mathfrak{N}$, $\mathfrak{L}_0=\langle 1\rangle$, $\hat{\mathfrak{L}}=\mathfrak{H}\times\mathfrak{N}-\mathfrak{N}$, $\mathfrak{L}=\mathfrak{N}(\hat{\mathfrak{L}})$, and $\mathfrak{L}/\mathfrak{N}$ is a Frobenius group with Frobenius kernel $\mathfrak{H}\times\mathfrak{N}/\mathfrak{N}$. Suppose that Hypothesis 5.1 is satisfied for all $L \in \hat{\mathfrak{L}}_{\pi}$. Assume that $\hat{\mathfrak{L}} \cap G^{-1}\mathfrak{N}G=\emptyset$ for all $G \in \mathfrak{G}$. If $\mathcal{X}$ is a faithful character of $\mathfrak{G}$ of degree less than $(|\mathfrak{H}| - 1)/2$ and if $\mathcal{X}$ is constant on $\varphi(L)$ for all $L \in \hat{\mathfrak{L}}_{\pi}$ then one of the following must be true.

(a) $\mathfrak{H}$ is a non-abelian p-group for some prime p with

$$a(\mathfrak{H}:\mathfrak{H}') < 4(\mathfrak{L}:\mathfrak{H} \times \mathfrak{N})^2$$

where a is the square free part of $(\mathfrak{H}:\mathfrak{H}')$.

(b) $\mathfrak{L} = \mathfrak{G}$. Then $\mathfrak{H}$ is normal in $\mathfrak{G}$.

(c) $\mathfrak{H}$ is an elementary abelian p-group for some prime p and no proper subgroup of $\mathfrak{H}$ is normal in $\mathfrak{L}$.

This theorem is a generalization of [10, Theorem 4.2], and its proof is entirely similar to the proof of that theorem, being based on Lemma 3.1, Theorem 4.2, Propositions 5.3 and 5.4 and other analogues of results in [10].

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