ON THE CONTINUATION OF SOLUTIONS OF A CERTAIN NON-LINEAR DIFFERENTIAL EQUATION[1]

by

C. V. Coffman

and

D. F. Ullrich[2]

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[2]This author is presently at North Carolina State University, Raleigh, North Carolina.

I. Introduction

Let $p(x)$ be positive and continuous on $[0, \infty)$ and let n be a positive integer; it is known that the differential equation

$$(1) \qquad y'' + p(x)\, y^{2n+1} = 0,$$

can have solutions defined on a bounded interval which do not adroit an extension to the interval $[0, \infty)$. An example of this singular behavior has been given by Hastings [1], and another example, which is perhaps conceptually somewhat simpler, is given below. In this note we will also establish certain conditions on $p(x)$ which are sufficient to guarantee that all solutions of (1) can be extended to $[0, \infty)$. Two such conditions for a more general equation have been given in [1]. For equation (1) these reduce respectively to the conditions i) that $p(x)$ be piecewise monotone, and ii) that $p(x)$ satisfy a locally uniform Lipschitz condition. The main result of this note states that all solutions of (1) will exist on $[0, \infty)$ if $p(x)$ is locally bounded variation. This condition on $p(x)$ is implied either by condition i) or by condition ii). We remark that piecewise monotonicity is still sufficient if $p(x)$ is only assumed to be non-negative.

The proof of our theorem involves the use of a bound for solutions of (1) in terms of the total variation of $p(x)$. Indeed, suppose that $p(x)$ is of bounded variation on $[0,x]$, with, say, $T(x)$ = total variation of $p(x)$ on $[0,x]$. Then, if $y(x)$ is a solution of (1) defined on $[0,x_0)$, and if $T(x) < \infty$ for each $x < x_0$, we have

$$\log |y(x)| \leq C_1 T(x) + C_2, \quad 0 \leq x < x_0, \tag{2}$$

where the positive constants C_1 and C_2 depend only on n, a lower bound for $p(x)$ on $[0,x_0]$, and on $y(0)$ and $y'(0)$. An inequality such as (2) also is valid in the linear case ($n = 0$) but the inequality is more significant when $n > 0$, as is indicated by the following facts. If $p(x)$ is continuous on $[0,x_0]$ and $T(x) < \infty$ for $x < x_0$ while $T(x) \to \infty$ as $x \to x_0$, then in the linear case the left hand side of (2) remains bounded from above as $x \to x_0$. On the other hand, as our example will show, if $p(x)$ has these same properties and $n > 0$, then (1) can have solutions $y(x)$ for which

$$\limsup_{x \to x_0^-} (T(x))^{-1} \log |y(x)| > 0. \tag{3}$$

II. Example.

In this section we construct an example of equation

$$y'' + p(x)\, y^3 = 0 \tag{4}$$

in which $p(x)$ is locally of bounded variation everywhere in $[0,\infty)$

with the exception of one point, x_0, and such that at least one solution has $[0,x_0)$ as its maximal interval of existence. This shows, incidentally, that global existence of solutions of (1) can be destroyed by a pathology of the coefficient at a single point, and in fact even when the coefficient differs from a constant by an arbitrarily small amount.

It should be remarked that a solution of (4) or of (1) on the interval $[0,x_0)$ can fail to have a continuation to the right of x_0 only if the solution changes sign infinitly many times as x approaches x_0 from the left. Indeed since a solution y(x) of (1) always satisfies $yy'' \leq 0$, an elementary argument shows that for a solution y defined on the interval $[0,x_0)$, and having only finitly many zeros there, both y' and y will possess finite limits as $x \to x_0-$. We note that a solution of (1) which is bounded in a finite interval can have at most finitly many zeros there. This follows from the Sturm comparison theorem.

The example of (4) which we construct below can be regarded as resulting from a perturbation of the coefficient in the autonomous equation

$$\frac{d^2}{dt^2} u + C^2 u^3 = 0 . \tag{5}$$

For convenience we choose the positive constant C so that (5) has a solution u(t) satisfying

$$u(0) = u(1) = 1, \quad \frac{d}{dt} u(0) = \frac{d}{dt} u(1) = 0, \tag{6}$$

and having exactly two zeros in (0,1). That C can be so chosen follows in an elementary way from the fact that all solutions of equation (5) are oscillatory and periodic, for any non-zero value of C.

We require the following result.

Lemma 1. *For each positive integer* n, *there exists a continuous function* $q_n(t)$ *on* $[0,1]$ *with*

(7) $$q_n(0) = q_n(1) = 0,$$

and such that the differential equation

(8) $$\frac{d^2U}{dt^2} + (C^2 + q_n(t))U^3 = 0,$$

has a solution $U_n(t)$ *satisfying*

(9) $$\frac{dU_n(0)}{dt} = \frac{dU_n(1)}{dt} = 0, \quad U_n(0) = 1, \; U_n(1) = \left(\frac{n+1}{n}\right)^2,$$

and having at least two zeros in $(0,1)$. *In addition the* $q_n(t)$ *can be chosen in such a way that each is of bounded variation on* $[0,1]$ *with*

(10) $$\int_0^1 |dq_n(t)| \le Kn^{-1}, \qquad n = 1, 2, \ldots,$$

for an appropriate constant K.

We now proceed with the construction of our example, deferring the proof of Lemma 1 to the end of this section.

Let the sequence $\{a_n\}$ be defined by setting

$$a_1 = 0, \quad a_n = \sum_{k=1}^{n-1} \frac{1}{k^2}, \qquad n > 1.$$

For $0 \le x < \frac{\pi^2}{6}$ ($= \lim_{n\to\infty} a_n$) we define the coefficient p(x) in (4), and a solution y(x) of (4), in the following way. For each n = 1,2,... put

(11) $$p(x) = C^2 + q_n(n^2(x - a_n)), \quad \text{for } a_n \le x \le a_{n+1},$$

and define

(12) $$y(x) = n^2 u_n(n^2(x - a_n)), \quad \text{for } a_n \le x \le a_{n+1}.$$

It follows from (7) that p(x) is continuous on $[0, \frac{\pi^2}{6})$ and from (9) that y(x) is at least of class C^1 on $[0, \frac{\pi^2}{6})$. From (11) and (12) it follows that y(x) satisfies (4) on each of the intervals $[a_n, a_{n+1})$, and, therefore, since it is of class C^1 on $[0, \frac{\pi^2}{6})$, it must in fact be of class C^2 and satisfy (4) everywhere in $[0, \frac{\pi^2}{6})$. Inequality (10) implies that $q_n(t) \to 0$ uniformly on [0,1] as $n \to \infty$; consequently, (11) implies that $p(x) \to C^2$ as $x \to \frac{\pi^2}{6}$. We can, therefore, extend p(x) continuously to $[0,\infty)$ by setting $p(x) = C^2$ for $x \ge \frac{\pi^2}{6}$. On the other hand, the solution y(x) of (4) which we have constructed has at least two zeros in each of the intervals (a_n, a_{n+1}), and from (9) and (12) we see that

(13) $$y(a_n) = n^2, \quad n = 1, 2, \ldots;$$

thus, y(x) cannot be continued beyond $\frac{\pi^2}{6}$.

We shall now show that y(x) satisfies (3), with $x_0 = \frac{\pi^2}{6}$ and where T(x) denotes, as in section 1, the total variation of p(x) in $[0,x]$. In view of (10) and the definition of $p(x)$ we have

$$T(x) \le K \sum_{k=1}^{n} \wedge \; \wedge \le K_1 \log n \quad \text{for } 0 \le x < a_n .$$

(The total variation of $p(x)$ in $[a_n, a_{n+1}]$ is the same as the total variation of $q_n(t)$ in $[0,1]$.)

Since $\frac{\pi^2}{6} - a_n = \sum_{k=n}^{\infty} \frac{1}{k^2} \sim \text{const.} \times n^{-1}$, as $n \to \infty$,

(14) $\quad T(x) \le K_2 \left|\log\left(\frac{\pi^2}{6} - x\right)\right|, \quad \text{for } 0 \le x < \frac{\pi^2}{6}.$

Similarly, (13) implies that

(15) $$\limsup_{x \to \frac{\pi^2}{6} -} \left(\frac{\pi^2}{6} - x\right)^{2} y(x) > 0,$$

and combining inequalities (14) and (15) we finally obtain

$$\limsup_{x \to \frac{\pi^2}{6} -} (T(x))^{-1} \log |y(x)| > 0.$$

Proof of Lemma 1. We shall first define a function $U_n(t)$ of class C^∞ on $[0,1]$ which satisfies (9) and has two zeros in $(0,1)$. The function $q_n(t)$ will then be defined by

(16) $$q_n(t) = -[C^2 + t_n r^3(t) \; f^{\wedge} U_n(t)].$$

Let $U_\infty(t)$ denote the solution of equation (5) which satisfies (6), and let t_0 $(0 < t_0 < 1)$ be such that $U_\infty(t)$ is positive in $[t_0, 1]$:

Then, $U_n(t)$ will be represented in the form

$$
(17)\qquad
\begin{aligned}
U_n(t) &= U_{00}(t) & 0 \le t \le t_0, \\
U_n(t) &= (2n+1)/n^2 + U_{00}(t) - \int_t^1 (t-s) f_n(s)\, ds, & t_0 \le t \le 1.
\end{aligned}
$$

The continuity of U_n, $\frac{d}{dt} U_n$ and $\frac{d^2}{dt^2} U_n$ requires that

$$(18)\qquad \int_{t_0}^{1} (t_0 - s)\, f_n(s)\, ds = (2n+1)/n^2$$

$$(19)\qquad \int_{t_0}^{1} f_n(s)\, ds = 0$$

and

$$(20)\qquad f_n(t_0) = 0.$$

On the other hand, if (18), (19), and (20) hold, and if $q_n(t)$ and $U_n(t)$ are defined by (16) and (17) respectively, then $U_n(t)$ is of class C^2 on [0,1] and satisfies (8). In order that (7) hold it suffices to have

$$(21)\qquad f_n(1) = -c^2\left[\left(\frac{n+1}{n}\right)^6 - 1\right].$$

If we assume f_n to have the form

$$(22)\qquad f_n(t) = \alpha_n (t - t_0)^3 + \beta_n (t - t_0)^2 + \gamma_n (t - t_0),$$

then (20) holds automatically and (18, (19), and (21) become equivalent to the linear system

$$
(23)\quad \begin{cases}
\frac{1}{4}\alpha_n z^4 + \frac{1}{3}\beta_n z^3 + \frac{1}{2}\gamma_n z^2 = 0, \\
\frac{1}{5}\alpha_n z^5 + \frac{1}{4}\beta_n z^4 + \frac{1}{3}\gamma_n z^3 = -(2n+1)/n^2 \\
\alpha_n z^3 + \beta_n z^2 + \gamma_n z = -C^2\left[\left(\frac{n+1}{n}\right)^6 - 1\right],
\end{cases}
$$

where $z = 1 - t_o$. Since $z \neq 0$, the determinant of (23) is easily seen to be non-zero. Therefore, if f_n is taken to be of the form (22) then α_n, β_n and γ_n can be chosen in such a way that (18), (19), (20), and (21) hold; in fact, α_n, β_n, and γ_n are uniquely determined, and for a suitable constant A we have

$$(24)\qquad \max(|\alpha_n|, |\beta_n|, |\gamma_n|) \le An^{-1}, \quad n = 1, 2, \ldots .$$

Thus, by setting

$$h_n(t) = u_n(t) - u_\infty(t),$$

it follows from (22) and (24) that for n = 1,2,..., and for a suitable constant B

$$\max\left(|h_n(t)|, \left|\frac{d}{dt}h_n(t)\right|, \left|\frac{d^2}{dt^2}h_n(t)\right|\right) \le Bn^{-1}, \quad 0 \le t \le 1 .$$

Using this inequality in (16), it is easily shown that

$$\left|\frac{dq_n(t)}{dt}\right| \le Kn^{-1}, \quad 0 \le t \le 1, \; n = 1, 2, \ldots,$$

and, since $q_n(t)$ has a continuous derivitive, (10) follows.

The proof of Lemma 1 is now complete.

III. Theorem.

The main result of this note is the following.

Theorem. Let n be a positive integer and let $p(x)$ be positive, continuous, and locally of bounded variation on $[0,\infty)$. Then for arbitrary real numbers a and b and for any $x_0 \in [0,\infty)$ the initial value problem

$$y(x_0) = a, \quad y'(x_0) = b \tag{25}$$

has a unique solution which exists on $[0,\infty)$.

The proof of the Theorem depends upon the following lemma.

Lemma 2. Let n be a positive integer, and let $p(x)$ be positive and of class C^1 on an interval $[x_0,x_1)$. Assume there exists a non-negative function $g(x)$ such that

$$p'(x)/p(x) \le g(x) \quad \text{on } [x_0,x_1) \tag{26}$$

and

$$\int_{x_0}^{x} g(x)\,dx < \infty, \qquad \text{for } x < x_1 .$$

Then the solution $y(x)$ of (1) which satisfies (25) will exist on $[x_0,x_1)$, and satisfy the inequality

$$\Phi(x) \le \Phi(x_0) \exp \int_{x_0}^{x} g(x)\,dx, \quad x_0 \le x < x_1 . \tag{28}$$

where

$$(29) \qquad \Phi = (y')^2 + \frac{1}{n+1} p(x)\, y^{2n+2} .$$

Proof, Let $y(x)$ be a solution of (1) satisfying (25) and assume that $y(x)$ exists on an interval $[x_0, x_2)$, $x_0 < x_2 < x_1$. If Φ is defined in terms of y by (29), then clearly Φ is of class C^1 on $[x_0, x_2)$; in fact we have

$$\Phi'(x) = \frac{1}{n+1} p'(x)\, y^{2n+2}(x), \qquad x_0 \le x < x_2 .$$

Using (26) we obtain

$$\Phi'(x) \le g(x)\Phi(x), \qquad x_0 \le x < x_2 ,$$

and integration of this differential inequality gives (28) for $x_0 \le x < x_2$. It follows now from (27) and (28) that both y' and y are bounded on $[x_0, x_2)$, this implies that the solution $y(x)$ can be continued to the right of x_2. Since the point $x_2 \in [x_0, x_1)$ was arbitrary, the proof of Lemma 2 is complete.

Proof of the Theorem, Let $x_1 > x_0$ and assume that $p(x) \ge m > 0$ on $[x_0, x_1]$. It is possible to approximate $p(x)$ uniformly on $[x_0, x_1]$ by a sequence of functions $\{p_\nu(x)\}$, where each $p_\nu(x)$ is a class C^1 on $[x_0, x_1]$ and satisfies

$$p_k(x) \ge m \qquad \text{on } [x_0, x_1]$$

and $$\int_{x_o}^{x_1} |p'_k(x)|\, dx \le T,$$

where T is the total variation of p(x) on $[x_o, x_1]$. For each k = 1,2,... let $y_n(x)$ denote the solution of

$$y'' + p_n(x)\, y^3 = 0$$

which satisfies the initial condition (25). Since $p'_k(x)/p_k(x) \le m^{-1} |p'_n(x)|$, it follows from Lemma 2 that each of the $y_n(x)$ exists on $[x_o, x_1]$ and furthermore that for k = 1,2,..

$$|y_k(x)| \le |y_k(x_o)|\, e^{T/m}, \quad x_o \le x \le x_1 .$$

As $k \to \infty$ the $y_k(x)$ tend uniformly to a solution y(x) of (1) satisfying the initial conditions (25). This shows that (1)-(25) has a solution on $[x_o, x_1)$ for arbitrary $x_1 > x_o$; a similar argument shows that the solution exists on $[0, x_o]$. The uniqueness of the solution of (1)-(25) follows from standard results, since the term $p(x)\, y^{2n+1}$ satisfies a Lipschitz condition in y.

References

1. Stuart P. Hastings, Boundary value problems in one differential equation with a discontinuity, Journal of Differential Equations, 1, (1965), 346-369.

Carnegie Institute of Technology