FULL ORDERS AND THEIR FAMILIES OF CONVEX SUBGROUPS ON LOCALLY NILPOTENT GROUPS

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I. Introduction

If a group G has a full order relation, <, such that for each a,b,c € G if a < b then both ac < be and ca < cb, we say that (G, <) is an <u>f,o. group</u>, that it is <u>fully ordered</u> by <, and that < is a <u>full order</u> of G. If G is fully ordered by < and C 5 G is a subset such that when a,b € *C,* then all x between a and b are also in C, we say that C is <u>convex</u> with respect to < or if there is no ambiguity simply that C is convex. We denote by $C_<$ the family of all sub groups of G convex with respect to <. The family $C_<$ is called the <u>convex family</u> of (G, <) .

Several authors have investigated properties of those families of subgroups of a group G which are the convex families determined by the various full orders of G. Among the investigators K. Iwasawa [3J has given conditions characterizing such families of subgroups. Another set of conditions on a family of subgroups of a group G which is necessary and sufficient that G admits a full order is stated in Fuchs [1] pp. 51, 52, and attributed by him to L. S. Rieger and V. D. Podderyugin.

In this paper we will investigate the convex families of full orders in torsion free locally nilpotent (TFLN) groups. Our main result (theorem 2) is that for an arbitrary full order < on the TFLN group G, the convex family, $\mathcal{C}_{<G}$ is a central system. We remark that there are several examples known of full orders on non-TFLN groups which are not even normal systems. We use our result to obtain a characterization of those families of subgroups of TFLN groups which are convex families. This characterization is simpler than the general characterization given by Iwasawa. Secondly, we show that the Rieger-Podderyugin conditions (theorem 1) do not characterize convex families, a question raised in Fuchs [1].

II. Results

A family of subgroups $\mathfrak{F}$, of a group G is a <u>normal system</u> if each $F \in \mathfrak{F}$ is normal in G, if $\mathfrak{F}$ is a chain with respect to set inclusion, and if $\mathfrak{F}$ is closed with respect to arbitrary unions and intersections. A <u>jump</u> in $\mathfrak{F}$ is a pair of elements $C, D \in \mathfrak{F}$ such that $C \subset D$ and no element of $\mathfrak{F}$ is properly between C and D. Letting Z(H) denote the center of any group H, we say that $\mathfrak{F}$ is a <u>central system</u> in group G if {1} and G are in $\mathfrak{F}$, if $\mathfrak{F}$ is a normal system of G, and if for each jump C,D in $\mathfrak{F}$, $D/C \subseteq Z(G/C)$.

<u>Theorem 1</u>: (Rieger, Podderyugin as per Fuchs [3]).
A group G is orderable if and only if it contains a family, $\mathfrak{F}$, of subgroups satisfying (i) - (v):

(i) 3 is a chain containing {1} and G and is closed under arbitrary unions and intersections.

(ii) If C e 3 then for each g € G, $g^{-1}Cg$ € 3.

(iii) If C,D is a jump in 3 then C is normal in D and D/C is isomorphic to a subgroup of the additive group of the real numbers.

(iv) If C,D is a jump in 3 then (N(C), N(C), D) $\subseteq$ c where N(C) is the normalizer of C in G,[2]

(v) If C e 3 and S(a) is the normal subsemigroup of G generated by a and if C H S(a) is non-empty then some conjugate of a is in C.

Furthermore there will be a full order, $<_3$ of G such that 3 £ C^.

Until now it was not known whether a full order < of G existed such that 3 = $C_<$. However, by the example below this question is answered in the negative.

We now consider TFLN groups. It has been known for some time that all TFLN groups can be fully ordered (see Neumann (5) ; Graham [2} has another proof) • Another result in Graham [j2] is the following,

Lemma: If G is TFLN and < is a full order of G then $C_<$ is a normal system of G.

Proof: By straightforward calculation if C € C_and g e G then $g^{-1}Cg$ e C also. Thus $<f^{1}Cg$ => C or $g^{-1}Cg$ S C, To show that

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2 If x,y e G then (x,y) = $x^{-1}y^{-1}xy$. If S and T are subgroups of G then (S>T) = gp((s,t) | s e S and t e T) . Finally (S,T,U) meanjs ((S,T),U) .

$g^{-1}Cg = C$ for all $C \in \mathcal{C}$ and $g \in G$ we assume otherwise. Without loss of generality we assume that there is a $g \in G$ and a $C \in \mathcal{C}$ such that $g^{-1}Cg \neq C$. Then there is an $a \in C$ such that $g^{-1}ag \notin C$, so for arbitrary positive n we have

(1) $$g^{-n}ag^{n} \notin g^{-n}Cg^{n}.$$

Let $C_m = g^{-m}Cg^{m}$, let $(a,g;0) = a$, and let $(a,g;n+1) = ((a,g;n),g)$. Note that $C_{i+1} \neq C_i$ and $g^{-1}ag = a(a,g;1)$. By induction on n the identity (2) below can be established, where $\prod_i (a,g;i)$ is to be interpreted as some finite product of commutators $(a,g;i)$ in which none of the i's is greater than k. For example $g^{-2}ag^{2} = (a,g;1)^2(a,g;2)$.

(2) $$g^{-n}ag^{n} = \Big(\prod_{i\le n-1}(a,g;i)\Big)(a,g;n).$$

Since $a(a,g;1) \in C_1$ it follows by induction on m and by (2) that $(a,g;m) \in C_m$. However G is locally nilpotent so the group generated by a and g is nilpotent, say of class k. Then all commutators of length k + 1 and greater are the identity so from (2) we get

$$g^{-k-1}ag^{k+1} = \Big(\prod_{i\le k}(a,g;i)\Big)(a,g;k+1) = \prod_{i\le k}(a,g;i) \in C_k .$$

But this contradicts (1) taking n = k, and the lemma is established.

Theorem 2: If G is TPLN and fully ordered by < then $\mathcal{C}_<$ is a central system.

Proof: Since {1} and G are convex they are in $\mathcal{C}$. By the lemma $\mathcal{C}$ is a normal system. Now let $C < D$ be a jump in $\mathcal{C}$. All that remains is to show that $D/C \subseteq Z(G/C)$. First note that if for each $g \in G$ the mapping $\hat{g}: G/C \to G/C$ induced by conjugation by g is the identity map on D/C, then for each d € D (dC)^ = g~^{1}dgC = d(d,g)Cj hence it follows that (d,g) = 1 mod C so that D/C c z(G/C).

Thus the problem reduces to showing that for each g € G, ĝ|D/C is the identity map. Suppose on the contrary for some g *e* G, ĝ|D/C is not the identity map. Then there is a d *e* D such that (d^g) *f* 1 mod C„ Since G is TFLN one of (d^g;m) = 1, so for some first m, say n + *2,* (d,g; n + 2) =1 mod C. Then (d,g;n) *f* (d,g; n + 1) mod C and neither is congruent to 1 mod C# Let b = (d,g;n).

Since (d,g; n + 1)C^ = (d,g; n + 1) (d,g; n + 2)C = (d^g; n + 1) C and bC§ = b(d,g; n + 1)C we have for all integers r >_ 0 that bCĝ^ = b(d,g; n + l)^{r}C. If r < 0 then bC = bC§~^{r} ^^{r} = (b(d,g; n + 1)$^{-r}$C)ĝ^{r} = (bC)^^{r}(d,g; n + 1) ~^{r}C = (bC^^{r}) (d,g; n + 1)^{r}C so for all r = 0, ±1^ ±2,...^ we have

(3) $$bC\hat{g}^{r} = b(d,g;\ n + 1)^{r}C = b(b^g)^{r}C.$$

Now < on G induces a full order on G/C since C is convex* We will denote this induced order by < also* Furthermore, for each r^ ^^{r} is an order preserving automorphism of G/C.

Also (b,g)C^ = (d,g; n + 1) C^ =? (d^g; n + 1) C = (b,g)C hence for all *r,* (b,g)C^^{r} = (b,g)C. Now by the choice of b, C ^ (b,g)C ^ bC. Therefore either bC < (b,g)C or bC > (b,g)C. If bC < (b,g)C then bC^^{r} < (b,g)C§^{r} for all r. But bC§^{r} = b(b,g)^{r}C = bC((b,g)C)^{r} < (b^gjc so bC < ((b,g)C)^{1} "^{r} for all rational integers r. This is impossible since neither is the identity and since D/C is an Archimedean ordered group. A similar contradiction is reached if we assume that bC > (b,g)C.

Therefore the assumption that ^JD/C is not the identity leads to a contradiction and the theorem is established.

Using the conditions in theorem 1 with the added information from theorem 2 we can now characterize those systems of subgroups of TPLN group G which are the convex families with respect to the various full orders of G,

Theorem 3: If G is TFLN the system 0 of subgroups of G is the convex family of subgroups with respect to some full order of G if and only if:

(i) C is a central system of G.

(ii) If C,D is a jump in C the D/C is isomorphic to a subgroup of the additive group of real numbers.

Proof: If C is the convex family of G with respect to some full order of G then (i) holds by theorem 2 and (ii) holds since in the induced order on D/C, D/C is an Archimedean ordered group, hence isomorphic to a subgroup of the additive reals. (See Fuchs QJ P* 45. The theorem is due 0. Holder.)

On the other hand if (jL) and (ii) hold for C then one can define a full order on G as in Neumann [6J, theorem 2.3. (Strictly speaking the fuller generality of Graham [2J, theorem 23 is needed.) This full order is constructed so that the induced order on each jump factor D/C of C is Archimedean, hence no new convex subgroups arise. Thus C turns out to be the convex family of the newly defined full order of G.

III. Example

Let F be the free group on x and y, P_4 the fourth term in the lower central series of F and $G = F/F_4$. Group G is the free nilpotent group of class 3 on two generators* Such groups are discussed in Kurosh [4J. In G let 3 be the family {G, G_{2J} [1}}, where G_2 is the second term in the lower central series, namely (G,G). Since 3 is a series of length 2 it cannot be a central series (G is nilpotent of class 3), so by theorem 2 it cannot be the convex family with respect to any full order of G, However we will now show that it does satisfy the Rieger-Podderyugin conditions. Thus these conditions do not characterize the convex families of a group which admits full orders.

On to the conditions of theorem 1:

(i) Immediate,

(ii) Since G_2 is normal in G, this condition is satisfied.

(iii) $^G/^G 9$ anc* $^G 7$^^ =: $^G 2$ are eas^ly shown to be Abelian, They are known to be torsion free and finitely generated, hence they are isomorphic to subgroups of the additive reals,

(iv) Since $N(G_2)$ * G and since for the lower central series $\{G_k)$ of any group $(G_{\pm J}\ G_j) \subseteq G_{\pm + j}$, we have

$(N(G_2), N(G_2), G)$ £ G_3 £ G_2, and $(N(l), N(l), G_2)$ £ $_G$^ = {1}.

(v) Finally suppose that there is an a *e* G such that S(a) (1 G_i. *f^''.^'j* i = 1,2,4# Then we must show some conjugate of a is in G_i. We actually show that in each of the three cases a itself is in G_i.

If $i = 1$, then $G_i = G_1 = G$ and $a \in G$.

If $i = 2$ and $S(a) \cap G_i \neq \emptyset$, then for some finite non-empty set $\{x_i \mid i = 1,2,\dots,r\} \subseteq G$ $\prod x_i^{-1} a x_i \in G_2$, since all elements of $S(a)$ are of this form. But then $\prod_i x_i^{-1} a x_i = \prod_i (a(a,x_i)) \in G_2$. However $(a,x_i) \in G_2$ for each i so $\prod_1^r (a) = a^r \in G_2$. But G/G_2 is torsion free whence $a \in G_2$.

If $i = 4$ then $G_i = \{1\}$. Then if $S(a) \cap \{1\} \neq \emptyset$ there is a finite non empty set $\{y_i \mid i = 1,\dots,s\} \subseteq G$ such that $\prod y_i^{-1} a y_i = 1$. Thus $\prod (a(a,y_i)) = 1$. Since each $(a,y_i) \in G_2$ we have $\prod_1^s a(a,y_i) \equiv a^s \equiv 1 \bmod G_2$. As before G/G_2 is torsion free so $a \in G_2$. But then $(a,y_i) \in G_3$ for each i so $\prod a(a,y_i) \equiv a^s \equiv 1 \bmod G_3$. Since G_2/G_3 is also torsion free it follows that $a \in G_3$. Then $(a,y_i) \in G_4 = \{1\}$ for each i so finally $\prod a(a,y_i) = a^s = 1$. But G is torsion free so $a = 1$ and we are done.

References

1. FUCHS, L., 'Partially Ordered Algebraic Systems', Addison-Wesley, Reading, Mass, 1963.

2. GRAHAM, G. P., 'On Dimension Subgroups. Some Ordering Properties of Torsion Free Locally Nilpotent GroupsJ, Doctoral Dissertation, University of Wisconsin, 1965.

3. IWASAWA, K., '0n Linearly Ordered Groups', Journal of the Mathematical Society of Japan, vol. 1 (1948) pp. 1-9.

4. KUROSH, A. G#, 'The Theory of Groups', vol. 3, Chelsea, New York, New York, 1960.

5. NEUMANN, B. H., *0n ordered groups', American Journal of Mathematics, vol. 71 (1949) pp. 1-18.

6. NEUMANN, B. H., 'An Embedding Theorem for Algebraic Systems', Proceedings of the London Mathematical Society (3) 4 (1954) pp. 138-153.