A Necessary Condition for
Hyperbolicity of a Polynomial
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By

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1. Introduction. A polynomial

$$P(\zeta) = \sum_{k=0}^{m} P_k(\zeta), \ \zeta \in C_{n+1}, P_k(\lambda\zeta) = \lambda^k P_k(\zeta) \qquad (1.1)$$

with complex constant coefficients is called hyperbolic in the sense of Gårding [1], with respect to the direction $N \in R_n$, if and only if $P_m(N) \neq 0$ and the roots τ of $P(\tau N + \xi)$ have bounded imaginary parts for real $\xi \in N^{\perp}$. It follows that P_m is hyperbolic, if P is, and has only real roots. It is natural to ask for which lower order perturbations of the principal part P_m the polynomial P is hyperbolic. In two dimensions this question was answered completely by A. Lax [4]. She showed that it is necessary and sufficient that if P_m has a root of multiplicity ν at the point $\tau_0 N + \xi_0$, then each polynomial P_{m-k}, $k < \nu$, has a root of multiplicity $\nu - k$ at $\tau_0 N + \xi_0$. Kasahara and Yamaguti [3] showed that the necessary condition carries over into higher dimensions provided that by multiplicity one means the number of times a given factor appears in the factorization of P_m. In this paper we shall show that the necessary condition is valid in any number of dimensions using a much stronger definition of multiplicity (which for the purpose of distinction we call order). Our definition of order is the common one associated with the Taylor's expansion of a function of several variables.

Definition of Multiplicity and Order. Let $P(\tau, \xi)$ be a polynomial of degree m in τ whose coefficients are of class C^m with respect to ξ in some subdomain of R_n. The multiplicity of P at (τ_0, ξ_0) is the least integer ν such that

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$$\left(\frac{\partial}{\partial \tau}\right)^{\nu} P(\tau_o, \xi_o) \neq 0.$$

The order of P at (τ_o, ξ_o) is the least integer μ such that

$$\left(\frac{\partial}{\partial \xi}\right)^{\alpha} \left(\frac{\partial}{\partial \tau}\right)^{k} P(\tau_o, \xi_o) \neq 0$$

for some α, k with $|\alpha| + k = \mu$.

It is clear that the order of P is always less than or equal to its multiplicity. In one dimension the two concepts are always the same; hence they are also the same for homogeneous polynomials in two dimensions. For this reason no confusion can arise from interchanging the words order and multiplicity in the theorem of A. Lax.

The question arises as to when the order and the multiplicity are the same. Let us write

$$P(\tau, \xi) = \prod_{k=1}^{m} (\tau - \tau_k(\xi)).$$

If the root $\tau_o = \tau_k(\xi_o)$ has multiplicity ν and if all of the roots $\tau_j(\xi)$ are $\nu - 1$ times differentiable functions of ξ, then any derivative

$$\left(\frac{\partial}{\partial \tau}\right)^{k} \left(\frac{\partial}{\partial \xi}\right)^{\alpha} P(\tau_o, \xi_o), \quad |\alpha| + k \leq \nu - 1,$$

is a sum of products each of which contains the factor zero. Hence in this case the order is equal to the multiplicity. Unless the irreducible factors of P have distinct, and hence C^m, roots, there seems to be no a priori reason why this should be so. One of the conclusions of this paper is that for homogeneous hyperbolic polynomials the order and the multiplicity are always the same.

For an interesting complement to the theory of this paper the reader is referred to Gårding [2].

2. Statement of Results. As mentioned in the introduction, the main purpose of this paper is to extend the necessary condition

of A. Lax to higher dimensions. Our proof is based on the following somewhat stronger result which may also be of independent interest.

Theorem 2.1. Let $P(\tau) = \tau^m + a_1(\xi)\tau^{m-1} + \cdots$ be a polynomial of degree m in τ whose coefficients are of class C^m with respect to ξ in a neighborhood $\xi = 0$, $\xi \in R_n$. If the roots $\tau_k(\xi)$ of P satisfy the order relation $\operatorname{Im} \tau_k(\xi) = O(|\xi|)$ as $\xi \to 0$ and if $P(\tau, 0)$ has a root τ_o of multiplicity ν, then P has a zero of order ν at $(\tau_o, 0)$.

Our generalization of the Lax necessary condition then is:

Theorem 2.2. Suppose that the polynomial P is hyperbolic with respect to the direction N and P_m has a zero of multiplicity ν at the point $\tau_o N + \xi_o$, $\xi_o \in N^{\perp}$. Then for each k, $0 \leq k < \nu$, the polynomial P_{m-k} has a zero of order $\nu - k$ at $\tau_o N + \xi_o$.

We emphasize that the above theorem gives new information even if there are no lower order terms present. One might ask whether the roots of a homogeneous hyperbolic polynomial do, in fact, have any a priori smoothness. The best general result that we have been able to prove in this direction is:

Theorem 2.3. Let P_m be hyperbolic with respect to N and suppose that the roots $\tau_1(\xi), \ldots, \tau_m(\xi)$ of P_m are indexed in non-decreasing order, $\xi \in N^{\perp}$. Then for each $\xi_o \in N^{\perp}$, there exists a constant K such that

$$|\tau_k(\xi) - \tau_k(\xi_o)| \leq K|\xi - \xi_o|, \quad \xi \in N^{\perp}.$$

The constant K is uniform over compact subsets of any set where $\tau_k(\xi)$ has constant multiplicity.

There remains the difficult question of deciding whether or not the necessary condition is also sufficient. The following theorem gives some evidence in support of the conjecture that it is.

Theorem 2.4. The necessary condition of Theorem 2.2 is also sufficient for second order operators.

3. Some Combinatorial Lemmas. The proof of our basic theorem, Theorem 2.1, is rather long so we shall first prove some lemmas to facilitate its proof. We begin by introducing some notation.

A multi-index α is a vector $(\alpha_1,\ldots,\alpha_r)$ where each coordinate α_j is an integer, $1 \le \alpha_j \le n$. The number of components r is denoted by $|\alpha|$. If $\beta = (\beta_1,\ldots,\beta_s)$ is another multi-index, by $\alpha\beta$ we mean the index $(\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_s)$. Let F be a sufficiently differentiable function of $x = (x_1,\ldots,x_n)$ and let $y = (y_1,\ldots,y_n)$ be an n-vector. The symbols F^α and y_α are used to denote the derivative $\partial^r F/\partial x_{\alpha_1}\ldots\partial x_{\alpha_r}$ and the product $y_{\alpha_1}\ldots y_{\alpha_r}$ respectively. (Note that we are using the classical convention rather than the more recent one of letting $y_\alpha = y_1^{\alpha_1}\ldots y_n^{\alpha_n}$). A homogeneous polynomial of degree m may be written

$$P(y) = \sum_{|\alpha|=m} a^\alpha y_\alpha$$

where the coefficients a^α are symmetric within permutations of the components of α. At times it will be convenient, for example, to re-write

$$P(y) = \sum_{|\alpha|=j,}\ \sum_{|\beta|=k,}\ \sum_{|\gamma|=m-j-k} a^{\alpha\beta\gamma}\, y_\alpha y_\beta y_\gamma$$

where j and k are fixed integers, $0 \le j + k \le m$. The symmetry of the coefficients allows us to combine α and γ into a single index δ, $|\delta| = m - k$, and to re-write

$$P(y) = \sum_{|\beta|=k,}\ \sum_{|\delta|=m-k} a^{\beta\delta}\, y_\beta y_\delta .$$

We shall also find it convenient to perform the above indicated manipulations on derivatives.

For functions ϕ of one variable τ, the k-th derivative will be indicated by $\phi^{(k)}$. When considering functions of several variables $F(\tau,x)$, where the first variable τ is singled out, the expression $F^{(k)\alpha}$ means the mixed derivative $(\partial/\partial\tau)^k(\partial/\partial x)^{\alpha}F$.

The following lemma illustrates the use of the product notation for indices.

<u>Lemma 3.1</u>. If $F: R_n \to R_1$ is of class C^{r+1} and $x(\sigma): R_1 \to R_n$ is of class C^2, then

$$\sum_{|\alpha|=r} F^{\alpha}(x(\sigma))\frac{d}{d\sigma}(x'_{\alpha}) = r\sum_{\substack{|\alpha|=r-1\\|\beta|=1}} F^{\alpha\beta}(x(\sigma))x'_{\alpha}x''_{\beta}.$$

Proof. It follows from the rule for differentiating a product that

$$\sum_{|\alpha_j|=1} F^{\alpha_1\cdots\alpha_r}\frac{d}{d\sigma}(y'_{\alpha_1}\cdots y'_{\alpha_r}) = \sum_{k=1}^{r}\sum_{\substack{|\alpha_j|=1\\j\neq k}} F^{\alpha_1\cdots\alpha_k\cdots\alpha_r}y'_{\alpha_1}\cdots y''_{\alpha_k}\cdots y'_{\alpha_r}.$$

The proof is completed by using the symmetry of F^{α}, setting $\alpha_k = \beta$ and setting the product of the remaining indices equal to α.

The above lemma is used to obtain a formula for the derivatives of a class of composite functions.

<u>Lemma 3.2</u>. Let $\phi(\sigma) = F(x(\sigma))$ where F is a polynomial in $x\epsilon R_n$ and $x(\sigma)$ is a quadratic in σ. Then

$$\phi^{(k)}(\sigma) = \sum_{j=0}^{[k/2]} \frac{k!}{j!2^j(k-2j)!} \sum_{\substack{|\alpha|=k-2j\\|\beta|=j}} F^{\alpha\beta}(x)x'_{\alpha}x''_{\beta} \qquad (3.1)$$

<u>Proof</u>. We proceed by induction. The assertion is clearly true for $k = 0$. Supposing (3.1) to be true, we differentiate both sides, using Lemma 3.1 together with the fact that x''_{β} is constant, to obtain

$$\phi^{(k+1)}(\sigma) = \sum_{j=0}^{[k/2]} \frac{k!}{j!2^j(k-2j)!} \sum_{\substack{|\alpha|=k-2j \\ |\beta|=j,|\gamma|=1}} F^{\alpha\beta\gamma}(x)\, x'_\alpha x'_\beta{}' x'_\gamma$$

$$+ \sum_{j=0}^{[k/2]} \frac{k!}{j!2^j(k-2j-1)!} \sum_{\substack{|\alpha|=k-2j-1 \\ |\beta|=j,|\gamma|=1}} F^{\alpha\beta\gamma}(x)\, x'_\alpha x'_\beta{}' x'_\gamma .$$

In the first term on the right we replace $\alpha\gamma$ by a single index (of length $k - 2j + 1$) and in the second we replace $\beta\gamma$ by a single index. The summation index j in the second term is then replaced by $j - 1$. After simplifying, the expression (3.1) is obtained with k replaced by $k + 1$. This completes the proof.

The next lemma relates certain bilinear forms in the derivatives of a polynomial to quadratic forms in its roots.

<u>Lemma 3.3.</u> Suppose that $P(\tau) = \tau^m + \ldots$ is a polynomial of degree m in τ with roots $r_k + i\, s_k$, $k = 1,2,\ldots,m$. Let $F(x) = x_1 \ldots x_m$ and $Y_k = (\tau - r_k)^2 + s_k^2$. Then

$$\sum_{j=0}^{k} \frac{(2k)!\; 2^{2j}}{(k-j)!(2j)!} \sum_{\substack{|\alpha|=2j \\ |\beta|=k-j}} F^{\alpha\beta}(Y)\, s_\alpha$$

$$= \mathrm{Re} \sum_{j=0}^{2k} (-1)^{j+k} \binom{2k}{j} P^{(j)}(\tau)\overline{P^{(2k-j)}(\tau)} \tag{3.2}$$

and

$$\sum_{j=0}^{k} \frac{(2k+1)!\, 2^{2j+1}}{(k-j)!(2j+1)!} \sum_{\substack{|\alpha|=2j+1 \\ |\beta|=k-j}} F^{\alpha\beta}(Y)\, s_\alpha$$

$$= \mathrm{Im} \sum_{j=0}^{2k+1} (-1)^{j+k} \binom{2k+1}{j} P^{(j)}(\tau)\overline{P^{(2k+1-j)}(\tau)} . \tag{3.3}$$

<u>Proof</u>. Let $\phi(\sigma) = |P(\tau + i\sigma)|^2$ and note from the factored form of P that $\phi(\sigma) = F(x(\sigma))$ where $x_k(\sigma) = (\tau - r_k)^2 + (\sigma - s_k)^2$.

Since $x_k(0) = Y_k$, $x_k'(0) = -2s_k$ and $x_k''(0) = 2$, it follows from Lemma 3.2 that

$$\phi^{(k)}(0) = \sum_{j=0}^{[k/2]} \frac{k!(-2)^{k-2j}}{j!(k-2j)!} \sum_{\substack{|\alpha|=k-2j \\ |\beta|=j}} F^{\alpha\beta}(Y)s_\alpha .$$

On the other hand it is a consequence of Leibniz' formula for the derivative of a product that

$$\phi^{(k)}(0) = \sum_{j=0}^{k} (-1)^{k-j}(i)^k \binom{k}{j} P^{(j)}(\tau)\, \overline{P^{(k-j)}(\tau)} .$$

By equating the above two expressions and separating the even and odd cases we obtain

$$\sum_{j=0}^{k} \frac{(2k)!2^{2k-2j}}{j!(2k-2j)!} \sum_{\substack{|\alpha|=2k-2j \\ |\beta|=j}} F^{\alpha\beta}(Y)s_\alpha$$

$$= \sum_{j=0}^{2k} (-1)^{k+j} \binom{2k}{j} P^{(j)}(\tau)\, \overline{P^{(2k-j)}(\tau)} , \tag{3.4}$$

and

$$\sum_{j=0}^{k} \frac{(2k+1)!}{j!(2k+1-j)!} \sum_{\substack{|\alpha|=2k+1-2j \\ |\beta|=j}} F^{\alpha\beta}(Y)s_\alpha$$

$$= -i \sum_{j=0}^{2k+1} (-1)^{k+j} \binom{2k+1}{j} P^{(j)}(\tau)\, \overline{P^{(2k+1-j)}(\tau)} . \tag{3.5}$$

We next replace the summation index j in the sums on the left sides of the inequalities (3.4) and (3.5) by $k - j$. The proof is then completed by noting that the left sides of (3.4) and (3.5) are real.

Our basic technical lemma is the following corollary of Lemma 3.3.

Lemma 3.4. Suppose that P, F and Y are as defined in Lemma 3 and set

$$Q_k = \frac{(2k)!}{k!} \cdot \prod_{|\alpha|=k} F^{\alpha}(Y) \tag{3.6}$$

Then we have

$$Q_k = \operatorname{Re} \sum_{j=0}^{2k} (-1)^{j+k} \binom{2k}{j} P^{(j)}(\tau)\overline{P^{(2k-j)}(\tau)} + O\Big(\sum_{j=1}^{k} Q_{k+j}|s|^{2j}\Big) \tag{3.7}$$

and

$$\operatorname{Im} \sum_{j=0}^{2k+1} (-1)^{j+k} \binom{2k+1}{j} P^{(j)}(\tau)\ \overline{P^{(2k+1-j)}(\tau)} = O\Big(\sum_{j=0}^{k} Q_{j+k+1}|s|^{2j+1}\Big) . \tag{3.8}$$

Here $|s| = (s_1^2 + \ldots + s_m^2)^{1/2}$.

Proof. Since $F^{\alpha}(x) = F(x)/x_{\alpha}$ or $F^{\alpha}(x) = 0$ and $Y_k \geqq 0$, it follows that

$$F^{\alpha}(Y) \le K\ Q_{|\alpha|} \tag{3.9}$$

where K depends only on m. (3.7) is then proved by substituting (3.9) into those terms on the left side of (3.2) where $j \ge 1$ and (3.8) is proved by substituting (3.9) into the left side of (3.3).

The application of Lemma 3.4 which we have in mind involves polynomials $P(\tau,\xi)$ in τ whose coefficients depend on $\xi = (\xi_1, \ldots, \xi_n) \in R_n$ and whose roots have imaginary parts which are of the order $O(|\xi|)$ as $\xi \to 0$. For such polynomials the order terms in (3.7) and (3.8) are respectively at least $O(|\xi|^2)$ and $O(|\xi|)$. We next study the question of refining these error estimates in case certain derivatives of P are known to vanish at a point.

It is convenient to define

$$B_p(\tau,\xi) = \sum_{j=0}^{p} (-1)^{j+[p/2]} \binom{p}{j} P^{(j)}(\tau,\xi)\overline{P^{(p-j)}(\tau,\xi)}$$

and to set $\rho = (|\xi|^2 + \tau^2)^{1/2}$.

Lemma 3.5. Assume that $P(\tau,\xi) = \tau^m + \ldots$ is a polynomial of degree m in τ whose coefficients are of class C^m in a neighborhood of $\xi = 0$ and suppose that, for integers λ,ν which satisfy $\lambda + 1 \le \nu \le m$, we have

$$P^{(k)\alpha}(0,0) = 0, \ |\alpha| \le \lambda - 1, \ k + |\alpha| \le \nu - 1.$$

If $p \le 2\nu$, then

$$B_p(\tau,\xi) = O(|\xi|^\lambda) + O(\rho^{\lambda+1}) + O(\rho^{2\nu-p}), \quad \rho \to 0. \tag{3.10}$$

If $p \le \nu$ and $\nu - p \le \lambda$, then

$$B_p^{(k)\alpha}(0,0) = 0 \quad \begin{cases} k + |\alpha| < \lambda + \nu - p & (3.11) \\ k + |\alpha| = \lambda + \nu - p, \ |\alpha| < \lambda & (3.12) \end{cases}$$

and

$$B_p^{(\nu-p)\alpha}(0,0) = 2(-1)^{[p/2]} \operatorname{Re} P^{\alpha}(0,0) P^{(\nu)}(0,0), |\alpha| = \lambda. \tag{3.13}$$

Proof. The derivative $B_p^{(k)\alpha}(0,0)$ is a linear combination of terms of the form

$$P_p^{(j+i)\beta}(0,0) \ P^{(p-j+k-i)}(0,0) \tag{3.14}$$

where $0 \le j \le p$, $0 \le i \le k$ and $|\beta| + |\gamma| = |\alpha|$. At least one of the terms in the above product vanishes if

$$|\beta|, |\gamma| \le \lambda - 1 \ \text{ and } \ k + |\alpha| < 2\nu - p. \tag{3.15}$$

Otherwise we would have $j + i + |\beta| \ge \nu$ and $p - j + k - i + |\gamma| \ge \nu$ which implies the absurdity $k + |\alpha| \ge 2\nu - p$.

Now if $k + |\alpha| < 2\nu - p$ and $2\nu - p \le \lambda$ then $|\alpha| \le \lambda - 1$ so by (3.15), (3.14) we have $B_p^{(k)\alpha}(0,0) = 0$; hence

$$B_p(\tau,\xi) = O(\rho^{2\nu-p}). \tag{3.16}$$

On the other hand, if $2\nu - p > \lambda$, $k + |\alpha| \le \lambda$ and $|\alpha| \le \lambda - 1$, (3.15) is again satisfied. It follows that all derivatives of B_p up to order λ vanish at (0,0) except possibly $B_p^{\alpha}(0,0)$, $|\alpha| = \lambda$. Hence

$$B_p(\tau,\xi) = O(|\xi|^\lambda) + O(\rho^{\lambda+1}). \qquad (3.17)$$

The proof of (3.10) is completed by combining (3.15) and (3.16).

We turn now to the proof of (3.12) and (3.13). The inequalities $k + |\alpha| \leq \lambda + \nu - p$ and $\lambda \leq \nu - 1$ imply that $k + |\alpha| < 2\nu - p$. Hence if $|\beta|, |\gamma| \leq \lambda - 1$, it follows from (3.14) and (3.15) that (3.12) is satisfied. It also shows that the only contribution to $B_p^{(p-\nu)\alpha}(0,0), |\alpha| = \lambda$, arises from (3.14) with $|\beta| = 0$ or $|\gamma| = 0$. If $|\gamma| = 0$ the expression (3.14) will be zero unless $i + j = \nu$ and if $|\beta| = 0$ it will be zero unless $i + j = 0$. Thus the truth of (3.13) is verified.

It remains to prove (3.11). The inequalities $k + |\alpha| < \lambda + \nu - p$ and $\nu - p \leq \lambda$ show that $|\alpha| < 2\lambda$; hence either $|\beta| \leq \lambda - 1$ or $|\gamma| \leq \lambda - 1$. If both are $\leq \lambda - 1$, the argument of the preceding paragraph shows that (3.14) vanishes. If $|\beta| \geq \lambda$ we would have $p - j + k - i + |\gamma| \geq \nu$ which together with $|\gamma| = |\alpha| - |\beta| \leq |\alpha| - \lambda$ implies $p + k + |\alpha| - \lambda \geq \nu$ contrary to (3.11). A similar argument shows that (3.14) vanishes if $|\gamma| \geq \lambda$. This completes the proof of Lemma 3.5.

Lemma 3.6. If in addition to the hypotheses of Lemma 3.5, the roots $r_k(\xi) + i\, s_k(\xi)$ of P satisfy the order relation $s_k(\xi) = O(|\xi|)$ as $\xi \to 0$, then

$$Q_k = \text{Re } B_{2k}(\tau,\xi) + O(|\xi|^{\lambda+2}) + O(\rho^{\lambda+3}) + O(\rho^{2\nu-2k}). \qquad (3.18)$$

and

$$\text{Im } B_{2k+1}(\tau,\xi) = O(|\xi|^{\lambda+1}) + O(\rho^{\lambda+2}) + O(\rho^{2\nu-2k-1}). \qquad (3.19)$$

Proof. We first prove (3.18) by induction on k. It is clearly true when $k = \nu$, for all it says in this case is that Q_ν is bounded near (0,0). Assuming that (3.18) is true for $k+1,\ldots,\nu$, we obtain from (3.10) and (3.18) that

$$Q_{k+j} = O(|\xi|^\lambda) + O(\rho^{\lambda+1}) + O(\rho^{2\nu-2k-2j})$$

when $j \geq 1$ and $k + j \leq \nu$. On the other hand it is a consequence of (3.7) that

$$Q_k = \text{Re } B_{2k} + O(\sum_{j=1}^{\nu-k} Q_{k+j}) + O(|\xi|^{2\nu-2k}).$$

The proof of (3.18) is completed by combining the above two results. The identity (3.19) is now easily proved by substituting (3.10) into (3.18) and then using the resulting estimate in (3.8). This completes the proof of Lemma 3.6.

The next lemma formalizes the arguments which will be used several times in the proof of Lemma 3.8.

Lemma 3.7. Suppose that $a_k(\xi)$, $k = q, q + 1, \ldots, w$, are homogeneous functions (not necessarily polynomials) of degree k in ξ, $\xi \in R_n$, and that a_q is not identically zero. If for (τ, ξ) in a neighborhood of the origin we have

$$\sum_{k=q}^{w} a_k(\xi)\tau^{w-k} + O(\rho^{w+1}) \geq 0,$$

then q and w are even and $a_q(\xi) \geq 0$.

Proof. Replace (τ, ξ) by $(\epsilon\tau, \epsilon\xi)$, $\epsilon > 0$, divide by ϵ^w and let $\epsilon \to 0$ to obtain

$$\sum_{k=q}^{w} a_k(\xi)\tau^{w-k} \geq 0, \tag{3.20}$$

which, by homogeneity is then valid for all (τ, ξ). If w were not even, the opposite inequality would be obtained by replacing (τ, ξ) by $(-\tau, -\xi)$, contradicting the fact that a_q is not identically zero. Now divide (3.20) by τ^{w-q} and let $\tau \to +\infty$ to obtain $a_q(\xi) \geq 0$. That q is even is again a consequence of homogeneity. This completes the proof.

The next lemma shows that the vanishing of certain derivatives of a polynomial whose roots have small imaginary parts implies the vanishing of more derivatives at the same point.

Lemma 3.8. If in addition to the hypotheses of Lemmas 3.5 and 3.6 we have $P^{(\nu)}(0,0) \neq 0$, then

$$P^{(k)\alpha}(0,0) = 0, \ |\alpha| \leq \lambda, \ k + |\alpha| \leq \nu - 1.$$

Proof. It is sufficient to prove the above result when $k = 0$. For by the theorem of Gauss and Lucas, Polya and Szegö [6], p. 89, problem 31, the roots of the derivatives of a polynomial lie in the convex hull of the derivatives of the polynomial; in particular, the imaginary parts of the roots of the polynomials $P^{(k)}$ are of the order $O(|\xi|)$ as $\xi \to 0$. The general result is then obtained by applying the special case to the polynomials $P^{(j)}$, $j \leq \nu - 1$, with ν replaced by $\nu - j$.

By hypothesis $P^{\alpha}(0,0) = 0$ when $|\alpha| \leq \lambda - 1$, hence we have only to prove that $P^{\alpha}(0,0) = 0$ for $|\alpha| = \lambda$. The proof consists of several steps. The proof that $\operatorname{Re} P^{\alpha}(0,0) = 0$, $|\alpha| = \lambda$ is broken down into the three cases: (I) ν even, λ even, (II) ν even, λ odd, and (III) ν odd. The proof that $\operatorname{Im} P^{\alpha}(0,0) = 0$, $|\alpha| = \lambda$, is separated into two cases: (IV) ν even and (V) ν odd.

Proof of Case I. Since ν is even let $\mu = \nu/2$. Now apply Lemma 3.6 with $k = \mu$. The order term $O(\rho^{2\nu-2k})$ may be dropped since $\nu \geq \lambda + 1$. Using (3.18) and substituting the results of (3.10)-(3.13) into the Taylors expansion of $B_{2\mu}$ it is seen that

$$Q_{\mu} \leq \frac{2(-1)^{\mu}}{\lambda!} \operatorname{Re} \sum_{|\alpha|=\lambda} P^{\alpha}(0,0)\, \xi_{\alpha} \overline{P^{(\nu)}(0,0)} + O(\rho^{\lambda+1}).$$

Since $Q_{\mu} \geq 0$, Lemma 3.7 implies that

$$0 \leq (-1)^{\mu} \operatorname{Re} \sum_{|\alpha|=\lambda} P^{\alpha}(0,0)\, \xi_{\alpha} \overline{P^{\nu}(0,0)} . \tag{3.21}$$

Next apply Lemma 3.6 with $k = \mu - 1$, again using (3.10)-(3.13), to obtain

$$Q_{\mu-1} \le \frac{2(-1)^{\mu-1}}{(\lambda+2)!} \sum_{|\alpha|=\lambda} P^{\alpha}(0,0)\overline{P^{(\nu)}(0,0)}\,\tau^2$$

$$+\ a_{\lambda+1}(\xi)\tau + a_{\lambda+2}(\xi) + O(|\xi|^{\lambda+2}) + O(\rho^{\lambda+3}),$$

where a_k is homogeneous in ξ of degree k. It is clear from the proof of Lemmas 3.5 and 3.6 that the term $O(|\xi|^{\lambda+2})$ is independent of τ; hence it may be absorbed into $a_{\lambda+2}(\xi)$. Another application of Lemma 3.7 then shows that

$$0 \le (-1)^{\mu-1}\ \mathrm{Re} \sum_{|\alpha|=\lambda} P^{\alpha}(0,0)\,\xi_\alpha \overline{P^{\nu}(0,0)}\,.$$

By combining the above result with (3.21), and using the fact that $P(\tau,0)$ has real coefficients we then have

$$\mathrm{Re} \sum_{|\alpha|=\lambda} P^{\alpha}(0,0)\,\xi_\alpha = 0;$$

hence $P^{\alpha}(0,0) = 0$ for each $\alpha, |\alpha| = \lambda$.

Proof of Case II. Proceeding as above, we obtain (3.21). Since λ is odd it follows immediately that $P^{\alpha}(0,0) = 0$, $|\alpha| = \lambda$.

Proof of Case III. Since ν is odd let $\nu = 2\mu + 1$. Again using Lemmas (3.5) and (3.6), we obtain

$$Q_{\mu} \le \frac{2(-1)^{1}}{\lambda!}\ \mathrm{Re} \sum_{|\alpha|=\lambda} P^{\alpha}(0,0)\,\xi_\alpha \overline{P^{(\nu)}(0,0)}\,\tau$$

$$+\ a_{\lambda+1}(\xi) + O(\rho^{\lambda+2}).$$

The inequality $Q_{\mu} \ge 0$ and Lemma 3.7 show that $P^{\alpha}(0,0) = 0$ for $|\alpha| = \lambda$.

Proof of Case IV. Let μ be defined by $\nu = 2\mu$. By applying (3.19) of Lemma 3.6 with $k = \mu$ and then using the results of Lemma 3.5 we obtain

$$\frac{(-1)^{\mu}}{(\lambda+1)!}\ \mathrm{Im} \sum_{|\alpha|=\lambda} P^{\alpha}(0,0)\,\xi_\alpha \overline{P^{(\nu)}(0,0)}\,\tau + a_{\lambda+1}(\xi)$$

$$= O(|\xi|^{\lambda+1}) + O(\rho^{\lambda+2}).$$

By incorporating the term $O(|\xi|^{\lambda+1})$ into $a_{\lambda+1}(\xi)$ and applying Lemma 3.7, we see that $\operatorname{Im} P^{(\alpha)}(0,0) = 0$ (recall that $P^{(\nu)}(0,0)$ is real).

Proof in Case V. Let $\nu = 2\mu + 1$ and apply (3.19) of Lemma 3.6 with $k = \mu$. Arguing as above it is shown that

$$\frac{(-1)^{\mu}}{\lambda!} \operatorname{Im} \sum_{|\alpha|=\lambda} P^{\alpha}(0,0)\, \xi_{\alpha} P^{(\nu)}(0,0) = O(\rho^{\lambda+1});$$

hence $P^{\alpha}(0,0) = 0$, $|\alpha| = \lambda$. This completes the proof of Lemma 3.8.

4. Proof of Theorems. Theorem 2.1 is proved by induction using the induction hypothesis (assuming $\tau_o = 0$)

$$I(\lambda): \quad P^{(k)\alpha}(0,0) = 0, \quad |\alpha| \le \lambda - 1, \; k + |\alpha| \le \nu - 1.$$

$I(1)$ is simply the statement that P has a zero of multiplicity ν at $(0,0)$. The induction step is provided by Lemma 3.8.

To prove Theorem 2.2, we assume, as we may, that $N = (1,0,\ldots,0)$. Suppose that the polynomial

$$P(\tau,\xi) = \sum_{k=0}^{m} P_k(\tau,\xi)$$

is hyperbolic and that P_m has a zero of multiplicity ν at the point (τ_o,ξ_o). Consider the polynomial

$$\tilde{P}(\tau,\xi,\epsilon) \equiv \epsilon^m P(\tau/\epsilon,\xi/\epsilon) = \sum_{k=0}^{m} \epsilon^k P_{m-k}(\tau,\xi).$$

If we write P_m in factored form

$$P_m(\tau,\xi) = \prod_{k=1}^{m} (\tau - \tau_k(\xi))$$

we see that the roots σ_k of $\tilde{P}$ are $\epsilon\tau_k(\xi/\epsilon)$. Since P is hyperbolic we then have $|\operatorname{Im} \tau_k| \le M$; hence $\operatorname{Im} \sigma_k = O(\epsilon)$. We notice that $\tilde{P}(\tau,\xi_o,0) = P_m(\tau,\xi_o)$ so $\tilde{P}$ has a zero of multiplicity ν at $\xi = \xi_o$, $\epsilon = 0$. Hence, by Theorem 2.1, $\tilde{P}(\tau,\xi,\epsilon)$ has a zero of order ν at $(\tau_o,\xi_o,0)$; that is

$$(\frac{\partial}{\partial\epsilon})^r \tilde{P}^{(k)\alpha}(\tau_o,\xi_o,0) = 0,\ k + |\alpha| + r < \nu.$$

But this implies that

$$r!\ P^{(k)\alpha}_{m-r}(\tau_o,\xi_o) = 0,\ k + |\alpha| < \nu - r,$$

which is just what is meant by saying that P_{m-r} has a zero of order $\nu - r$ at (τ_o,ξ_o). This completes the proof of Theorem 2.2.

We turn now to the proof of Theorem 2.3. Let the multiplicity of P_m at (τ_o,ξ_o) be ν. By Theorem 2.2, P_m then has a zero of order ν at (τ_o,ξ_o). It is then a consequence of Lemmas (3.5) and (3.6) that

$$Q_{\nu-1} = O((\tau - \tau_o)^2 + |\xi - \xi_o|^2) \tag{4.1}$$

near (τ_o,ξ_o). The assumption that the roots $\tau_1(\xi),\ldots,\tau_m(\xi)$ are in non-decreasing order implies that they are continuous. Let us now re-order them so that $\tau_1(\xi),\ldots,\tau_\nu(\xi)$ are the roots which coalesce at ξ_o. Now $Q_{\nu-1}$ is a sum of products of factors $|\tau - \tau_k(\xi)|^2$ taken $m - \nu + 1$ at a time. One of these products is

$$|\tau - \tau_1(\xi)|^2 \prod_{j=\nu+1}^{m} |\tau - \tau_j(\xi)|^2. \tag{4.2}$$

By continuity there exists an $m > 0$ such that

$$\prod_{j=\nu+1}^{m} |\tau \quad \tau_j(\xi)|^2 \geq m \tag{4.3}$$

for τ,ξ in a neighborhood of (τ_o,ξ_o). We deduce from (4.1), (4.2) and (4.3) that

$$|\tau - \tau_1(\xi)|^2 = O((\tau - \tau_o)^2 + |\xi - \xi_o)^2),(\tau,\xi) \quad (\tau_o,\xi_o)$$

The required result is obtained by setting $\tau = \tau_o = \tau_1(\xi_o)$. Since the above argument can be repeated with $\tau_1(\xi)$ replaced by $\tau_k(\xi)$, $k \leq \nu$ we have

$$|\tau_k(\xi) - \tau_o(\xi)| \leq K|\xi - \xi_o|.$$

That the constant K is uniform over compact subsets of a set where P_m has multiplicity ν is a consequence of the uniformity of the O constant in (4.1) and of m in (4.3).

We next prove Theorem 2.4. Let

$$P(\tau,\xi) = P_2(\tau,\xi) + P_1(\tau,\xi) + P_o(\tau,\xi)$$

where P_k is homogeneous of degree k in (τ,ξ) and $P_2(\tau,\xi)$ is hyperbolic. Let

$$L_1(P_2;\epsilon,\xi) = (\tau - \tau_1(\xi))^2 + (\tau - \tau_2(\xi))^2.$$

By the theory of symmetric functions, or for that matter by direct computation, $L_1(P_2;\tau,\xi)$ is a polynomial in (τ,ξ). Since it is clearly homogeneous it is a quadratic form in (τ,ξ);

$$L_1(P_2;\tau,\xi) = Q(\tau,\xi), \quad \tau\epsilon R_1, \xi\epsilon R_n.$$

Now choose a coordinate system such that

$$Q(\tau,\xi) = \eta_1^2 + \ldots + \eta_r^2, \quad r \leq n + 1.$$

It follows that, in this coordinate system

$$P_1(\tau,\xi) = \sum_{j=1}^{r} a_j\eta_j.$$

Otherwise P_1 would be non-zero at a point where P_2 has a double zero. Hence

$$|P_1(\tau,\xi)|^2 \leq K\, L_1(P_2;\tau,\xi),$$

for some constant K. It then follows from the result of McCarthy and Pederson [5] that P is hyperbolic. This completes the proofs.

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