MISCELLANEOUS MATHEMATICAL NOTES

by

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and

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In this report we present six brief notes as follows:

Notes 2, 3, and 5 contain, to the authors' knowledge, new results or interpretations of rather simple nature. Notes 1, 4, and 6 provide simple proofs of well-known theorems. Note 3 is a revised (and corrected) version of an earlier report under the same title which appeared as a University of New Mexico, Department of Mathematics, TR 20.

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A BLITZ PROOF OF THE CONTRACTIVE MAPPING THEOREM

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The Contractive Mapping Theorem asserts: If (X,d) is a complete metric space, and $f:X \to X$ is a contractive function with contraction constant $k < 1$ (i.e., for all $x,y \in X$, $d(f(x),f(y)) \leq kd(x,y)$) then:

i. f has a unique fixpoint ξ,

ii. for any $u \in X$, $\{f^n(u)\} \to \xi$,

iii. $d(\xi, f^n(u)) \leq k^n d(u,f(u))/(1 - k)$.

It is easy to verify that contractivity of f itself is not quite necessary for the success of the proof. It suffices that some iterate of f, say $g = f^p$ be contractive (k)--and this is what we assume below--while f may be even discontinuous. In this case parts i and ii of the Theorem still hold true, while in place of the approximation theorem iii, we get a modified statement.

iii' $\quad d(\xi, f^{pn}(u)) \leq k^n d(u,f^p(u))/(1 - k)$.

The proof is very simple although the details might fill a page. (For the case $p = 1$ see for example [1], Chapter 2. The case $p > 1$ does not seem to appear in the literature, but an indication of the proof will be found in [2].) Here we give a very brief proof which produces a little bit more and shows how the Contractive Mapping Theorem follows from the Cantor Intersection Theorem.

We consider first the case of a bounded metric space, i.e. $\operatorname{diam} X = \Delta$. Let $S_1 = \{x | x = f(x)\}$, $S_p = \{x | x = g(x)\}$, the sets of fixpoints of f and $g = f^p$, respectively. By induction we show that $\{f^n(X)\}$ and $\{g^n(x)\}$ are descending chains of sets and that diam $g^n(X) \leq \Delta k^n \to 0$. Then $\overline{g^n(X)}$ has the same properties and from the Cantor Intersection Theorem it follows that $\bigcap_{n=0}^{\infty} \overline{g^n(X)} = \{\xi\}$, a singleton. Thus, using the continuity of g, we get

$$g(\{\xi\}) = g(\bigcap_{n=0}^{\infty} \overline{g^n(X)}) \subset \bigcap_{n=0}^{\infty} g(\overline{g^n(X)}) \subset \bigcap_{n=0}^{\infty} \overline{g^{n+1}(X)} = \{\xi\}.$$

Consequently

$$\xi \in S_p \subset \bigcap_{n=0}^{\infty} g^n(X) \subset \bigcap_{n=0}^{\infty} \overline{g^n(X)} = \{\xi\},$$

from which it follows that $S_p = \{\xi\}$, a singleton.

Finally we observe that $g(f(\xi)) = f(g(\xi)) = f(\xi)$, so that $f(\xi) \in S_p$, i.e. $\xi = f(\xi)$, $\xi \in S_1$. Since $S_1 \subset S_p$ we conclude that $S_1 = \{\xi\}$. This yields part i of the Theorem.

Since $S_1 \subset \bigcap_{n=0}^{\infty} f^n(X) = \bigcap_{n=0}^{\infty} g^n(X) = S_1$ we get the following strengthening of part ii of the Theorem: <u>If</u> $x_n \in f^n(X)$, $n = 0,1,2,\ldots$, <u>then</u> $\{x_n\} \to \xi$.

Since $f^{pn+s}(X) \subset g^n(X)$ for all $s \geq 0$, and $\xi \in g^n(X)$, while diam $g^n(X) \leq \Delta k^n$, we get the following modification of part iii! of the Theorem: <u>If</u> $x \in f^{pn+s}(X)$ <u>then</u> $d(\xi,x) \leq \Delta k^n$.

In an unbounded space the situation is quite different, since the diameters of $g^n(X)$ do not tend to zero. (For example, on a Banach space $g = \frac{1}{2}I$ is contractive but $g^n(X) = X$ for all n.) In this case, for any z, let $Y(z) = B_z(d(z,g(z))/(1 - k))$, the closed ball with center z and radius $d(z,g(z))/(1 - k)$. Since $g(Y(z)) \subset Y(z)$ and diam $Y(z) = \Delta \leq 2d(z,g(z))/(1 - k)$, we can apply the results of the preceding discussion to the restriction of g to $Y(z)$. After proving that the fixpoint of g, and thus of f is unique on X, we conclude that part i of the Theorem is proven, while part ii is now modified to: <u>If</u> $x_n \in f^{pn}(Y(z))$ <u>then</u> $\{x_n\} \longrightarrow \xi$. Since $f^{pn+s}(Y(z))$ is not necessarily included in $g^n(Y(z))$, we can only conclude in place of iii': <u>If</u> $x_n \in f^{pn}(Y(z))$ <u>then</u> $d(\xi, x_n) \leq \Delta k^n \leq 2k^n d(z, f^p(z))/(1 - k)$.

<u>References</u>

[1] A. N. Kolmogorov and S. V. Fomin, 'Elements of the theory of Functions and Functional Analysis', Translation of the 1st edition (1954) by L. F. Boron, Graylock Press, Richester (1957).

[2] I. I. Kolodner, 'Fixed Points', American Mathematical Monthly, <u>71</u> (1964) p. 906.

THE COMPACT GRAPH THEOREM

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The closed graph theorem is a well known and very appreciated theorem of linear analysis. Apparently little known (and perhaps not at all noticed) but still useful is the simple compact graph theorem which is proven below. Before stating the theorem, consider a typical question which is answered by it. Let $f:[0,1] \rightarrow R$ be a function whose graph, $G(f) = \{(x,f(x)) \mid x \in [0,1]\}$ is a closed and bounded set. Is it true that $G(f)$ is a connected set? The answer is yes since, as will be evident, f is continuous.

THEOREM: Let X and Y be topological spaces and let $f:X \rightarrow Y$ be a function with domain X, range $R(f) \subset Y$ and graph $G(f) \subset X \times Y$. Then:

i. X and $R(f)$ are compact if $G(f)$ is compact;

ii. $G(f)$ is compact if X is compact, f is continuous and Y is a Hausdorff space.

iii. f is continuous if $G(f)$ is compact and X is a Hausdorff Space.

PROOF: Let p_1 and p_2 be the coordinate projections of $X \times Y$ on X and Y. They are both continuous.

i. $X = p_1(G(f))$ and $R(f) = p_2(G(f))$ are both compact if $G(f)$ is compact since they are images of a compact set under continuous functions.

ii. Since f is continuous and Y is a Hausdorff space, $G(f)$ is closed. Since X is compact and f is continuous, $R(f)$ is compact and so $X \times R(f)$ is compact. Thus $G(f) \subset X \times R(f)$ is a closed subset of a compact set, and thus it is compact.

iii. Consider the restriction of p_1, $q = p_1 | G(f) : G(f) \to X$. Since $(x, f(x)) \neq (y, f(y))$ implies $x \neq y$, q is one-to-one and onto and q^{-1} exists. Assume now that X is a Hausdorff space and that $G(f)$ is compact. Then q is a homeomorphism, and thus q^{-1} is continuous. Since $f = p_2 \circ q^{-1}$, f is continuous.

ON COMPLETENESS OF PARTIALLY ORDERED SETS AND FIXPOINT THEOREMS FOR ISOTONE MAPPINGS

I. I. Kolodner

1. If P is a conditionally complete lattice and f is an isotone mapping on the <u>interval</u> $\langle a,b \rangle = \{x \in P, a \leq x \leq b\}$ into itself, then f has a fixpoint. See, e.g., [1], pp. 53-54. In applications one frequently has to work with posets which are not lattices; a normed linear space in which the cone of strictly positive elements is open is an example of such a poset. Consequently it is desirable to establish fixpoint theorems for isotone mappings under less stringent conditions than those assumed in the classic theorem quoted above. In this note we explore some such possibilities.

It is obvious that some sort of completeness assumption on the poset P is required in order to achieve a sufficiently general fixpoint property. For example, if

$$f:[-1,0) \cup (0,1] \to [-1,0) \cup (0,1], \; f(x) = x/2,$$

then f is isotone (under the usual ordering of reals) but has no fixpoint. As we shall see below, the reason for this is that some chains in the poset fail to have a supremum, and some fail to have an infimum.

2. Concerning the poset $P = (P,\leq)$, we will use below the following hypotheses:

H_1. P is a <u>lattice</u>.

H_2. P is <u>order complete</u>; that is, each bounded subset of P has a supremum and an infimum. (Alternate name: P

is a conditionally complete lattice.)

H_3. P is <u>chain complete</u>; that is, each bounded chain in P has a supremum and an infimum.

H_4. P is <u>enumerably chain complete</u>; that is, each bounded enumerable chain in P has a supremum and an infimum.

Clearly $H_2 \Rightarrow H_3 \Rightarrow H_4$ and $H_2 \Rightarrow H_1$. On the other hand, there are posets satisfying H_3 or H_4 which are not lattices. For example, if in R^2 one defines the order $\geq$ by

$x > y$ iff $x_1 > y_1$ and $x_2 > y_2$, $x \geq y$ iff $x > y$ or $x = y$, (here $x = (x_1,x_2)$, $y = (y_1,y_2)$) then $(R^2,\geq)$ is not a lattice, but it is chain complete.

As a further illustration consider the three posets $P_i = (L_1[0,1],\leq_i)$, $i = 1,2,3$ where the relations $\leq_i$ are defined by:

$f \leq_1 g$ if $f(x) = g(x)$ or $f(x) < g(x)$ on $[0,1]$,

$f \leq_2 g$ if $f(x) \leq g(x)$ on $[0,1]$,

$f \leq_3 g$ if $f(x) \leq g(x)$ on $[0,1]$ ae.

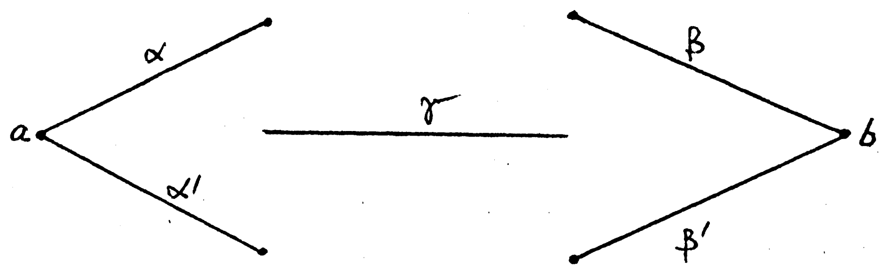
P_1 is not a lattice, but is chain complete. P_3 is order complete; see [2], p. 302, Theorem 22. P_2 is a lattice which has property H_4 as follows from the bounded convergence theorem. However, it fails to be chain complete. To prove this we first show that P_2 is not order complete. Let $A \subset [0,1]$ be a non-measurable set, and let $S = \{X_{\{z\}} | z \epsilon A\}$. ($X_U$ is the characteristic function of U.) Then the only suitable candidate for $\sup S$ is X_A which is not a measurable function, and so P_2 is not order complete. Our assertion will now follow from Theorem 1 below.

THEOREM 1. H_1 <u>and</u> H_3 <u>imply</u> H_2.

$f(\sup C) \in \underline{F}$; since $\sup C \leq f(\sup C)$, $C \cup \{f(\sup C)\}$ is a chain in $\underline{F}$. As C is a maximal chain, we conclude that $f(\sup C) \in C$, i.e., $f(\sup C) \leq \sup C$. Thus $f(\sup C) = \sup C$.

4. The classic proof of Theorem 2 under hypothesis H_2 proceeds along the same lines. However, instead of picking a maximal chain in $\underline{F}$ in order to pry into F, one generally considers $\underline{F}$ itself which has a supremum under the more stringent hypothesis. One finds then that $\sup \underline{F} \in F$ and it follows trivially that $\sup F = \sup \underline{F}$. Similarly, by working with $\overline{F}$ one finds that $\inf \overline{F} \in F$ and $\inf \overline{F} = \inf F$. Under hypothesis H_3, $\sup C \in F$ exhibited in the proof of Theorem 2 turns out to be a maximal element of F. Similarly, one can construct minimal elements in F by considering maximal chains in $\overline{F}$. In general, however, F will not have a supremum or an infimum under hypothesis H_3.

Observe that it suffices to assume in place of H_3 that either every chain in P with an upper bound has a supremum, or that every chain in P with a lower bound has an infimum. On the other hand, none of these conditions is necessary in order to insure that every isotone function on $< a,b >$ has a fixpoint. Consider, for example, the poset described in the figure below. The order is defined



by ($x < y$ iff x is to the left of y) except that elements of

α are not comparable with elements of α' and likewise, the elements of β are not comparable with elements of β'. Then the subset γ is a chain without supremum or infimum, and thus none of the conditions we mentioned are satisfied. Let f be any isotone self-mapping of this poset; we will show that it has a fixpoint. Either $a = f(a)$, or $b = f(b)$, or $a < f(a) \leq f(b) < b$. In the latter case, f maps $< f(a), f(b) > = A$ into itself. Since A is a complete lattice $f|A$ has a fixpoint (in A) and so f also has a fixpoint.

5. Under hypothesis H_2 for P, F is an order complete subset of P but not a sublattice of P. (If $A \subset F$ then sup $A \in \underline{F}$ in view of P^3: One then verifies that $\inf\{x \in P | x \geq f(x) \geq \sup A\}$ is the supremum of A in F and similarly one shows that $\sup\{x \in P | x \leq f(x) \leq \inf A\}$ is the infimum of A in F.) Under hypothesis H_3 for P, F need not have the property H_3. It may be then interesting to seek a completeness criterion for P which is inherited by F. Under hypothesis H_4 for P, F could be empty. However, we shall show that property H_4 is shared by F if one imposes the following restriction on the isotone functions:

H_5. f is <u>continuous</u> <u>on</u> <u>enumerable</u> <u>chains</u>; that is, whenever A is an enumerable chain in $< a,b >$, and sup A, inf A, sup f(A), inf f(A) exist then

$$f(\sup A) = \sup f(A), \quad f(\inf A) = \inf f(A).$$

We note that since f(A) is an enumerable chain whenever A is an enumerable chain, the assumption H_4 will imply the existence of inf A, sup A, inf $f(A)$ and sup $f(A)$. Also, in view of P^4, it suffices to assume in H_5 that $f(\sup A) \leq \sup f(A)$ and $f(\inf A) \geq \inf f(A)$.

THEOREM 3. _If_ H_4 _and_ H_5 _hold then_ f _has a fixpoint. The set of fixpoints_ F _has property_ H_4 _and_ $\sup F = \inf F_b$, $\inf F = \sup F_a$. _Furthermore, if_ P _is a lattice then_ F _is a lattice, but not necessarily a sublattice of_ P.

PROOF: Because of P^1, $\underline{F} \neq \emptyset$. Consider F_x for $x \epsilon \underline{F}$. In view of P^5, F_x is an enumerable chain in $\underline{F}$. In view of H_4, $\sup F_x$ exists. P^5 shows now that $\sup F_x = \sup f(F_x)$ and H_5 shows that $\sup F_x = f(\sup F_x)$. Thus $\sup F_x$ is a fixpoint of f and so $F \neq \emptyset$.

Next, let $C \subset F$ be an enumerable chain. H_4 implies that $\sup C \epsilon < a,b >$ exists. Now $f(C) = C$; thus $\sup C = \sup f(C) = f(\sup C)$ in view of H_5, and therefore $\sup C \epsilon F$. Similarly, $\inf C \epsilon < a,b >$ exists and $\inf C = \inf f(C) = f(\inf C)$.

Next, $\sup F_a \epsilon F$ and $\inf F_b \epsilon F$. Since for all $x \epsilon F$, $a \leq x \leq b$, $f^n(a) \leq f^n(x) = x = f^m(x) \leq f^m(b)$ for all $n,m = 1,2,\ldots$ Thus $\sup F_a \leq x \leq \inf F_b$ and it follows that $\sup F_a = \inf F \leq \sup F = \inf F_b$.

Finally assume that P is a lattice and let $x \epsilon F$ and $y \epsilon F$. In view of P^3, $x \cup y \epsilon \underline{F}$, and it follows from the first part of the proof that $\sup F_{x \cup y} \epsilon F$. Obviously, $\sup F_{x \cup y}$ is an upper bound in F for x and y. Suppose next that $z \epsilon F$ and that z is an upper bound for x and y. Then $x \cup y \leq z$, so that for all $n = 1,2,\ldots$, $f^n(x \cup y) \leq f^n(z) = z$. Thus $\sup F_{x \cup y} \leq z$. Thus $\sup F_{x \cup y}$ is the supremum in F of $\{x,y\}$. Similarly one shows that $\inf F_{x\cap y}$ is the infimum in F of $\{x,y\}$.

Note that if in place of H_5 we assume that f is continuous on chains, then F has property H_3.

References

[1] Garrett Birkhoff, '<u>Lattice Theory</u>', American Mathematical Society Colloquium Publications, vol. XXV, New York, 1948.

[2] N. Dunford and J. T. Schwartz, '<u>Linear Operators</u>', Part I, Interscience Publishers, New York, 1958.

THE JORDAN CURVE THEOREM FOR PIECEWISE SMOOTH CURVES

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It is the purpose of this note to provide an elementary proof of the Jordan Curve Theorem for the class of piecewise smooth curves. This is just the class considered by Ahlfors [1]. The only tools which we use are the notions of compactness and continuity together with the concept of the index of a closed curve relative to a point. In particular, one who is familiar with the material in the first four chapters of Ahlfors is adequately prepared to read this note.

DEFINITIONS AND NOTATION

An arc is said to be smooth if it has a C^1 parametrization. A piecewise smooth arc is one which is obtained by joining end to end a finite number of smooth arcs. If an arc C is parametrized by $z = \phi(t)$, $a \leq t \leq b$, and $S \subset [a,b]$, we shall denote by $C\{S\}$ the image of S under ϕ. At other times, if $z,\zeta \in C$, we shall use $C[z,\zeta]$ to denote a portion of C joining z to ζ. By C' we mean the complement of the curve C with respect to the plane.

PRELIMINARIES. We begin by introducing, for each smooth arc C, a class of arcs C_ϵ which plays the role of the lines parallel to a given segment. These arcs will be used to connect points close to C by an arc which does not intersect C.

LEMMA 1. *Let* $C: z = \phi(t)$, $0 \leq t \leq L$, *be a simple smooth arc parametrized by arclength. Define, for each real* ϵ, C_ϵ

to be that arc parametrized by $z = \phi_\epsilon(t) \equiv \phi(t) + i\epsilon\phi'(t)$, $0 \le t \le L$. *There exists a* $d > 0$ *such that* $C_\epsilon \cap C = \emptyset$ *when* $0 < |\epsilon| < d$.

Proof: We begin by showing that portions of C and C_ϵ, corresponding to sufficiently small neighborhoods of the parametric interval $[0,L]$, are disjoint. Let $t, \tau \in [0,L]$. We then have, after some manipulation,

$$\phi_\epsilon(t) - \phi(\tau) = (t - \tau + i\epsilon)\phi'(t) + \int_\tau^t (\phi'(s) - \phi'(t))ds.$$

By uniform continuity, there exists a $\delta > 0$ such that $|\phi'(s) - \phi'(t)| < 1/2$ if $|s - t| < \delta$. Hence, if $|t - \tau| < \delta$, we have

$$|\phi_\epsilon(t) - \phi(\tau)| \ge |t - \tau + i\epsilon||\phi'(t)| - |t - \tau|/2.$$

Now $|\phi'(t)| = 1$ since t represents arclength. Consequently

$$\phi_\epsilon(t) \ne \phi(\tau) \text{ if } |t - \tau| < \delta \text{ and } \epsilon \ne 0. \qquad (1)$$

We next prove that, for ϵ sufficiently small, each point on C has a neighborhood which is disjoint from C_ϵ. To this end we choose points $0 = t_0 < t_1 < \ldots < t_n = L$ such that $|t_k - t_{k-1}| < \delta/4$. It is then a consequence of (1) that

$$C\{|t - t_k| \le \delta/4\} \cap C_\epsilon\{|t - t_k| \le \delta/2\} = \emptyset. \qquad (2)$$

The point sets $C\{|t - t_k| \le \delta/4\}$ and $C\{|t - t_k| \ge \delta/2\}$ are disjoint and compact since C is simple and the continuous image of a compact set is compact. Hence they have a positive distance d_k. The fact that the portions of C and C_ϵ, corresponding to the set $|t - t_k| \geq \delta/2$, have a distance at most $|\epsilon|$ then shows that

$$C_\epsilon\{|t - t_k| \geq \delta/2\} \cap C\{|t - t_k| \leq \delta/4\} = \emptyset \text{ if } |\epsilon| < d_k. \quad (3)$$

By combining (2) and (3) it is easily seen that $C\{|t - t_k| \leq \delta/4\} \cap C_\epsilon = \emptyset$ if $|\epsilon| < d_k$. It follows that $C \cap C_\epsilon = \emptyset$ if $|\epsilon| < d = \min\{d_k\}$. This completes the proof.

We next use a standard variational argument to show that any point sufficiently close to an interior point of C lies on one of the curves C_ϵ.

LEMMA 2. Let C and C_ϵ be as defined in Lemma 1. If $z \notin C$ is closer to C than it is to either end point of C, then there exists a $t_o \in (0,L)$ and an $\epsilon_o \neq 0$ such that $z = \phi(t_o) + i\epsilon_o\phi'(t_o)$, that is $z \in C_{\epsilon_o}$.

Proof: Since z is closer to C than it is to $\phi(0)$ or $\phi(L)$, there exists a $t_o \in (0,L)$ such that $|z - \phi(t_o)| = \text{dist}\{z,C\}$. Using the definition of distance and the identity $z - \phi(t) = z - \phi(t_o) + \phi'(t_o)(t - t_o) + o(t - t_o)$, we have

$$|z-\phi(t_o)|^2 \leq |z-\phi(t)|^2 = |z-\phi(t_o)|^2 + 2\,\mathrm{Re}(z-\phi(t_o))\overline{\phi'(t_o)}(t-t_o) + o(t-t_o).$$

It then follows from the fact that $t - t_o$ can be either positive or negative that $2\,\mathrm{Re}(z - \phi(t_o))\overline{\phi'(t_o)} = 0$. But this is equivalent to $z - \phi(t_o) = i\epsilon_o\phi'(t_o)$ for some real $\epsilon_o \neq 0$. This completes the proof.

The previous two lemmas allow us to say that two points $z \in C_\epsilon$ and $\zeta \in C_\eta$ are on the same or opposite sides of C according to whether ϵ and η have the same or opposite signs.

Note that we have not excluded the possibility of a point being on both sides of C. Fortunately, this is not important for our purpose. Once the Jordan Curve Theorem has been proved, it is an easy exercise to show that if $z \in C_\epsilon$, $0 < |\epsilon| < d$, then z is only on one side of C.

The previous two lemmas will be used to show that any two points sufficiently close to and on the same side of C can be joined by a curve in C'. In order to prove the same result for points on opposite sides of C we shall need another lemma.

LEMMA 3. _Let_ C _be as in Lemma_ 1. _There exists a_ $d > 0$ _such that the 'half neighborhood'_ $z = \phi(L) + \epsilon\phi'(L)e^{i\theta}$, $0 < \epsilon < d$, $-\pi/2 \leq \theta \leq \pi/2$, _is disjoint from_ C.

Proof: There exists a δ such that $|\phi'(s) - \phi'(L)| < 1/2$ if $|s - L| < \delta$. We then have

$$\phi(L) + \epsilon\phi'(L)e^{i\theta} - \phi(t) = \int_t^L \big(\phi'(s) - \phi'(L)\big)ds + \phi'(L)[(L-t) + \epsilon e^{i\theta}].$$

It then follows from the triangle inequality and the fact that $|\phi'(L)| = 1$ that the right side of the above equality is greater in absolute value than

$$|L - t + \epsilon e^{i\theta}| - (L - t)/2$$

if $L - t < \delta$. The above expression is easily seen to be positive for $\epsilon > 0$ and $-\pi/2 \leq \theta \leq \pi/2$. Hence the 'half neighborhood' is disjoint from $C\{L - \delta \leq t \leq L\}$. Let d be the distance from $\phi(L)$ to $C\{0 \leq t \leq L - \delta\}$. If $0 < \epsilon < d$, then the 'half neighborhood' is disjoint from C. This completes the proof.

LEMMA 4. Let C be as in Lemma 1, A a compact set and z, ζ two points of $(C \cup A)'$ each of which is closer to an interior point of C than it is to A or an endpoint of C. If (i) $C \cap A = \phi(0)$ or (ii) $C \cap A = \phi(0) \cup \phi(L)$ and z, ζ are on the same side of C, then z can be joined to ζ by an arc in $(A \cup C)'$.

Proof: Let $z_1 = \phi(t_1)$ and $\zeta = \phi(\tau_1)$ be points on C which minimize the respective distances from z and ζ to C. By Lemma 2, z and ζ can be connected in $(A \cup C)'$ to points $z_2 = \phi(t_1) + i\epsilon\phi'(t_1)$ and $\zeta_2 = \phi(\tau_1) + i\eta\phi'(\tau_1)$ for all small ϵ, η. Assume that $\eta = \pm\epsilon$. If case (i) is in force we suppose that $t_1 < \tau_1$. The arc $C[\phi(t_1), \phi(L)]$ is disjoint from A and hence has a positive distance δ from it. Let $|\epsilon|$ be less than δ and the d's of Lemmas 1 and 3. If $\eta = \epsilon$ then, by Lemma 1, the arc $C_\epsilon\{t_1 \leq t \leq \tau_1\}$ serves to join z_2 and ζ_2 in $(C \cup A)'$. If $\eta = -\epsilon$ we may join z_2 and ζ_2 to the points $\phi(L) + i\epsilon\phi'(L)$ and $\phi(L) - i\epsilon\phi'(L)$ by the curves $C_\epsilon\{t_1 \leq t \leq L\}$ and $C_{-\epsilon}\{t_1 \leq t \leq L\}$. It then follows from Lemma 3 that these two points can be joined by an arc in $(C \cup A)'$. The proof of (ii) is similar and will be omitted.

<u>THE JORDAN ARC THEOREM</u>. Once one has proved either the Jordan Curve Theorem or its companion the Jordan Arc Theorem, the proof of the other one is relatively simple. Lemmas 1, 2, and 3 are now used to give a simple proof of the Jordan Arc Theorem for our special class of curves.

THEOREM 1. <u>The complement of a simple piecewise smooth arc</u> C <u>is an open connected set having</u> C <u>as its boundary</u>.

Proof: It follows from the fact that C is compact that C' is open and that the boundary of C' is contained in C. Lemma 1 shows that each smooth point of C is a boundary point of C'. Since the boundary of any set is closed, the 'corner points' of C are also in the boundary of C'.

It remains to show that C' is connected. We proceed by induction on the number of smooth segments of C to show that C' is arcwise connected. Suppose then that C is a simple smooth arc. If $z,\zeta\epsilon C'$ we may join them by a smooth arc Γ which does not pass through either end point of C. If Γ does intersect C we may, because of the continuity of the parametrization of Γ, join z and ζ in C' to points z_1 and ζ_1 which are arbitrarily close to interior points of C. By Lemma 4 z_1 can be connected to ζ_1 by a curve in C'. Thus any two points in C' can be connected by a curve in C'. Hence C' is arcwise connected.

Suppose now that Theorem 1 is true for arcs having n smooth segments. If C_{n+1} has $n + 1$ smooth segments, let C_n denote the first n segments and C the last. If $z,\zeta\epsilon C'_{n+1}$ then, by our induction hypothesis, z and ζ can be joined by an arc Γ in C_n'. We may assume that Γ does not pass through an end point of C since removing a point from an open connected set does not disconnect it. We may then join z and ζ in C_{n+1}' to points z_1 and ζ_1 which are arbitrarily close to interior points of C. It then follows from Lemma 4, with $A = C_n$, that z_1 can be connected to ζ_1 by an arc in C_{n+1}'. This completes the proof.

THE JORDAN CURVE THEOREM. We are now in a position to state and prove the main theorem of this note.

THEOREM 2. The complement of a simple closed piecewise smooth curve C consists of two components E and I each having C as its boundary. Moreover, the index of C is equal to zero in E and, if C is oriented properly, is equal to one in I.

Proof: We first show that C' consists of at most two components. Since C is compact, there exists a point ζ which lies outside of a disk containing C in its interior. Let E denote the set of points which can be joined to ζ by a curve in C'. E is clearly connected since any two of its points can be connected to ζ by a curve in E. Let $I = C' - E$. If $I \neq \emptyset$, let Γ be the simple piecewise smooth curve obtained by removing an open smooth segment γ from C. By Theorem 1, any point $z_1 \in I$ can be joined to ζ by a curve $\Gamma_{z_1} \subset \Gamma'$. This curve necessarily crosses γ for otherwise z would be in E. As in the proof of Theorem 1, we now choose points z_1', ζ' arbitrarily close to interior points of γ such that $\Gamma_{z_1}[z_1, z_1']$ and $\Gamma_{z_1}[\zeta', \zeta]$ are in C'. We claim that z_1' and ζ' are on opposite sides of γ for otherwise by Lemma 4 z could be joined to ζ by a curve in C'. Let z_2 be another point in I and let z_2' play the role analogous to z_1'. The point z_2' must be on the same side of γ as z_1' for otherwise z_2 could be connected to ζ by a curve in C'. Since z_1' and z_2' are on the same side of γ, it follows from Lemma 4 that z_1 and z_2 can be connected by a curve in C'. Hence I is connected.

We next consider the difference between the index of C at two points on opposite sides of a smooth portion of C. If $z_o = \phi(t_o)$ is such a point, it follows from Lemma 1 that, as long as ϵ retains its sign, $z_\epsilon = \phi(t_o) + i\epsilon\phi(t_o)$ is in the same component of C'. It follows that

$$\Delta = n(C, z_o + i\epsilon\phi'(t_o)) - n(C, z_o - i\epsilon\phi'(t_o))$$

is constant for all small $\epsilon > 0$. By using the continuity of ϕ' at t_o, it is easily shown that

$$\left(\frac{1}{\phi(t) - z_o - i\epsilon\phi'(t_o)} - \frac{1}{\phi(t) - z_o + i\epsilon\phi'(t_o)}\right)\phi'(t)$$

$$= \frac{2i\epsilon}{(t - t_o)^2 + \epsilon^2} + E$$

where given an $\eta > 0$ there exists a $\delta > 0$ such that

$$|E| < \eta \frac{\epsilon}{[(t - t_o)^2 + \epsilon^2]} \quad \text{if } |t - t_o| < \delta.$$

If $\tilde{C}$ denotes the portion of C corresponding to $|t - t_o| > \delta$, we then have

$$\Delta = \frac{1}{2\pi i} \int_{\tilde{C}} \left(\frac{1}{z - z_o - i\epsilon\phi'(t_o)} - \frac{1}{z - z_o + i\epsilon\phi'(t_o)} \right) dz$$

$$+ \frac{1}{2\pi i} \int_{t_o-\delta}^{t_o+\delta} \left(\frac{2i\epsilon}{(t - t_o)^2 + \epsilon^2} + E \right) dt.$$

The first integral tends to zero as $\epsilon \to 0$ since its integrand is continuous at $\epsilon = 0$. In the second integral we substitute $t - t_o = \epsilon s$ and then let $\epsilon \to 0$. We then obtain

$$|\Delta - 1| < \eta.$$

But since Δ is an integer we must have $\Delta = 1$. It follows that C' has at least two components. But we already know that C' has at most two components; hence I is not empty. The above argument also shows that every smooth point of C is a boundary point of both E and I. That the 'corners' are boundary points follows from the fact that the boundary is a closed set. Since in E (the unbounded component of C') the index of C is zero, it follows that in I it is ± 1, hence by re-orienting, if necessary, we may assume that it is 1. This completes the proof.

Reference

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A NOTE ON FIRST ORDER SEMI-LINEAR PARTIAL DIFFERENTIAL EQUATIONS: AN EXERCISE IN COMPOSITION

I. I. Kolodner and R. N. Pederson

The theories of initial value problems for linear partial differential equations and for non-linear ordinary differential equations have reached a fair degree of sophistication. By comparison, the corresponding theory for non-linear partial differential equations is in a state of infancy. It is, therefore, of interest to classify those equations in the latter category for which one can obtain information by using properties of solutions of equations in the first two categories.

The authors [1] have recently obtained estimates for solutions of semi-linear parabolic equations

$$(1) \qquad \frac{\partial u}{\partial t} - Lu = g(t,u), \; u(x,0) = f(x),$$

where g is concave in u, by combining properties of solutions of the linear partial differential equation

$$(2) \qquad \frac{\partial u}{\partial t} - Lu = 0, \; u(x,0) = f(x),$$

and the ordinary differential equation

$$(3) \qquad \frac{dv}{dt} = g(t,v), \; v(0) = q.$$

It was shown that, under mild regularity assumptions, if $u(x,t;f)$ and $v(t;q)$ are the solutions of (2) and (3), then the compositions $u(x,t;v(t;f(\cdot)))$ and $v(t;u(x,t;f))$ are pointwise lower and upper bounds to the solution of (1).

Perhaps a few remarks on the above notation are in order. The symbol $u(x,t;f)$ represents the value of a function which depends on the point (x,t) and on the function f; that is, it is a functional of f. The expression $v(t;f(\cdot))$, for fixed t, denotes a function, with the appropriate domain of definition, and therefore is a fit candidate for substitution into $u(x,t;\cdot)$. On the other hand, $v(t;q)$ is a function of the pair t and q. The substitution of $u(x,t;f)$ into $v(t;\cdot)$ is therefore a meaningful operation.

It is the purpose of this note to point out that for the semi-linear first order equations in n-space dimensions, $x = (x_1,\ldots,x_n)$,

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} a_i(x,t) \frac{\partial u}{\partial x_i} = g(t,u), \quad u(x,0) = f(x), \tag{4}$$

either of the above indicated procedures yields the solution of (4) exactly. Furthermore, no assumption of concavity of g is needed; it is only required that solutions of (3) be unique. The proof makes use of the fact from the theory of characteristics, see [2], that the solution of

$$\frac{\partial w}{\partial t} + \sum_{i=1}^{n} a_i(x,t) \frac{\partial w}{\partial x_i} = 0, \quad w(x,0) = f(x), \tag{5}$$

$a_i, f \epsilon C^1$, is given by $w(x,t;f) = f(\varphi(x,t))$ where $\varphi(x,t)$ is the function whose i'th component φ_i is the solution of the problem (5) with initial data

$$f(x) = x_i.$$

It is then easily seen that $w(x,t;v(t,f(\cdot)))$ and $v(t;w(x,t;f))$ are both equal to $v(t;f(\varphi(x,t)))$. Indeed, the prescription for computing $w(x,t;v(t;f(\cdot)))$ is to substitute $\varphi(x,t)$ into $v(t;f(\cdot))$, the result being $v(t;f(\varphi(x,t)))$. A less subtle argument gives the same result for $v(t;w(x,t;f))$. That $v(t;f(\varphi(x,t)))$ is the solution of (4) is a simple exercise in differentiation together with an application of the uniqueness theorem for solutions of (4).

Another way of looking at the result of the previous paragraph is that the operation of solving (5) with the solution of (3) as initial condition is identical with the operation of solving (3) with the solution of (5) as initial condition. That is, the indicated composition is commutative.

References

[1] I. I. Kolodner and R. N. Pederson, Bounds for Solutions of Semi-Linear Parabolic Equations, To appear in the Journal of Differential Equations.

[2] R. Courant and D. Hilbert, Methods of Mathematical Physics, II Interscience Publishers, New York, 1962.

A SIMPLE PROOF OF THE SCHRÖDER-BERNSTEIN THEOREM.

I. I. Kolodner

The objective of the present note is to clarify the proof of the Schröder-Bernstein theorem given in Suppes ([1], p. 95-96). While in principle our construction does not involve anything beyond what already appears in [1], we stress here the fact that the proof is achieved by finding a fixpoint of a certain function. This function turns out to be an isotone self-mapping of a complete lattice, and thus has a fixpoint.

A clarification of this type seems not to have attracted the attention of other expositors. The proof in [1] follows an ex-machina pattern: an equation is written, one does this and that, and presto--the conclusion follows. The same attitude is taken by Fraenkel ([2], p. 102-103) on which the proof of [1] is based, and the same is indicated by Dieudonné ([3], p. 10 exercise 4, and p. 13 exercise 3) through hints.

The Schröder-Bernstein theorem asserts: If the functions $f_1 : X_1 \to X_2$ and $f_2 : X_2 \to X_1$ are one-to-one, then there exists a function $g : X_1 \to X_2$ which is one-to-one and onto.

The beginning of our proof is standard: the assertion follows if one can produce subsets $A_1 \subset X_1$ and $A_2 \subset X_2$ such that

$$(1) \qquad f_1(A_1) = A_2' \quad \text{and} \quad f_2(A_2) = A_1'.$$

(Primes denote complements.) Indeed, the restrictions $f_1 | A_1 : A_1 \to A_2'$ and $f_2 | A_2 : A_2 \to A_1'$ are then one-to-one and onto functions and thus $g : X_1 \to X_2$ defined by

$$(2) \qquad g(x) = \begin{cases} f_1(x), & \text{if } x \in A_1, \\ (f_2|A_2)^{-1}(x), & \text{if } x \in A_1', \end{cases}$$

has the desired property.

The functions f_1 and f_2 induce associated set functions $F_1: P(X_1) \to P(X_2)$ and $F_2: P(X_2) \to P(X_1)$ on the corresponding power sets, where $F_i(U) = \{f_i(x) | x \in U \in P(X_i)\}$, $i = 1,2$. Let $C_i: P(X_i) \to P(X_i)$, $i = 1,2$ be the complement function, i.e., $C_i(U) = U' \subset X_i$. Then condition (1) implies that the pair (A_1, A_2) will be a suitable pair of sets if and only if it is a solution of the pair of conditional equations

$$(3) \qquad F_1(x) = C_2(y), \; F_2(y) = C_1(x).$$

Thus it suffices to show that (3) has a solution.

Eliminating y from equations (3) we now get a <u>single fixpoint equation</u> on $P(X_1)$:

$$(4) \qquad x = K(x), \text{ where } K = C_1 \circ F_2 \circ C_2 \circ F_1 : P(X_1) \to P(X_1).$$

It is easily checked that if U is a solution of (4) then the pair $(U_1, (C_2 \circ F_1)(U))$ is a solution pair of the system (3). Thus our objective is now to show that K has a fixpoint.

To solve (4) observe that F_1 and F_2, being set functions induced by point functions, are isotone (i.e., order preserving) under the orderings of $P(X_1)$ and $P(X_2)$ by set inclusion. The complements C_1 and C_2 are antitone (i.e., order inverting) functions. Thus K, being a composition of two isotone and two antitone functions, is an isotone function.

At this point we may invoke a well known fixpoint theorem for isotone functions since $(P(X_1), \subset)$ is a complete lattice; see e.g., [4], p. 54. However, the matter is so simple that we hardly need a reference to finish the proof. Let $S = \{x \in P(X_1) | x \subset K(x)\}$; S is not empty, since $\emptyset \subset K(\emptyset)$. We shall show that $U = \cup\{x \in S\}$ is a fixpoint of K. If the element $a \in U$ then $a \in x$ for some $x \in S$ and so $a \in x \subset K(x)$; since K is isotone and $x \subset U$, it follows that $K(x) \subset K(U)$ whence $a \in K(U)$, or $U \subset K(U)$. Since K is isotone, this yields $K(U) \subset K(K(U))$ showing that $K(U) \in S$. From the definition of U we now conclude that $K(U) \subset U$, and consequently $U = K(U)$, as asserted.

Coming back to the original question, we see that $A_1 = U$, $A_2 = (C_2 \circ F_1)(U)$ is a pair satisfying the conditions (1).

References

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[4] Garrett Birkhoff, '<u>Lattice Theory</u>', American Mathematical Society Colloquium Publications, Vol. XXV, New York, 1948.