EXISTENCE AND NONEXISTENCE IN THE LARGE OF SOLUTIONS TO QUASILINEAR WAVE EQUATIONS

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1. Introduction

This paper concerns solutions of the non-linear hyperbolic equation

$$Q^2(w_x)w_{xx} = w_{tt}. \tag{1.1}$$

Using methods developed by Ludford [5], Zabusky [8] studied this equation for Q having the special form

$$Q^2(\xi) = (1 + \epsilon\xi),$$

and for the boundary conditions,

$$w(0,t) = w(L,t) = 0, \quad t > 0, \tag{1.2}$$

$$w(x,0) = w_0(x), \quad 0 \leq x \leq L, \tag{1.3}$$

$$w_t(x,0) = 0, \quad 0 \leq x \leq L. \tag{1.4}$$

Zabusky examined this problem as a model for a non-linear string. He observed that any solution must eventually break down in the sense that some second derivative becomes infinite. This result as well as the techniques used to prove it, namely the method of Riemann invariants, have analogs in Ludford's work [5] on the one-dimensional motion of a polytropic gas.‡

Using methods which differ from those of Ludford and Zabusky, Lax [3] extended the breakdown result to a general

‡During the process of revision our attention was called to work of R. E. Meyer [6]. The class of equations considered there does not include (1.1).

positive function Q subject to the condition

$$|Q'(\xi)| \geq m > 0. \tag{1.5}$$

Here we study (1.1) under the following assumptions on Q and boundary conditions on w:

$$Q(\xi) > 0 \quad \text{and } Q(0) = 1, \tag{Q_1}$$

$$Q'(\xi) \lesseqgtr 0 \qquad \text{for} \quad \xi \gtrless 0, \tag{Q_2}$$

$$w(0,t) = w(L,t) = 0,$$
$$w(x,0) \equiv 0,\ w_t(x,0) = w_1(x),\ 0 \leq x \leq L. \tag{1.6}$$

In order to simplify the calculations we impose convexity conditions on $w_1(x)$ (see conditions on $f(x)$ in Section 2), but it is pointed out in Section 3 how our results can be extended to broader classes of functions $w_1(x)$.

It is interesting to compare equation (1.1) with the usual linear wave equation in which Q is a constant. In that case the conditions (1.2), (1.3) and (1.4) correspond to a unique solution existing for all time. This solution and its derivatives up to any order k can be bounded in terms of bounds for derivatives of w_o having corresponding order. Moreover all solutions are periodic and possess a common period depending only on the constants Q and L. For the non-linear equation our results like those of Lax and Zabusky show that global solutions for all x, $0 \leq x \leq L$, almost never exist. This of course eliminates the possibility of periodic solutions.

However certain boundedness results for the linear wave equation are maintained. It turns out that wherever the solution exists w_t is bounded by the same bound as for the linear case (see (2.8a)), while w_x may or may not be bounded depending on the nature of Q.

A physical application of (1.1) occurs in the theory of one-dimensional motions of a gas in material coordinates. In that theory Q^2 is related to the sound speed and Lax's assumption (1.5) is satisfied. On the other hand the equation also serves to describe one-dimensional motions in elasticity. This is pointed out in [1] for shearing motions in general incompressible isotropic simple elastic materials in finite strain. Application to longitudinal vibrations in elastic materials satisfying certain special constitutive assemptions is discussed in [2] §§97-98. In both of these applications condition (1.5) is unsatisfactory, while (Q_1) and (Q_2) are quite natural.

One purpose of this paper is to extend the Ludford-Zabusky-Lax result to general Q's satisfying (Q_1) and (Q_2). The techniques again involve Riemann invariants. In the course of our study it was observed that there are certain differences in behavior of solutions of (1.1) corresponding to differences in the nature of Q. These differences concern the regions of the x, t plane in which smooth solutions do exist, and a discussion of them forms the second aim of the paper.

The classical boundary value theory for equation (1.1) on a strip establishes the existence of a solution for all x in $0 \leq x \leq L$ for some interval $0 \leq t \leq T$. For a given x, of course, the solution may exist for a much longer time; the results described here concern the time intervals of existence for different x-values. It turns out that if Q satisfies appropriate conditions there can exist intervals of x in which the solution must exist for all time even though it does break down for some x-values outside these intervals.

The crucial condition on Q for existence of such x-intervals is convergence of the integrals

$$\int_0^{\infty} Q(\xi)\,d\xi = a_1, \quad \int_0^{-\infty} Q(\xi)\,d\xi = -a_2. \tag{1.7}$$

Our methods yield initial data leading to x-intervals of global existence only if either a_1 or a_2 is finite. A novel feature of our theorems is that they establish existence in arbitrarily large time intervals only for solutions w which have the property that the values of w_x become unbounded. (Our conditions on $w_1(x)$ will exclude the solution $w(x,t) \equiv 0$, for example).

Our purpose in considering the initial conditions (1.6) rather than those of Zabusky is as follows. The breakdown result remains true if one retains (1.3) and replaces (1.4) by the more general condition $w_t(x,0) = w_1(x)$. Moreover, if either a_1 or a_2 is finite then infinite time intervals of existence occur for large classes of initial data, cases in which $w_1(x) \equiv 0$ being an exception. Hence Zabusky's initial conditions are special in this regard. We have adopted (1.6) in order to illustrate the possibility of infinite intervals of existence and still keep the computations simple.

2. Statement of results

For orientation we begin our discussion with some standard facts about the method of Riemann invariants. Consider the hyperbolic quasi-linear system of two equations,

$$L[U] = U_t - A(U)U_x = 0, \qquad U = (u,v). \tag{2.1}$$

Suppose that U is a continuous function $U = \underline{\underline{U}}(R)$ of $R = (r,s)$. If the transformation $\underline{\underline{U}}$ is one-to-one and has nonvanishing Jacobian then it is possible to invert so as to obtain $R = \underline{\underline{R}}(U)$, and by introducing $R(x,t) = R(U(x,t))$ we may write (2.1) in the form

$$R_t - \underline{\underline{A}}(R)R_x = 0, \tag{2.2a}$$

where

$$\underline{\underline{A}}(R) = (\nabla\underline{\underline{U}})^{-1} A(\underline{\underline{U}}(R))\nabla\underline{\underline{U}}, \qquad \nabla\underline{\underline{U}} = \begin{pmatrix} u_r & u_s \\ v_r & v_s \end{pmatrix}. \tag{2.2b}$$

When $\underline{\underline{U}}(R)$ is chosen so that $\underline{\underline{A}}(R)$ is diagonal (this is always possible after multiplication by integrating factors) then the functions r,s are called Riemann invariants.

The method of Riemann invariants consists in interchanging the roles of R and X = (x,t) in (2.2). This means we assume that R=R(X) is one-to-one and has nonvanishing Jacobian. Then X= $\underline{\underline{X}}(R)$ and (2.2) becomes

$$x_r + d_2 t_r = 0, \quad x_s + d_1 t_s = 0, \tag{2.3a}$$

with

$$\underline{\underline{A}}(R) = \begin{pmatrix} d_1 & 0 \\ 1 & d_2 \end{pmatrix}, \quad d_1 \neq d_2. \tag{2.3b}$$

In (2.3) d_1 and d_2 are functions of r and s so that these equations are linear.

The idea now is to solve (2.3) for $X = \underline{\underline{X}}(R)$, invert to obtain R = R(X) and then obtain a solution of (2.1) by means of the formula

$$U(X) = \underline{\underline{U}}(R(X)). \tag{2.4}$$

The following well known and easily proved result shows the sense in which this is a valid procedure. Note that "region" always means open region.

<u>Lemma 1</u>. <u>Let</u> X(R) <u>denote a solution of</u> (2.3). <u>If</u> Ω' <u>is a region of the</u> R <u>plane in which the mapping</u> X = X(R) <u>is one-to-one onto a region</u> $\underline{\underline{\mathcal{D}}}$ <u>and</u> $J(X) = \det \nabla_R X \neq 0$, <u>then</u> (2.4) <u>yields a solution of</u> (2.1) <u>on</u> $\underline{\underline{\mathcal{D}}}$.

We emphasize here that the theorem requires both that the mapping be globally one-to-one and that $J \neq 0$. The latter condition yields only local invertibility. It is the global property which is hard to prove in general. Observe that equations (2.2), (2.3) and the transformation formulas

$$dr = \frac{t_s dx - x_s dt}{J}, \quad ds = \frac{-t_r dx + x_r dt}{J}, \tag{2.5}$$

yield immediately the following relations:

$$J = (d_1 - d_2) t_r t_s, \tag{2.6a}$$

$$j = \det(\nabla_x R(X)) = J^{-1} = (d_2 - d_1) r_x s_x. \tag{2.6b}$$

These formulas yield immediately the following supplementary result.

<u>Lemma 2. Let</u> $X(R)$ <u>and</u> Ω' <u>satisfy the conditions of Lemma 1 and assume in addition that</u> $X = X(R)$ <u>is continuous on</u> $\overline{\Omega'}$. <u>Then if</u> t_r <u>or</u> t_s <u>approaches</u> 0 <u>as</u> $R \to R_o \in \partial\Omega'$, <u>the function</u> $U(X)$ <u>defined by</u> (2.4) <u>satisfies</u> $u_x(X(R)) \to \infty$ <u>or</u> $v_x(X(R)) \to \infty$ <u>as</u> $R \to R_o$.

In order to apply these lemmas we must translate boundary and initial conditions on U as a function of X into conditions on X as a function of R. The resulting <u>linear</u> problem for $X(R)$ is then solved in a portion Ω of the R plane. For that part of Ω for which $X(R)$ is one-to-one and $J \neq 0$ we then obtain a solution of (2.1).

It turns out in the problems we consider that the non-vanishing of J is sufficient to guarantee that $X(R)$ is globally one-to-one. Thus, by (2.3a) and (2.6a) we can obtain existence theorems for (2.1) by solving linear equations for $X(R)$ and showing that the derivatives t_r and t_s do not change sign. Note that the domains of existence one obtains are automatically determined by the linear problem for $X(R)$.

Let Ω_o be the image of the U-plane under the mapping $R = \underline{\underline{R}}(U)$, and let Ω be the sub-region in which $X(R)$ is determined. The initial curves on which the data for $X(R)$ are given will of course be in Ω_o. In the problems we consider these initial curves are of finite length and consequently Ω will be bounded. There are then two possibilities. First $\bar{\Omega} \subset \Omega_o$ in which case one gets bounds for U simply in terms of the size of Ω. The other case, however, is that $\bar{\Omega} \cap \partial \Omega_o$ may be non-empty. Then it will follow that U can become arbitrarily large in its domain of existence. This latter case corresponds to $a_1 < \infty$ or $a_2 < \infty$ in (1.7). We remark that when $a_1 = \infty$ and $a_2 = \infty$, as is the case in gas dynamics problems, Ω_o is all of the R-plane so that if one starts with finite data then U always possesses a priori bounds.

We now state our problem precisely. We formulate it as a system for $w_x = u$, $w_t = v$. We seek a solution $U(x,t) = (u(x,t),\ v(x,t))$ of the equation

$$U_t - A(U)U_x = 0 \qquad \text{(E)}$$

when

$$A(U) = \begin{pmatrix} 0 & 1 \\ Q^2(u) & 0 \end{pmatrix},$$

where Q satisfies conditions (Q_1) and (Q_2) of Section 1. U is to satisfy the conditions,

$$v(0,t) = v(L,t) \equiv 0, \qquad t \geq 0, \qquad \text{(A)}$$

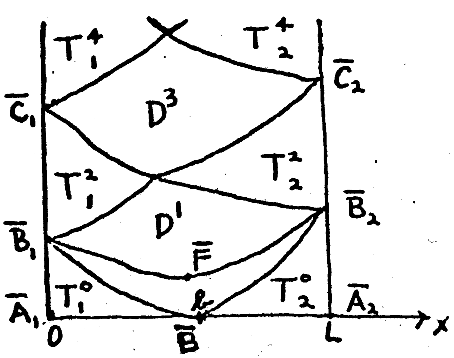
$$u(x,0) \equiv 0, \qquad 0 \leq x \leq L, \qquad \text{(B)}$$

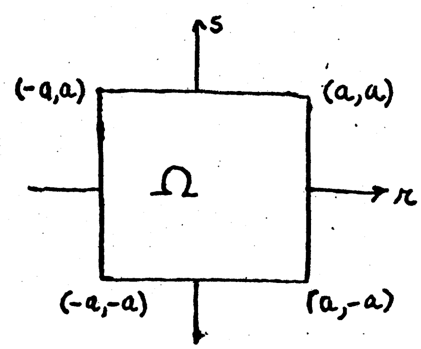
$$v(x,0) = f(x), \qquad 0 \leq x \leq L. \qquad \text{(C)}$$

It is easy to see that this is equivalent to (1.1) and (1.6). Concerning f(x) we assume that it is twice differentiable and concave and vanishes at x = 0 and x = L. Under these conditions it will have a unique maximum value a > 0 at some point b in (0,L). Some remarks on more general boundary conditions are contained in Section 3.

Let X(R) be a solution of (2.3) in a region Ω. We denote by $D(X,\Omega)$ the subregion of Ω in which $J(X) \neq 0$ and by $S(X,\Omega)$ the image of $D(X,\Omega)$ under X. $U(X,\Omega)$ denotes the function (2.4). Our results can be now stated with reference to the figures below. The situations mentioned in the introduction can be dealt with as three separate cases.

<u>Case</u> 1. $a < \min(a_1,a_2) \leq \infty$.





<u>Figure</u> 1.

Theorem 1. *If Q and f satisfy the conditions of Case 1 there exist functions X^1 and X_i^o, $i = 1,2$ defined on Ω such that:*

(1) $S(X_i^o, \Omega)$ *contains some triangle of the form T_i^o of Figure 1, and the associated mapping is one-to-one.*

(2) $S(X^1, \Omega)$ *contains some region of the form B_1BB_2F of Figure 1.*

(3) *The function* U *defined by*

$$U(X) = \begin{cases} U_i^o(X) & \text{in} \quad T_i^o \quad i = 1,2 \\ U^1(X) & \text{in} \quad B_1BB_2F \end{cases},$$

is a smooth solution of (E) *satisfying* (A),(B),(C).

Remark 1. The curves separating the T_i^o from D^1 are characteristics of (E) which are images under an X_i^o of one of the sides of Ω.

Remark 2. As we have indicated before, we obtain bounds for the solutions immediately. For equation (E) the Riemann invariants are

$$r = v + M(u), \quad s = v - M(u), \quad M(u) = \int_o^u Q(\xi)\,d\xi. \tag{2.7}$$

Since by Figure 1 $|r| \leq a$ and $|s| \leq a$ we have then

$$|v| \leq a, \quad |u| \leq m(a), \tag{2.8a}$$

where $m(a)$ is defined by

$$\int_o^{m(a)} Q(\xi)\,d\xi = a \tag{2.8b}$$

Remark 3. It can happen that $D(X^1,\Omega) = \Omega$, $S(X^1,\Omega) = D^1$ and that the mapping $\Omega \to D^1$ is one-to-one. Then of course the solution U is extended to all of D^1. One can then proceed to T_i^2 in a fairly obvious fashion (see Section 5). The process can be continued to $D^3, T_i^4, \ldots$ until some second derivative becomes infinite. The linear problems to be solved to obtain these successive continuations are all in the same square Ω so that the bounds (2.8) obtain as long as the solution exists. The interesting fact is that one need not actually solve any further problems. All the remaining solutions, which we denote here by X^{2n+1} and X_i^{2n} $i = 1,2,$ can be obtained from the functions X_i^o and X^1. This is the content of the next theorem and is our statement of the idea of Ludford and Zabusky.

Theorem 2. Under the hypotheses of Theorem 1 there exist linear combinations $\Psi(R)$, $\Psi'(R)$ and $\Psi_i'(R)$, $i = 1,2$, of the functions $X_i^o(R)$, $X^1(R)$ and $X^1(R^*)$ $[R^* = (-s, -r)]$, such that the successive solutions $X^{2n+1}(R)$ $X_i^{2n}(R)$ of (2.3) on Ω can be expressed in the form

$$X^{2n+1} = n\Psi + \Psi', \quad X_i^{2n} = n\Psi + \Psi_i', \quad n = 1,2.. \quad . \tag{2.9}$$

The crucial fact about the representation (2.9) is the following:

Lemma 3. The function $\det \nabla_R \Psi$ changes sign in Ω.

It follows from Lemma 3 that $\det \nabla_R X^{2n+1}$ must change sign in Ω for n sufficiently large. Hence it is 0 someplace in Ω and by Lemma 2 the solution must break down. This yields

Theorem 3. If Q and f satisfy the conditions for Case 1 some second derivative of the solution must eventually become infinite.

Case 2. $a > \max(a_1, a_2)$

Note that in this case the image Ω_o of the U plane is the infinite strip $-2a_2 < f-s < 2a_1$.

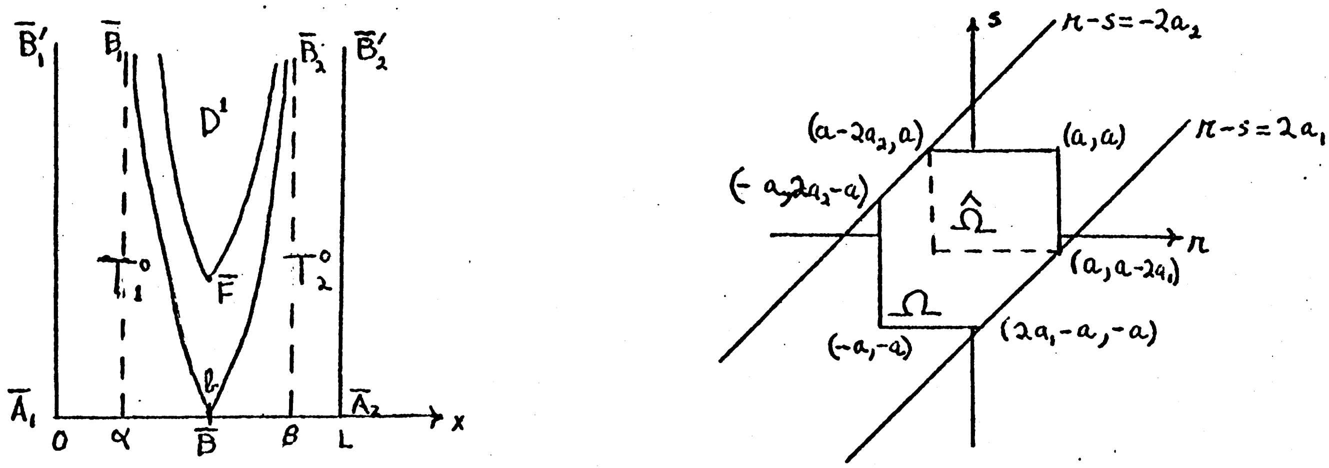


Figure 2.

Theorem 4. If Q and f satisfy the conditions for Case 2 there exist functions X_i^o defined on Ω and X^1 defined on a subset $\hat{\Omega}$ of Ω such that the conclusions of Theorems 1 and 3 remain true. The curves $\bar{B}\bar{B}_1$, $\bar{B}\bar{B}_2$ bounding T_1^o and T_2^o tend asymptotically to lines $x = \alpha$ and $x = \beta$, where $0 < \alpha < \beta < L$.

Remark 4. Observe that here we obtain global existence (i.e. for all time) in the intervals $[0,\alpha]$ and $[\beta,L]$.

Remark 5. Note by (2.7) and Figure 2 that the first inequality in (2.8a) continues to hold. However the second inequality is now meaningless, and indeed we shall find that u is unbounded in T_i^o, $i=1,2$. On the other hand for each $\tau \in (0,\infty)$ u is bounded in the truncated triangles $T_i^o \cap (t < \tau)$, $i=1,2$

Case 3. $\min(a_1,a_2) \le a \le \max(a_1,a_2)$.

For this case the geometrical situation is intermediate between those depicted in Figures 1 and 2. When the right hand inequality is strict one of the tiangles T_i^o is bounded, the other is unbounded. Thus global existence of U is obtained only in one interval (which is a degenerate one if equality holds on the left). When equality holds on the right, both of the triangles T_i^o are unbounded, but global existence still obtains in at most one nondegenerate interval. For brevity we will not indulge in a detailed analysis of Case 3.

3. The linear problems.

In this section we formulate the linear problems for X(R) in the R-plane. This means first the translation of conditions (A), (B), (C) into conditons on X. For the special choice of A(U) occuring in (E) equations (3.2) become

$$x_r - qt_r = 0, \; x_s + qt_s = 0 \qquad q = Q(M^{-1}(\tfrac{r-s}{2})), \tag{3.1}$$

with M(u) being defined as in (2.7). If we eliminate x from these equations we find,

$$t_{rs} = \rho(r-s)(t_r - t_s), \tag{3.2}$$

where $\rho(\xi) = Q'(M^{-1}(\xi/2))/4q^2(\xi)$. Note that conditions (Q_1)

and (Q_2) yield

$$\rho(\xi) \lesseqgtr 0 \quad \text{for} \quad \xi \gtreqless 0. \tag{3.3}$$

Equations (2.2) become in the present case,

$$r_t - qr_x = 0, \quad s_t + Qs_x = 0. \tag{3.4}$$

Condition (B) states that $t = 0$ corresponds to $u = 0$ or $r = s$. Hence

$$t_r(r,r) + t_s(r,r) = 0 \tag{3.5}$$

In addition $x = x(r,s)$ hence by (C)

$$1 = x_r r_x + x_s s_x = f'(x)(x_r + x_s) \quad \text{on} \quad r = s \tag{3.6}$$

Now $q = Q(0) = 1$ on $r = s$ hence by (3.1) and (3.6)

$$1 = f'(x)(t_r - t_s) \quad \text{on } r = s \tag{3.7}$$

Again we have $v = r = f(x)$ on $r = s$. Here we must use our assumptions on $f(x)$ for we wish to invert this equation. Let us denote the ascending and descending portions of the concave function $f(x)$ by $\varphi_1(x)$ and $\varphi_2(x)$ respectively. Thus

$$f(x) = \varphi_1(x) \quad 0 \leq x \leq b, \quad \varphi_1'(x) > 0 \quad 0 \leq x < b, \quad \varphi_1'(b) = 0 \tag{f_1}$$

$$f(x) = \varphi_2(x) \quad b \leq x \leq L, \quad \varphi_2'(x) < L \quad b < x \leq L, \quad \varphi_2'(b) = 0 \tag{f_2}$$

Now, corresponding to the interval $\bar{A}_1\bar{B}$ of Figure 1 we have by (f_1),

$$x(r,r) = \varphi_1^{-1}(r), \quad 0 \le r \le a. \tag{3.8}$$

Then by (3.1), (3.5) and (3.7)

$$t_r(r,r) = -t_s(r,r) = \frac{1}{2}\,[\varphi_1'(x(r))]^{-1} = \frac{1}{2}(\varphi_1^{-1}\,(r))', \; 0 \le r \le a. \tag{3.9}$$

On the interval $\bar{B}\bar{A}_2$ we have

$$x(r,r) = \varphi_2^{-1}\,(r), \quad 0 \le r \le a, \tag{3.10}$$

and

$$t_r(r,r) = -t_s\,(r,r) = \frac{1}{2}(\varphi_2^{-1}(r))', \quad 0 \le r \le a. \tag{3.11}$$

Consider now the first of conditions (A). Here $v = 0$ so that the image of $x = 0$ is $r = -s$. Thus $x(r,-r) = 0$ or $x_r(r,-r) - x_s(r,-r) = 0$. Hence by (3.1)

$$t_r(r,-r) + t_s(r,-r) = 0. \tag{3.12}$$

The second of conditions (A) yields the same condition for t.

Observe that conditions (3.9) imply that $t_r > 0, t_s < 0$ while in (3.11) the signs are reversed. We wish t to increase as we move into the R region in which (3.2) is to be solved.

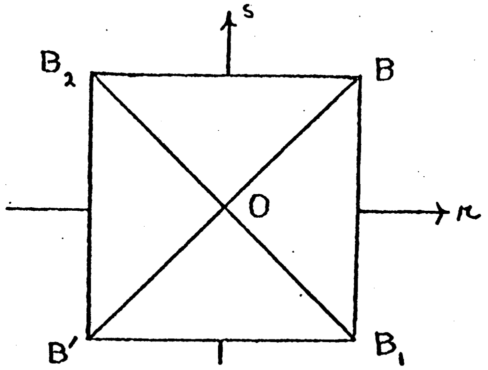


Figure 3.

What suggests itself for Case 1 is the following. We determine $t_1^o(r,s)$ as a solution of (3.2) in $\Omega_1 = OBB_1$ with (3.9) on OB and (3.12) on OB_1; $t_2^o(r,s)$ is a solution of (3.2) in $\Omega_2 = OBB_2$ with (3.11) on OB and (3.12) on OB_2. The images of the regions OBB_i will then be T_i^o.

The fact that q is a function only of $r-s$ means that the equation (3.2) remains invariant under the change $(r,s) \longrightarrow (-s,-r)$. Hence each of the above problems can be extended to a doubled triangle Ω_i'. Define functions $\Phi_i(r)$ on $-a \leq r \leq a$ by

$$\Phi_1(-r) = -\Phi_1(r), \quad \Phi_2(-r) = 2L - \Phi_2(r), \quad \Phi_i(r) = \frac{1}{2}\varphi_i^{-1}(r), 0 \leq r \leq a.$$

Then the above problems for t_i^o are equivalent to the following:

$$t_{i,rs}^o = \rho(r-s)(t_{i,r}^o - t_{i,s}^o) \quad \text{in} \quad \Omega_1 = B'BB_i, \tag{3.13}$$

$$t_{i,r}^o(r,r) = -t_{i,s}^o(r,r) = \Phi_i'(r) \quad \text{on} \quad B'B. \tag{3.14}$$

Remark 6. For future reference we observe that the t_i^o can be extended to the reflection of Ω_i in $r = s$ by simply solving the initial-value problem (3.13), (3.14) for (r,s) on the opposite side of $r = s$.

Once the t values are determined the x values can be found from (3.1), (3.8) and (3.10). The appropriate equations are,

$$x_i^o(r,s) = 2\Phi_i(r) - \int_r^s q(r - s')t^o_{i,s}(r,s')ds', \quad i=1,2. \quad (3.15)$$

Case 2 is treated in the same way except that the appropriate regions in the R plane are trapezoids instead of triangles as indicated in Figure 2.

In order to obtain next the solution $t^1(r,s)$ corresponding to D^1 we solve a characteristic initial-value problem for (3.2). We take as data for t^1 the values of t_1^o on $r = a$ and the values of t_2^o on $s = a$. Here we must distinguish between the two cases. In Case 1, t_1 is obtained in all of the region Ω of Figure 1 with

$$\begin{aligned} t^1(a,s) &= t_1^o(a,s), \quad -a \leq s \leq +a \\ t^1(r,a) &= t_2^o(r,a), \quad -a \leq r \leq +a \end{aligned} \quad (3.16)$$

In Case 2, t^1 is obtained in the sub-rectangle Ω_1, $r > a - 2a_2$, $s > a - 2a_1$ of Ω (see Figure 2) and (3.16) holds on $a - 2a_1 < s < a$, $a - 2a_2 < r < a$ respectively. The function x^1 is obtained again from (3.1) this time in the form

$$x^1(r,s) = x_2^o(r,a) - \int_a^s q(r-s')t_s^1(r,s')ds'. \quad (3.17)$$

The solutions corresponding to the succeeding regions T_i^{2n} and D^{2n+1} in Case 1 (there are none for Case 2) are obtained by essentially the same processes. For the t_i^{2n} we match the t values on one characteristic edge of Ω to those assumed by t^{2n-1} (obtained in the preceding step), and use conditions (A) to obtain (3.12) on $r = -s$. The solutions t_i^{2n} will alternately be obtained in $\Omega_3 = B_1B'B_2$ and $\Omega_4 = B_1BB_2$ (see Figure 3). For the t^{2n+1} we match t values with t_1^{2n} on one characteristic side of Ω, and with t_2^{2n} on another characteristic side of Ω.

We end this section with some comments concerning the solution of (E) for more general initial functions than we have specified.

Remark 7. Suppose that $u(x,0) \equiv 0$ but that the initial function $v(x,0)$ has several maxima and minima at, say, $x_1, \ldots, x_n$. Then one can simply solve the problem in sections as indicated in Figure 4. The linear problem corresponding to

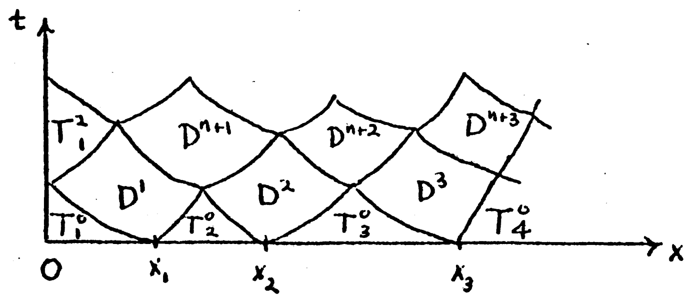


Figure 4.

T_1^o will be exactly as above. The problems corresponding to T_2^o, T_3^o, ... will be of the same type except that the data on $r = s$ will not be antisymmetric with respect to $r = 0$. The linear problems for D^1, D^2,... will be characteristic problems in rectangles with data taken from pairs of the triangles T_k^o. The continuations to T_1^2, D^{n+1}, D^{n+2},... are then dealt with in a strictly analogous way.

Remark 8. If both u and v are initially non-zero the general procedure is the same as that given in the present section except that the initial curves in the r,s plane are not straight lines. We shall not pursue this situation here, but it is possible to obtain results analogous to those we have given.

Remark 9. We want to comment on the special conditions of Zabusky and Lax, that is,

$$u(x,0) = f(x), \quad v(x,0) = 0. \tag{3.20}$$

This problem must be handled somewhat differently since we note that the image of the line $t = 0$ in the R-plane is $r = -s$, which is the same as the image of $x = 0$ or $x = L$ where $v \equiv 0$ also. Assume again that $f(x)$ has a single maximum at $x = b$. Then the appropriate procedure is as indicated in Figure 5. The solutions in T_1^0 and T_2^0 are obtained first by

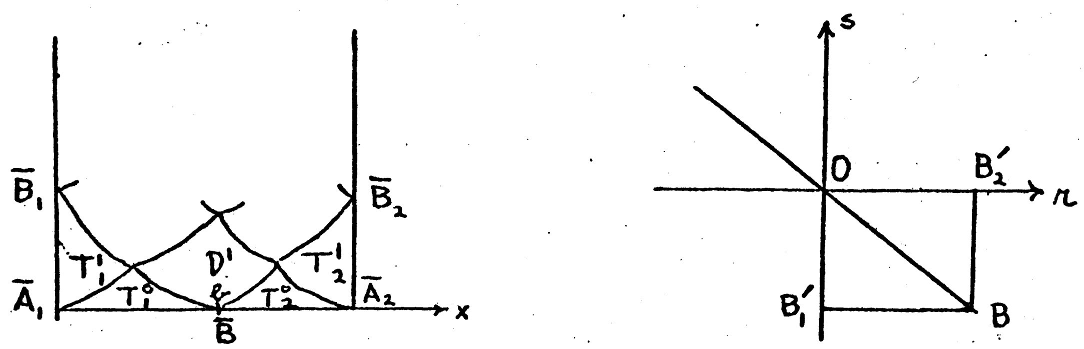


Figure 5.

solving initial-value problems in OBB_1' and OBB_2' respectively with data given along OB. B is the point $(\bar{r}, -\bar{r})$ where

$$\bar{r} = \int_0^{f(b)} Q(\xi)\, d\xi \, . \tag{3.21}$$

The solutions in D^1 and T_i^1 are then obtained in the same way as those in D^1 and T_i^2 in our problem.

The interesting difference between our problem and that of Zabusky is that in the latter problem, Case 2 does not lead to unbounded solutions. Indeed observe that in Case 2 $\bar{r} < a_1$ by (3.21). Hence the triangles OBB_1' and OBB'_2 lie entirely within $-2a_2 < f-s < 2a_1$ that is within Ω_o so that both u and v have a priori bounds. A consequence of these bounds is of course that the t(r,s) values remain bounded, so that one cannot infer the existence of a solution for all time for any interval of x.

4. Existence of Solutions in T_i^o and D^1

We consider the function t_1^o which in Case 1 satisfies (3.13) and (3.14) on the triangular region $\Omega_1 = B'BB_1$ (Figure 3), or in Case 2 satisfies these equations on the trapezoidal region $\Omega_1 = B'BB_1B_1'$ (Figure 6).

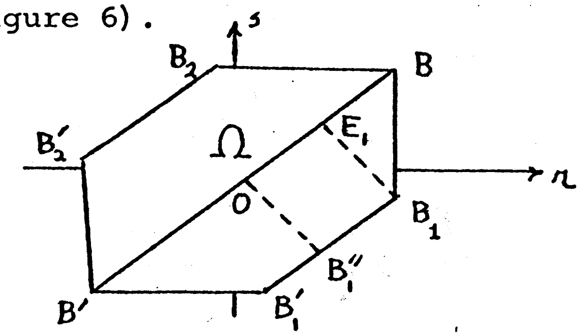


Figure 6.

That such a function exists and is uniquely determined is a standard result. We proceed to show that the function $x_1^o(r,s)$ defined by (3.15) satisfies (3.1). From (3.15) it follows directly that (3.1b) holds:

$$x_{1,s}^o = -qt_{1,s}^o.$$

Note moreover that

$$x_{1,r}^o = 2\Phi_1'(r) + t_{1,s}^o(r,r) - \int_r^s (q(r-s')t_{1,s}^o(r,s'))_r ds'. \tag{4.1}$$

By (3.14)

$$t_{1,s}^o(r,r) = -\Phi_1'(r), \tag{4.2}$$

while (3.2) may be wirtten in the form

$$(qt_{1,s}^o)_r = -(qt_{1,r}^o)_s. \tag{4.3}$$

Using (4.2) and (4.3) we may simplify (4.1) to

$$\begin{aligned} x_{1,r}^o &= \Phi_1'(r) + \int_r^s (q(r-s')t_{1,r}^o(r,s'))_{s'} ds' \\ &= q(r-s)t_{1,r}^o(r,s), \end{aligned} \tag{4.4}$$

which is (3.1a). Therefore since (3.8) and (3.9) clearly hold, the functions (x_1^o, t_1^o) as defined in (3.13), (3.14), and (3.15) constitute X_1^o. Moreover, as was observed earlier $t_1^o(r,s)$ and hence $x_1^o(r,s)$ may be extended to all of Ω by solving (3.13), (3.14) for (r,s) on the opposite side of $r = s$ and applying (3.15).

Now let Ω_1' denote the part of Ω_1 lying in $r + s > 0$. We want to show that $D(X_1^o, \Omega_1') = \Omega_1'$. T_1^o will then denote the image under X_1^o of Ω_1'.

Lemma 4. Let $k = \min \Phi_1'(\alpha)$ on $-a \leq \alpha \leq a$. Then in Ω_1',

$$t_{1,r}^o > 0 \quad t_{1,s}^o < 0, \quad q(t_{1,r}^o - t_{1,s}^o) > k. \tag{4.5}$$

We recall that $\varphi_1(x)$ is concave so that k is positive. We integrate (3.13) first with respect to s and then with respect to r and we obtain,

$$t_{1,r}^o(r,s) = \Phi_1'(r) - \int_s^r \rho(r-y)(t_{1,r}^o(r,y) - t_{1,s}^o(r,y))dy, \tag{4.6}$$

$$t_{1,s}^o(r,s) = -\Phi_1'(s) + \int_s^r \rho(x-s)(t_{1,r}^o(x,s) - t_{1,s}^o(x,s))dx. \tag{4.7}$$

Note that $\rho(r - y)$ and $\rho(x - s)$ are negative in Ω_1 and $\Phi'(r)$ is a positive even function. Hence, by successive approximations, $t_{1,r}^o > 0$ and $t_{1,s}^o < 0$.

Next we observe from (2.7) and the definition of ρ that

$$\rho = \frac{1}{2} \frac{d}{d(r-s)} \log q(r-s).$$

We denote $t_{1,r}^o - t_{1,s}^o$ by $P(r,s)$. Then if we subtract (4.7) from (4.6) we find,

$$P(r,s) = (\Phi_1'(r) + \Phi_1'(s)) - \frac{1}{2}\int_0^{r-s} \{\frac{d}{dz} \log q(z)\} (P(r,r-z) + P(z+s,s))dz. \tag{4.8}$$

Once more we apply successive approximations. We set

$$P^{o}(r,s) = \Phi_1'(r) + \Phi_1'(s),$$

and we verify by induction that

$$P^{n}(r,s) \geq k \sum_{j=0}^{n} \frac{(-1)^j}{j!} (\log q(r-s))^j.$$

The assertion is true for $n = 0$. Assume it is true for n. Then

$$P^{n+1}(r,s) \geq k - k \sum_{j=0}^{n} \frac{(-1)^j}{j!} \int_0^{r-s} \frac{d}{dz} \log q(z) \; (\log q(z))^j dz$$

$$= k \sum_{j=0}^{n+1} \frac{(-1)^j}{j} (\log q)^j.$$

On the other hand it is easy to see by a standard successive approximations analysis that the sequence $P^n(r,s)$ converges uniformly to some function $P(r,s)$ which thereby is the unique solution of (4.8). Hence we conclude that

$$P \geq ke^{-\log q} = k/q,$$

and this proves the last estimate of (4.5).

Lemma 4 and equation (2.6a) show that indeed $D(X_1^o,\Omega_1') = \Omega_1'$ and we proceed to the verification that $S(X^o,\Omega_1') = T_1^o$. As a first step we prove that $t_1^o(r,s)$ is positive in Ω_1'. We integrate (4.2), taking $t_1^o(r,r)$ to be zero and using (4.6). This yields,

$$t_1^o(r,s) = \Phi_1(r) - \Phi_1(s) - \int_s^r \int_s^x \rho(x-y)P(x,y)\,dy\,dx$$

$$\geq \Phi_1(r) - \Phi_1(s) - k\int_s^r \int_s^x \frac{\rho(x-y)}{q(x-y)}\,dy\,dx \qquad (4.9a)$$

$$= \Phi_1(r) - \Phi_1(s) + kJ(r-s),$$

where

$$J(z) = -z + \int_0^z \frac{d\tau}{q(\tau)} = -z + 2u(z). \qquad (4.9b)$$

The second equality in (4.9b) follows from the fact that by (2.7)

$$\frac{du(r-s)}{d(r-s)} = \frac{1}{2q}.$$

Note that $J'(z) = -1 + q(z)^{-1} > 0$ and $J(0) = 0$. Hence $J(z) > 0$ for $z > 0$. Thus

$$t_1^o(r,s) > \Phi_1(r) - \Phi_1(s) \quad \text{for} \quad r > s. \qquad (4.10)$$

Since $\Phi_1(r)$ is increasing it follows that $t_1^o > 0$ in Ω_1 hence certainly in Ω_1'.

Lemma 4 and equations (3.1) show that

$$x_{1,r}^o > 0, \quad x_{1,s}^o > 0 \quad \text{in } \Omega_1'. \qquad (4.11)$$

We recall that $t_1^o(r,s)$ was so constructed as to satisfy

$$t_1^o(r,s) = t_1^o(-s,-r), \tag{4.12}$$

that is $t_{1,r}^o(r,-r) + t_{1,s}^o(r,-r) = 0$. Hence by (3.1)

$$x_{1,r}^o(r,-r) - x_{1,s}^o(r,-r) = 0. \tag{4.13}$$

It follows that $x_1^o(r,-r)$ is constant and since $x_1^o(r,r) = 2\Phi(r)$ we have $x_1^o(0,0) = 0$ and hence $x_1^o(r,-r) \equiv 0$. But then by (4.11) $x_{1,r}^o + x_{1,s}^o > 0$ hence $x_1^o(r,s) > 0$ in Ω_1'.

We can now construct T_1^o. In Case 1 T_1^o is the region in the X plane bounded by the images of OB, OB_1 and BB_1 of Figure 3. Note that the image $\overline{B}\overline{B}_2$ of BB_1 is a characteristic of (2.1) and hence satisfies

$$\frac{dx}{dt} = q(u).$$

Since u decreases as we move down along BB_1 the curve $\overline{B}\overline{B}_1$ in the x - t plane is convex upward as shown in Figure 1 and $\overline{A}_1\overline{B}\overline{B}_1\overline{A}_1$ forms a closed curve. Then the remarks of the preceding paragraphs show that $S(X_1^o,\Omega_1') \subset T_1^o$ and by construction the boundary of Ω_1' is mapped onto the boundary of T_1^o.

We observe next that the mapping from Ω_1' to $S(X_1^o,\Omega_1')$ is one-to-one. Consider two points P and Q in Ω_1' say in the position below.

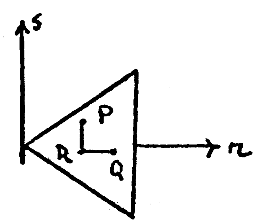


Figure 7.

We use the relations (4.5) and (4.11). Along PR and along RQ t increases hence $t_Q > t_P$. Similar considerations hold for other configurations and one proves $(x_p,t_p) = (x_Q,t_Q)$ only if $P = Q$. Thus $S(X_1^o,\Omega_1') = T_1^o$ and the mapping is one-to-one.

In Case 2 we need a slight modification. The region Ω_1' is now the trapezoid $BOB_1''B_1$ of Figure 6. We deduce, just as before, that the mapping X_1^o: $\Omega_1' \to S(X_1^o,\Omega_1')$ is one-to-one. We prove that $t_1^o \to \infty$ as $R \to R_o$ where R_o is on B_1B_1''. As $R \to R_o$ we have $r - s \to 2a$ hence $u \to \infty$ and hence by (4.9) $t_1^o \to \infty$. We now define T_1^o as the region in the X-plane bounded by the images of OB, BB_1 and OB_1'' under X_1^o.

Observe that Lemma 4 and (3.1) show that

$$x_r + x_s = q(t_r - t_s) > k \quad \text{in } \Omega_1'.$$

Since $x = 0$ on OB_1'' it follows that along any segment starting on OB_1'' and drawn parallel to OB x must increase from 0 to at least $k(a - a_1)$ on B_1E_1. Thus x values in B_1E_1B must be greater than $k(a - a_1)$. In particular if $\alpha = x(B_1)$ we must have $\alpha > k(a - a_1)$. We already know that $t \to \infty$ as $R \to B_1$ along BB_1 hence the image of BB_1 is the characteristic $\bar{B}\,\bar{B}_1$ of Figure 2. Consider the curves $x(r,s) = c$ for $c < k(a-a_1)$. These lie in the rectangle $OB_1''B_1E_1$. They start on OE_1 and end on $B_1''B_1$ and as $R \to B_1''B_1$ along them $t \to \infty$. Hence T_1^o contains at least the infinite strip $0 \leq x < k(a-a_1)$, $0 \leq t < \infty$. Hence we obtain T_1^o as in Figure 2 with $\alpha > k(a - a_1)$.

We know now that the mapping X_1^o is one-to-one hence we can define its inverse $R(X_1^o)$ and form $U_1^o(X) = \underline{U}(R(X))$. By Lemma 1 this satisfies equation (2.1). On $t = 0$ we have $r = s$, hence $u = 0$. Moreover for $0 \leq x \leq b$,

$$v(x,0) = r(x,0) = \Phi_1^{-1}(x) = \varphi_1(x).$$

On $x = 0$ we have $r = -s$ and hence $v = 0$. Thus U_1^o is the desired solution in T_1^o.

The proof for T_2^o proceeds in exactly the same way and we omit the details. The only change is that Lemma 4 becomes

$$t^o_{2,r}(r,s) < 0, \quad t^o_{2,s}(r,s) > 0. \tag{4.14}$$

Remark 10. If the function $Q(\xi)$ happens to be even the solution in T_2^o can be determined from that in T_1^o by the following simple procedure. Let $t_1^o(r,s;\Phi_1(\cdot))$ denote the solution $t_o^1(r,s)$ in Ω_1' as a functional of Φ_1. Then in Ω_2', the reflection of Ω_1' across $r = s$, one finds

$$t_2^o(r,s) = t_1^o(s,r;\overline{\Phi}_2(\cdot)), \quad \overline{\Phi}_2(r) = L - \Phi_2(r).$$

The proof of the first part of Theorems 1 and 4 is now complete. Moreover we have shown that in Case 2 U is indeed unbounded on T_i^o, $i = 1,2$. Next let us complete the justification of Remark 5 by verifying that u is bounded in each truncated triangle $T_i^o \cap (t < \tau)$. For brevity we again give an explicit analysis only for T_1^o. Here $u \geq 0$ by (2.7) while by (4.9) the condition $u \geq m$ is seen to comply that

$$t_1^o \geq k(-2a_1 + 2m).$$

It follows that in $T_1^o \cap (t < \tau)$ the solution satisfies

$$0 \leq u < a_1 + \frac{\tau}{2k} .$$

Let $t^1(r,s)$ denote the solution of (3.2) satisfying (3.16) in Ω (or $\hat{\Omega}$ in Case 2). If we could show that t_r^1 and t_s^1 do not change sign we would have an existence theorem for D^1. It can be shown by examples that these derivatives can change sign in Ω‡ hence the best that can be obtained in general is an existence theorem in the subregions of D^1 of Figures 1 and 2.

Lemma 5 *There exists a neighborhood of* $r = a$ *and* $s = a$ *in which* $t_r^1 < 0$, $t_s^1 < 0$.

Observe first that (4.5) and (4.14) yield $t_r^1(r,a) = t_{2,r}^o(r,a) < 0$ and $t_s^1(a,s) = t_{1,s}^o(a,s) < 0$. t_r^1 and t_s^1 become infinite near $r = a$ and $s = a$ respectively, and we must study this behavior. First consider the t_i^o. If we apply Riemann's method we obtain using (B) and (3.9),

$$t_1^o(r,s) = \int_s^r R(r',r',r,s)\Phi_1'(r')dr',$$

$$t_{1,r}^o(r,s)) = R(r,r,r,s)\Phi_1'(r) + \int_s^r R_r(r',r',r,s)\Phi_1'(r')dr',$$

. . . .

‡ This always happens on $\hat{\Omega}$ in Case 2 as is shown in Section 6.

$$t^o_{1,s}(r,s) = -R(s,s,r,s)\Phi_1'(s) + \int_s^r R_s(r',r',r,s)\Phi_1'(r')dr', \tag{4.15}$$

where R is the Riemann function. It is easy to see that $R(s,s,r,s) > 0$. Note that $\Phi_1'(r) \to \infty$ as $r \to a$ since $\varphi_1'(x) \to 0$ as $x \to b$. However, since $\Phi_1(a)$ exists the singularity must be integrable. Hence the integral term in (4.15) has a limit as $r \to a$ and we deduce that $t^o_{1,s}(a,s) \to -\infty$ as $s \to a$. Similarly $t^o_{2,r}(r,a) \to -\infty$ as $r \to a$.

Now apply Riemann's method to t^1. One finds

$$t^1(r,s) = R(a,s,r,s)t^o_1(a,s) + R(r,a,r,s)t^o_2(r,a) + \ldots, \tag{4.16}$$

where the dots indicate integrals over $t^o_1(a,s)$ and $t^o_2(r,a)$.†
Hence

$$t^1_r(r,s) \to -\infty \quad \text{as} \quad r \to a \quad \text{and} \quad t^1_s(r,s) \to -\infty \quad \text{as} \quad s \to a.$$

Lemma 5 now follows from the continuity of t^1_r and t^1_s.

It is easy to verify as in a previous calculation that x^1 as defined by (3.17) satisfies (3.1). Recall that x^1 is chosen so that $x^1(r,a) = x^o_2(r,a)$. But by (3.1)

$$x^1_s(a,s) = -q(a-s)t^1_s(a,s) = -q(a-s)t^o_s(a,s) = x^o_{1,s}(a,s),$$

and $x^1(a,a) = x^o_1(a,a) = b$ hence also $x^1(a,s) = x^o_1(a,s)$. This shows that the geometry in the X plane is as shown in Figures 1 and 2.

We set now $U^1 = \underline{\underline{U}}(R(X))$ in the region in which X^1 is one-to-one. It is easy to see that $U^1 = U^o_1$ or $U^1 = U^o_2$ on

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† See Section 6.

the appropriate boundaries. It is also true that the derivatives of U^1 match those of U_1^o or U_2^o. Observe that derivatives of U can be expressed in terms of derivatives of X. For example note that by (2.5) and (2.6)

$$r_x = J^{-1}t_s, \quad s_x = -J^{-1}t_r.$$

Hence by (2.7) and (3.1)

$$2v_x = r_x + s_x = (t_s - t_r)/2qt_r t_s.$$

On $r = a$ we have $t_{1,s}^o = t_s^1$. On the other hand by our analysis of (4.15) and (4.16) it follows that

$$v_x \rightarrow -1/2qt_s \quad \text{as} \quad r \rightarrow a.$$

This result holds for both v_1^o and v^1 and the corresponding t_s values are the same, consequently the v_x values agree. Similar facts hold for all the derivatives. This completes the proofs of Theorems 1 and 4.

5. The recursion formulas:

In Case 1 the problems for the t_i^{2n} and t^{2n+1} for $n \geq 1$ were described in Section 3. Explicitly, the conditions to be satisfied are

$$t_1^{2n}(r,(-1)^n a) = t^{2n-1}(r,(-1)^n a), t_{1,r}^{2n}(r,-r) + t_{1,s}^{2n}(r,-r) = 0, \tag{5.1}$$

$$t_2^{2n}((-1)^n a,s) = t^{2n-1}((-1)^n a,s), t_{2,r}^{2n}(r,-r) + t_{2,s}^{2n}(r,-r) = 0, \tag{5.2}$$

$$t^{2n+1}((-1)^n a,s) = t_1^{2n}((-1)^n a,s), \; t^{2n+1}(r,(-1)^n a) = t_2^{2n}(r,(-1)^n a). \tag{5.3}$$

Lemma 6. The t functions satisfy the following recursion formulas:

$$t_i^{2n}(r,s) = t^{2n-1}(r,s) + t^{2n-1}(-s,-r) - t_i^{2n-2}(r,s),$$
$$t^{2n+1}(r,s) = t_1^{2n}(r,s) + t_2^{2n}(r,s) - t^{2n-1}(r,s), \qquad n \geq 1. \tag{5.4}$$

The right sides of equations (5.4) are solutions of (3.2). Hence by the uniqueness theorems for (3.2) we need only verify that these functions satisfy (5.1)-(5.3) and this is easily done by induction.

Lemma 7. The x functions satisfy the formulas:

$$x_1^{2n}(r,s) = x^{2n-1}(r,s) - x^{2n-1}(-s,-r) - x_1^{2n-2}(r,s),$$
$$x_2^{2n}(r,s) = x^{2n-1}(r,s) - x^{2n-1}(-s,-r) - x_2^{2n-2}(r,s) + 2L, \tag{5.5}$$
$$x^{2n+1}(r,s) = x^{2n}(r,s) + x_2^{2n}(r,s) - x_1^{2n-1}(r,s), \quad n \geq 1.$$

These formulas may be proved as follows. One verifies by induction that equations (3.1) are satisfied and that the values match on the appropriate sides. We omit the details.

The recursion formulas (5.4) and (5.5) can be solved for all the t's and x's in terms of t_i^o and t^1 or x_i^o and x^1, respectively. This in turn produces a formula for the solutions U in terms of U_i^o and U^1. We write down the formula for the t's since we shall use it in the next section. The other formulas are similar.

Lemma 8. Set[‡]

$$\Psi(r,s) = t^1(r,s) + t^1(-s,-r) - t_1^o(r,s) - t_2^o(r,s).$$

Then for $n \geq 1$

$$t^{2n+1}(r,s) = n\Psi(r,s) + \begin{cases} t^1(-s,-r) & n \text{ odd} \\ t^1(r,s) & n \text{ even,} \end{cases} \tag{5.6}$$

$$t_1^{2n}(r,s) = n\Psi(r,s) + \begin{cases} t_2^o(r,s) & n \text{ odd} \\ t_1^o(r,s) & n \text{ even,} \end{cases} \tag{5.7}$$

$$t_2^{2n}(r,s) = n\Psi(r,s) + \begin{cases} t_1^o(r,s) & n \text{ odd} \\ t_2^o(r,s) & n \text{ even.} \end{cases} \tag{5.8}$$

These formulas can be verified by induction. We give a typical calculation. Suppose (5.7) and (5.8) are true for $n = 2k$ and (5.6) for $2k - 1$. Then by (5.4)

$$t^{4k+1} = 2k\Psi(r,s) + t_1^o(r,s) + 2k\Psi(r,s) + t_2^o(r,s) - (2k-1)\Psi(r,s) - t^1(-s,-r)$$

$$= (2k+1)\Psi(r,s) + t_1^o(r,s) + t_2^o(r,s) - t^1(-s,-r) - t^1(r,s) + t^1(r,s)$$

$$= 2k\Psi(r,s) + t^1(r,s),$$

which is (5.6).

These x and t functions will produce solutions U_i^{2n+1}

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‡ In this formula the functions $t_i^o(r,s)$ are to be considered as defined in all of Ω. We indicated in Remark 6 that the t_i^o could in fact be extended to all of Ω.

and U^{2n} in the x-t plane. One can verify as in earlier calculations that the various triangles fit together as shown in Figure 1. Moreover, as long as no r or s derivative of the t functions is 0 the mappings are one-to-one and one obtains the existence of a solution in the X-plane. Thus one could calculate the position of the breakdown in terms of solutions of linear problems.

6. Non-existence theorems:

In this section we prove Theorem 3 and its analog in Case 2. These state that the solution must ultimately break down. Consider Case 2. Here the proof is quite simple. Recall that $t^1(r,s)$ is a solution of a characteristic initial-value problem in the rectangle $\hat{\Omega} = EB_2BB_1$ of Figure 8 below. The lines BB_1 and EB_1 are $r = a$ and

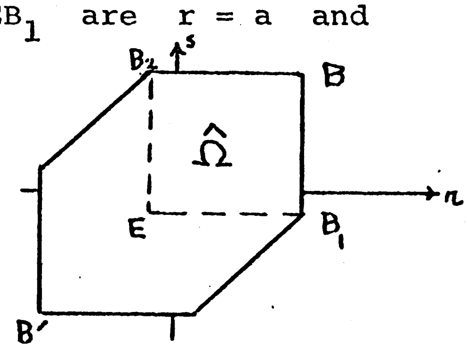

Figure 8

$s = a - 2a_1$ respectively. Suppose now that t_r^1 and t_s^2 do not change sign in the rectangle. Then in particular $t_r^1 < 0$ on EB_1 since we proved that it was negative near BB_2 and BB_1. Consequently if $r'' > r'$ we have

$$t^1(r',a-2a_1) > t^1(r'',a-2a_1).$$

But from (4.16) $t^1(r'',a-2a_1) \to +\infty$ as $r'' \to a$ and hence $t^1(r,a-2a_1) \equiv +\infty$ which violates (4.16).

The proof in Case 1 is much more complicated. We indicated in Section 2 that the essential step is the proof of Lemma 3. What we are going to show precisely is that

$$\Psi_r(a,-a) > 0, \quad \Psi_r(-a,a) < 0. \tag{6.1}$$

It follows that Ψ_r changes sign in Ω and hence by formulas (5.6) - (5.8) t_r^{2n+1} changes sign for n sufficiently large. Then by (2.6a) J will change sign.

The proof of formulas (6.1) requires a rather careful study of the representation of the solutions in terms of the Riemann function. The Riemann function $R(r,s;r',s')$ is a solution of the equation adjoint to (3.2) as a function of (r,s). As a function of the variables (r',s') it is the solution of the following characteristic initial-value problem:

$$R_{r's'} = \rho(r' - s')(R_{r'} - R_{s'}), \tag{6.2}$$

$$R_{r'} + \rho R = 0 \quad \text{on } s' = s, \tag{6.3}$$

$$R_{s'} - \rho R = 0 \quad \text{on } r' = r, \tag{6.4}$$

$$R(r,s;r,s) = 1. \tag{6.5}$$

Lemma 9. The Riemann function for (E) satisfies

$$R_{r'}(s,s,r',s') > 0, \quad R_{s'}(s,s,r',s') < 0 \text{ in } r' > s > s',$$

$$R_{r'}(s,s,r',s') < 0, \quad R_{s'}(s,s,r',s') > 0 \text{ in } r' < s < s'.$$

We prove this lemma in the same way as Lemma 4. From (6.3)-(6.5) we deduce that $R_{r'}(s,s,r',s) > 0$ and $R_{s'}(s,s,s,s') < 0$. Equation (6.2) yields,

$$R_{r'}(s,s,r',s) = R_{r'}(s,s,r',s) - \int_{s'}^{s} \rho(r'-y)(R_{r'}(s,s,r',y) - R_{s'}(s,s,r',y))dy,$$

$$R_{s'}(s,s,r',s') = R_{s'}(s,s,s,s') + \int_{s}^{r'} \rho(x-s')(R_{r'}(s,s,x,s') - R_{s'}(s,s,x,s'))dx.$$

The first part of Lemma 9 follows then by successive approximations and the second part by a similar calculation.

The Riemann representations for t_i^o and t^1 are

$$t_i^o(r',s') = -\int_{r'}^{s'} R(s,s,r',s')\Phi_i'(s)ds, \tag{6.6}$$

$$\begin{aligned} t^1(r',s') &= R(a,s',r',s)t_1^o(a,s') + R(r',a,r',s')t_2^o(r',a) \\ &+ \int_{s'}^{a} (R_s(a,s,r',s') + \rho(a-s)R(a,s,r',s'))t_1^o(a,s)ds \\ &+ \int_{r'}^{a} (R_r(r,a,r',s') - \rho(r-a)R(r,a,r',s'))t_2^o(r,a)dr. \end{aligned} \tag{6.7}$$

We have

$$\begin{aligned} t^o_{i,r'}(r',s') &= R(r',r',r',s')\Phi_i'(r') - \int_{r'}^{s'} R_{r'}(s,s,r',s')\Phi_i'(s)ds, \\ t^o_{i,s'}(r',s') &= -R(s',s',r',s')\Phi_i'(s') - \int_{r'}^{s'} R_{s'}(s,s,r',s')\Phi_i'(s)ds, \end{aligned} \tag{6.8}$$

and

$$t^1_{r'}(r',s') = R(r',a,r',s')t^o_{2,r}(r',a) + \dots ,$$

$$t^1_{s'}(-s',-r') = R(a,-r',-s',-r')t^o_{1,s}(a,-r') + \dots . \tag{6.9}$$

The dots in (6.9) indicate terms which remain bounded as $r' \to a$.

Now we are ready to calculate the function

$$\Psi_{r'}(r',s') = t^1_{r'}(r',s') - t^1_{s'}(-s',-r') - t^o_{1,r'}(r',s') - t^o_{2,r'}(r',s'), \tag{6.10}$$

at the point (-a,a). Note first that (6.8) and (6.9) lead to the following decomposition

$$\begin{aligned}\Psi_{r'}(r',s') &= R(r',a,r',s')R(r',r',r',a)\Phi_2'(r') \\ &+ R(a,-r',-s',r')R(-r',-r',a,-r')\Phi_1'(-r') \\ &- R(r',r',r',s')\Phi_1'(r') - R(r',r',r',s')\Phi_2'(r') + \dots,\end{aligned} \tag{6.11}$$

where the dots again indicate regular terms. Now from (6.3) - (6.5) we have

$$R(r',s,r',s') = \exp\{\int_s^{s'} \rho(r'-\tau)d\tau\},$$

$$R(r,s',r',s') = \exp\{\int_{r'}^{r} \rho(\tau-s')d\tau\}.$$

It follows that the coeffincients of $\Phi_1'(r')$ and $\Phi_2'(r')$ in (6.11) each sum to zero so that there are no singular terms as $r' \to a$. On writing out all regular terms in (6.10) we obtain

$$\begin{aligned}
\Psi_{r'}(-a,a) = {} & R(-a,a,-a,a)\overline{t^o_{2,r}}(-a,a) + (R_r + R_{r'}(-a,a,-a,a))t^o_2(-a,a) \\
& + R_{r'}(a,a,-a,a)t^o_1(a,a) \\
& - (R_r(-a,a,-a,a) - \rho(-2a)R(-a,a,-a,a))t^o_2(-a,a) \\
& + \int_{-a}^{a} (R_{rr'}(r,a,-a,a) - \rho(r-a)R_{r'}(r,a,-a,a))t^o_2(r,a)\,dr \\
& - R(a,a,-a,a)\overline{t^o_{1,s}}(a,a) + (R_s + R_{s'}(a,a,-a,a))t^o_1(a,a) \qquad (6.12) \\
& - R_{s'}(-a,a,-a,a)t^o_2(-a,a) \\
& + (R_s(a,a,-a,a) + \rho(2a)R(a,a,-a,a))t^o_1(a,a) \\
& - \int_{-a}^{a} (R_{rs'}(r,a,-a,a) - \rho(r-a)R_{s'}(r,a,-a,a))t^o_2(r,a)\,dr \\
& - \overline{t^o_{1,r}}(-a,a) - t^o_{2,r}(-a,a).
\end{aligned}$$

where the bars over $t^o_{2,r}$ and $t^o_{1,s}$ indicate that the singular terms $\pm R\Phi'_i$ are to be deleted.

The first integral in (6.12) is zero since as a function of r and s, R satisfies

$$R_r(r,a,r',a) - \rho(r-a)R(r,a,r',a) \equiv 0.$$

Moreover $t^o_i(a,a) = 0$. We substitute the expressions for $t^o_{1,r}$,

$t^o_{1,s}$, deleting the singular terms and obtain,

$$\Psi_{r'}(-a,a) = R_s(-a,a,-a,a)\,t^o_2(-a,a) + \int_{-a}^{a} R_{r'}(s,s,-a,a)\,\Phi'_1(s)\,ds$$

$$- \int_{-a}^{a} (R_{rs'}(r,a,-a,a) - \rho(r-a)R_{s'}(r,a,-a,a))\,t^o_2(r,a)\,dr. \qquad (6.13)$$

Now it is easily deduced from the ordinary differential equation (6.3) that

$$R_r(r,s,r',s) - \rho(r-s)\,R(r,s,r',s) = 0$$

Differentiating this identity with respect to s and setting $s = a$, $r' = -a$ we find that

$$R_{rs'}(r,a,-a,a) - \rho(r-a)R_{s'}(r,a,-a,a) = -R_{rs}(r,a,-a,a) -$$

$$- \rho'(r-a)R(r,a,-a,a) + \rho(r-a)R_s(r,a,-a,a). \qquad (6.14)$$

Now recall that in r and s R satisfies the equation adjoint to (3.2) that is,

$$R_{rs} + (\rho(r-s)R)_r - (\rho(r-s)R)_s = 0.$$

Hence (6.14) yields

$$R_{rs'}(r,a,-a,a) - \rho(r-a)R_{s'}(r,a,-a,a) = (\rho(r-a)R(r,a,-a,a))_r$$

Note also that by (6.5) and the differential equation (6.4) one finds

$$R_s(-a,a,-a,a) = -\rho(-2a)$$

Thus (6.13) becomes

$$\Psi_{r'}(-a,a) = -\rho(-2a)t_2^o(-a,a) + \int_{-a}^{a} R_{r'}(s,s,-a,a)\Phi_1'(s)\,ds$$

$$- \int_{-a}^{a} (\rho(r - a)R(r,a,-a,a))_r t_2^o(r,a)\,dr$$

$$= \int_{-a}^{a} R_{r'}(s,s,-a,a)\Phi_1'(s)\,ds + \int_{-a}^{+a} \rho(r - a)R(r,a,-a,a)t_{2,r}^o(r,a)\,dr.$$

$$(6.14)$$

Lemma 9 shows that $R_{r'}(s,s,-a,a) < 0$. $\Phi_1'(s)$ is positive and by (4.10) $t_{2,r}^o(r,a) < 0$. Also $r - a < 0$ so that $\rho(r - a) > 0$. It is easy to see that R is positive and hence all terms in (6.14) are negative. This proves the first inequality of (6.1) and the second is a very similar calculation. This completes the proof of Theorem 3.

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