BANACH SEQUENCE SPACES

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Let (R^2, N_1) be a two dimensional normed linear space. If N_1 satisfies the condition (a) below it is shown by adopting an iteration procedure that N_1 determines a Banach Sequence space B_{N_0}. This class of Banach spaces is a generalization of the classical l_p spaces $(1 \leq p \leq \infty)$. In this paper it is proposed to discuss the separability and reflexivity of these spaces intrinsically in terms of N_1.

In what follows N_1 is a norm on the coordinatized plane R^2 satisfying the condition

(a) if U is the unit ball of (R^2, N_1) and P is the positive quadrant of the plane then

$$\text{Convex hull } \{(1,0,(0,1),(0,0)\} \subset P \cap U \subset \text{Convex hull}\{(1,0), (0,0), (0,1), (1,1)\}.$$

Convex sets which satisfy the above inequality are known as fans and their relation to substitutive bases are discussed in Corson and Klee [1].

The norm N_1 may be utilized to define a norm N_{K-1} $(K > 2)$ on the K-dimensional space R^K by an iteration procedure as follows. If $(x_1, x_2, x_3) \in R^3$ let $N_2(x_1,x_2,x_3) = N_1(N_1(x_1,x_2), |x_3|)$. Since $P \cap U$ is a fan it is verified that $N_1(a_1, b_1) \leq N_1(a_2, b_2)$ if $0 \leq a_1 \leq a_2$ and $0 \leq b_1 \leq b_2$. As a consequence of this monotonicity it follows at once that N_2 is a norm on R^3. Proceeding inductively having defined the norm N_{K-1} on R^K let N_K be defined on R^{K+1} by setting $N_K(x) = N_1(N_{K-1}(x_1, \ldots, x_K), |x_{K+1}|)$ if $x = (x_1, \ldots, x_K, x_{K+1})$. Then N_K is a norm on R^{K+1}. With any sequence x let us denote

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the K-vector $(x_1, x_2, \ldots, x_K)$ by $x|K$. It is verified that $\{N_{K-1}(x|K)\}_{K \geq 2}$ is an increasing sequence. Let B denote the set of all sequences x such that $\{N_K(x|K+1)\}_{K \geq 1}$ is bounded. Then B is a linear space and the function $N_O(x) = \sup_{K \geq 2} \{N_{K-1}(x|K)\}$ is a norm on B. The normed linear space (B, N_O) is denoted by B_{N_O}.

Remark 1. If $N_1(x_1,x_2) = [|x_1|^p + |x_2|^p]^{1/p}$ for some p, $1 \leq p \leq \infty$ then B_{N_O} is the l_p sequence space and of course conversely.

Proposition 1. The normed linear space B_{N_O} is a Banach space.

Proof. Let $\{x^i\}_{i \geq 1}$ be a Cauchy sequence in B_{N_O}. Clearly $\{x_K^i\}$ is a Cauchy sequence of reals for a fixed $K \geq 1$. Hence $\{x^i\}$ converges coordinatewise to a sequence y. Further since $\{x^i\}$ is a Cauchy sequence $N_O(x^i) \leq K$ for all i and for some nonnegative real number K. Thus $N_{p-1}(y|p) = \lim_{i \to \infty} N_{p-1}(x^i|p) \leq N_O(x^i) \leq K$. Hence $y \in B_{N_O}$. From the definition of N_O it follows that for any $\epsilon > 0$ for large p, $N_O(y-x^i) \leq N_{p-1}((y - x^i)|p) + \epsilon$. Since $N_{p-1}((y - x^i)|p) \to 0$, as $i \to \infty$, $N_o(y - x^i) \to 0$ i.e. $x^i \to y$ in the space B_{N_O}. Hence B_{N_O} is a Banach space.

Before proceeding to discuss separability and reflexivity of the space B_{N_O} we establish a lemma.

Lemma 1. If $x \in B_{N_O}$ then x is a bounded sequence and $\sup_{i \geq 1} |x_i| \leq N_O(x)$. Further if $N_1(1,a) = 1$ for some $a > 0$ then the Banach space l_∞ is isomorphic to B_{N_O} while if $N_1(1,a) > 1$ for all $a > 0$ then every sequence in B_{N_O} is a null sequence.

Proof. Suppose $x \in B_{N_0}$. Since the condition (a) implies that the unit ball U of (R^2, N_1) is a subset of the unit ball of R^2 with the supremum norm it follows that $\sup(|x_1|,|x_2|) \leq N_1(x_1, x_2)$. Thus for every integer $p \geq 2$ $\sup_{1 \leq i \leq p}|x_i| \leq N_{p-1}(x_1, x_2,\ldots, x_p)$. Hence $\sup_{i\geq 1} |x_i| \leq N_0(x)$.

Next let for some $a > 0$ $N_1(1,a) = 1$. From the definition of N_0 it readily follows that $N_0(y) = 1$ where y is the sequence such that $y_1 = 1$ and $y_i = a$ for all $i > 1$. Hence if x is the constant sequence whose range is $\{a\}$ then $x \in B_{N_0}$. Thus every bounded sequence is in B_{N_0} i.e. the vector spaces l_∞ and B_{N_0} consist of the same elements. Since $\sup_{i\geq 1} |x_i| \leq N_0(x)$ for all $x \in B_{N_0}$ and B_{N_0} and l_∞ are Banach spaces, B_{N_0} is isomorphic to l_∞. In particular B_{N_0} is not separable.

Next let us assume $N_1(1,a) > 1$ for all $a > 0$. We note that non-zero constant sequences are not in B_{N_0}. For if such a sequence is in B_{N_0} then clearly the sequence I, the constant sequence with range $\{1\}$, is in B_{N_0}. Let $N_0(I) = m$. Since $N_1(1,1) > 1$, $m > 1$. It is verified by induction that

$$N_0(I|p + 1) \geq [N_1(1, 1/m)]^{p-2} N_1(1,1)$$

for $p \geq 2$. Thus $N_0(I)$ is infinite and $I \in B_{N_0}$. Next let $x \in B_{N_0}$. If possible for some $\epsilon > 0$ let there be a subsequence $\{x_{n_i}\}$ in x such that $|x_{n_i}| > \epsilon$ for all $i \geq 1$. Since $x \in B_{N_0}$, the sequence x in which all elements of x other than x_{n_i} are replaced by zero is also in B_{N_0}. By definition of the norm N_0 it then follows that the sequence y for which $y_i = x_{n_i}$ is in B_{N_0}. However $N_0(\epsilon I) \leq N_0(y)$. Thus $I\epsilon B_{N_0}$ which is a contradiction. Hence $x\epsilon B_{N_0}$ implies x is a null sequence.

Remark 2. From the above lemma it follows that either B_{N_0} is the same as the space l_∞ or $B_{N_0} \subset C_0$ where C_0 is the linear space of null sequences. Further since the condition (a) implies $|x_1| + |x_2| \geq N_1(x_1, x_2)$ for all $(x_1, x_2)\epsilon R^2$ it follows that $x\epsilon l_1$ implies $\sum_{i\geq 1}|x_i| \geq N_0(x)$ and $x\epsilon B_{N_0}$.

Thus if $B_{N_0} \neq l_\infty$ then $l_1 \subset B_{N_0} \subset C_0$ and the identity mappings i: $l_1 \rightarrow B_{N_0}$ and i: $B_{N_0} \rightarrow C_0$ are continuous. However it does not follow in this second case that B_{N_0} is separable (cf example at the end of this note).

We next proceed to show that the Banach space B_{N_0} is isometrically isomorphic to the conjugate of a Banach Space. In lemma 2 we obtain the adjoint of the norm N_{K-1} on R^K. Let M_1 be the adjoint norm of N_1. If U_1, U_∞ denote the unit balls of R^2 corresponding to the norms $|x_1| + |x_2|$, and $\mathrm{Sup}(|x_1|, |x_2|)$ then the unit ball U_{N_1} of (R^2, N_1) satisfies the inequality $U_1 \subset U_{N_1} \subset U_\infty$. Hence $U_1 \subset U_{M_1} \subset U_\infty$ by considering the polar sets. Thus (R^2, M_1) also satisfies the condition (a) and the iteration procedure determines the Banach Space B_{M_0}.

Lemma 2. The adjoint space of (R^p, N_{p-1}) is the space (R^p, M_{p-1}) where M_{p-1} is the p dimensional norm determined by M_1 through iteration procedure.

Proof. It suffices to show that if $(f_1, f_2, \ldots f_p)\epsilon R^p$ then

$M_{p-1}(f_1, f_2, \ldots, f_p) = \sup |\sum_{i=1}^{p} f_i x_i|$ where $N_{p-1}(x_1, x_2, \ldots, x_p) \leq 1$. The proof of this statement is by induction. Suppose that for some $K \geq 2$ every K-vector $(f_1, f_2, \ldots, f_K)$ satisfies $M_{K-1}(f_1, f_2, \ldots, f_K) = \sup |\sum_{i=1}^{K} f_i x_i|$ as $x = (x_1, x_2, \ldots, x_K)$ varies over the unit ball U_K of (R^K, N_{K-1}). Let $g = (g_1, g_2, \ldots, g_{K+1}) \in R^{K+1}$. By definition of M_1

$$M_K(g_1, g_2, \ldots, g_{K+1}) = M_1(M_{K-1}(g_1, g_2, \ldots, g_K), |g_{K+1}|)$$

$$\geq M_{K-1}(g_1, g_2, \ldots, g_K)\ N_{K-1}(x_1, x_2, \ldots, x_K)$$

$$+ |g_{K+1}|\ |x_{K+1}|$$

if $N_K(x_1, x_2, \ldots, x_{K+1}) \leq 1$. Thus by the induction hypothesis $M_K(g_1, g_2, \ldots, g_{K+1}) \geq |\sum_{i=1}^{K} g_i x_i| + |g_{K+1} x_{K+1}|$ since $(x_1, x_2, \ldots, x_K) \in U_K$. Hence $M_K(g_1, g_2, \ldots, g_{K+1}) \geq \|g\|$ where $\|\cdot\|$ is the adjoint norm of N_K.

Let us next show that $M_K(g_1, g_2, \ldots, g_K) \leq \|g\|$. Since M_{K-1} is the adjoint of N_{K-1} there exists a vector $x = (x_1, x_2, \ldots, x_K) \in U_K$ such that

$$M_{K-1}(g_1, g_2, \ldots, g_K) = \sum_{i=1}^{K} g_i x_i. \quad \text{Further}$$

$$M_K(g_1, g_2, \ldots, g_{K+1}) = M_1(M_{K-1}(g_1, g_2, \ldots, g_K), |g_{K+1}|)$$

$$= M_1(\sum_{i=1}^{K} g_i x_i, |g_{K+1}|).$$

Thus there exists a vector $(Z_1, Z_2) \in R^2$ with $Z_i \geq 0$ for $i=1,2$ and $N_1(Z_1, Z_2) \leq 1$ such that

$$M_K(g_1,g_2,\ldots,g_{K+1}) = \left(\sum_{i=1}^{K} g_i x_i\right) Z_1 + |g_{K+1}| Z_2$$

$$= \sum_{i=1}^{K} g_i x_i Z_1 + g_{K+1} \text{sign } g_{K+1} Z_2$$

Since $K+1 > 2$ and $(x_1,x_2,\ldots,x_K) \in U_K$

$N_K(x_1Z_1,\ x_2Z_1,\ldots,x_KZ_1,\ \text{sign } g_{K+1}Z_2)$

$$= N_1(N_{K-1}(x_1Z_1,x_2Z_1,\ldots,x_KZ_1),Z_2)$$

$$\leq N_1(Z_1,Z_2) \leq 1.$$ Hence

$M_K(g_1,g_2,\ldots,g_{K+1}) \leq \|g\|$. The proof is complete.

In the next theorem C_{N_0} is the closure of the linear subspace of finite sequences in B_{N_0}. Clearly the set of unit vectors $\{e^i\}_{i \geq 1}$ where $e^i_j = 1$ if $i = j$ and $e^i_j = 0$ if $i \neq j$ is a Schauder base for the space C_{N_0}.

Theorem 1. The Banach space B_{N_0} is isometrically isomorphic with the conjugate space of C_{M_0}.

Proof. Let $C^*_{M_0}$ denote the conjugate of C_{M_0} and let the adjoint norm be $\|\cdot\|$. Since $\{e^i\}_{i \geq 1}$ is a base of C_{M_0} it follows that $f \in C^*_{M_0}$ implies that $f(x) = \sum_{i \geq 1} f_i x_i$ where $f_i = f(e^i)$ and $x = \{x_i\}_{i \geq 1}$ is a sequence in C_{M_0}. Since for $z = (z_1,z_2,\ldots,z_p) \in U_{p-1}$, the unit ball of $(R^p,\ M_{p-1})$,

$$|f(z)| = \left|\sum_{i=1}^{p} f_i z_i\right| \leq \|f\|$$ it follows that

$$\left|\sum_{i=1}^{p} f_i z_i\right| \leq \|f\| \text{ for all } z \in U_{p-1}.$$

Thus $N_{p-1}(f_1,f_2,\ldots,f_p) \leq \|f\|$ for all $p \geq 2$.

Hence the sequence $\{f_i\}_{i \geq 1} \epsilon B_{N_O}$ and $N_O(\{f_i\}_{i \geq 1}) \leq \|f\|$. Since each function $f \epsilon C^*_{M_O}$ determines a unique sequence $\{f_i\}_{i \geq 1}$ where $f_i = f(e^i)$ the mapping σ: $C^*_{M_O} \rightarrow B_{N_O}$ defined by $\sigma(f) = \{f_i\}_{i \geq 1}$ is a linear operator and (1) $N_O(\sigma(f)) \leq \|f\|$. The mapping σ is onto. For if the sequence $f = \{f_i\}_{i \geq 1}$ is in B_{N_O} then consider the linear functional g on C_{M_O} defined by $g(x) = \sum_{i \geq 1} f_i x_i$. Now if the sequence x is such that $M_O(x) \leq 1$ then $M_{p-1}(x|p) \leq 1$ for every p-vector $x|p$. Thus $|\sum_{i=1}^{p} f_i x_i| \leq N_{p-1}(f_1, f_2, \ldots, f_p)$. Hence

$|g(x)| \leq \sup_{p \geq 2} N_{p-1}(f_1, f_2, \ldots, f_p) = N_O(f)$. Thus

$\sup_{M_O(x) \leq 1} |g(x)| \leq N_O(f)$. Hence $g \epsilon C^*_{M_O}$, $\sigma(g) = f$ and (2) $\|g\| \leq N_O(f) = N_O(\sigma(g))$. Thus σ is onto B_{N_O} and from inequalities (1) and (2) it follows that σ is an isometry. Thus B_{N_O} is isometrically isomorphic to $C^*_{M_O}$.

Corollary. The Banach space B_{N_O} is separable if and only if $\{e^i\}_{i \geq 1}$ is a base for the space.

Proof. We note that as a consequence of the monotonicity of M_O i.e. $x, y \epsilon C_{M_O}$ and $x_i \geq y_i \geq 0$ imply $M_O(x) \geq M_O(y)$ for every $x \epsilon C_{M_O}$ the series $x = \Sigma x_i e^i$ is subseries convergent. Thus $\{e^i\}_{i \geq 1}$ is an unconditional base for the space C_{M_O}. Since $CM^*_O = B_{N_O}$, B_{N_O} is separable if and only if the sequence biorthogonal to $\{e^i\}_{i \geq 1}$ is a base for the space B_{N_O} as a consequence of Theorem 5 on page 77, Day [2].

Remark 3. As a consequence of the above corollary it follows that the following statements are equivalent. (1) B_{N_O} is separable (2) $B_{N_O} = C_{N_O}$ i.e. $\{e^i\}_{i\geq 1}$ is a base of B_{N_O} (3) B_{N_O} has a base. Further if B_{N_O} is separable then $B^*_{N_O}$ is isometrically isomorphic to B_{M_O}.

The next theorem provides a characterization of reflexive Banach spaces B_{N_O}.

Theorem 2. The Banach space B_{N_O} is reflexive if and only if B_{N_O} and B_{M_O} are separable.

Proof. Suppose B_{N_O} and B_{M_O} are separable. Then $B^{**}_{N_O} = (B_{M_O})^* = B_{N_O}$ by Remark 3. As already noted in proving the preceding corollary, B_{N_O} has an unconditional base. Further $B^{**}_{N_O} = B_{N_O}$ implies that $B^{**}_{N_O}$ is separable. Hence by Theorem 4 on p. 76 of [2], B_{N_O} is reflexive. Conversely suppose B_{N_O} is reflexive. Then the closed subspace C_{N_O} is also reflexive. Hence the base $\{e^i\}_{i\geq 1}$ of C_{N_O} is boundedly complete by a theorem of James. (cf Theorem 3 on page 71, Day [2]). Thus if $x = \{x_i\}_{i\geq 1} \in B_{N_O}$ then since $N_O(\sum_{i=1}^{p} x_i e^i) \leq N_O(x)$ for $p \geq 1$ and since $\{e^i\}_{i\geq 1}$ is a boundedly complete base of C_{N_O} it follows that $x \in C_{N_O}$. Thus $B_{N_O} = C_{N_O}$ and B_{N_O} is separable. Since $C^*_{N_O} = B_{M_O}$ and C_{N_O} is reflexive it follows that B_{M_O} is also separable.

We are not able to obtain a complete characterization for the separability of the Banach space B_{N_O} intrinsically in terms of N_1. However, we establish three theorems. The first one provides a sufficient condition for the existence of a base in B_{N_O} (equivalently for the separability of B_{N_O}) in terms of the norm N_2. The second and third provide a necessary and a sufficient condition for the existence of a base (equivalently for the separability of B_{N_O}) in terms of the norm N_1.

example of a two dimensional norm N_1 such that each sequence in the corresponding Banach space B_{N_0} is a null sequence, yet B_{N_0} fails to be separable.

Theorem 3. If for some $\delta > 0$, $0 \leq x_i \leq \delta$ $i = 1,2,3$ implies $N_1(N_1(x_1,x_2), x_3) \geq N_1(x_1,N_1(x_2,x_3))$ and $N_1(1,a) > 1$ for all $a > 0$ then the sequence $\{e^i\}_{i\geq 1}$ is a base for the Banach space B_{N_0}.

Proof. Since $N_1(1,a) > 1$ for $a > 0$ lemma 1 implies that the sequences in B_{N_0} are null sequences. Suppose $\{e^i\}_{i\geq 1}$ fails to be a base for B_{N_0}. Then there exists a sequence $x = \{x_i\}_{i\geq 1}$ and a sequence of integers p_i such that $0 \leq x_i \leq \delta$ and for some $\epsilon > 0$ for all $i \geq 2$

$$N_{p_i-1}(x_{q_i+1}, x_{q_i+2}, \ldots, x_{q_i+p_i}) \geq \epsilon \quad \text{where } q_i = \sum_{t=1}^{i-1} p_t .$$

Since $N_1(N_1(x_1,x_2), x_3) \geq N_1(x_1,N_1(x_2,x_3))$ it is verified inductively that

$$N_{\sum_{i=1}^{n} p_i - 1}\left(x_1, x_2, \ldots, x_{\sum_{i=1}^{n} p_i}\right)$$

$$\geq N_{n-1}(z_1, z_2, \ldots, z_n) \tag{A}$$

where for $1 \leq i \leq \gamma - 1$

$$z_i = N_{p_i-1}(x_{q_i+1}, x_{q_i+2} \ldots, x_{q_i+p_i})$$

where $q_i = \sum_{t=1}^{i-1} p_t$.

By the choice of the sequence x, $z_i \geq \epsilon$ for $1 \leq i \leq r-1$. Thus the inequality (A) implies

$N_0(x) \geq N_{r-2}(\xi)$ where ξ is the (r-1) - vector with each coordinate $\xi_i = \epsilon$. Since this inequality is satisfied for large r the constant sequence with range $\{\epsilon\}$ is in B_{N_0} which is a contradiction. The proof is complete.

Next we proceed to obtain a necessary condition and a sufficient condition for the nonexistence of a basis in B_{N_0} in terms of the asymptotic behavior of the function $\alpha(s) = N_1(1,s) - 1$ as $s \to 0+$. We assume in the rest of this paper that $N_1(1,a) > 1$ for $a > 0$ so that every sequence in B_{N_0} is a null sequence. We start by establishing a useful lemma. With a little abuse of notation we denote the norm of an n-vector $(x_1,\ldots,x_n)$ by $N_0(x_1,\ldots,x_n)$ instead of $N_{n-1}(x_1,\ldots,x_n)$.

Lemma 3. The Banach space B_{N_0} does not admit a base if and only if for some $\epsilon > 0$ there exists for each pair of positive numbers δ,η a finite sequence $x_1,x_2,\ldots,x_n$ such that

(1) $0 < x_i < \eta\epsilon \quad 1 \leq i \leq N$

(2) $N_0(x_1,\ldots,x_n) = \epsilon$

(3) $N_0(\epsilon,x_1,\ldots,x_n) \leq (1+\delta)\epsilon$.

Proof. Suppose N_0 satisfies the above property. With a fixed choice of $\delta_o > 0$ select positive sequences $\{\delta_i\}_{i\geq 1}$ and $\{\eta_i\}_{i\geq 1}$ such that

(4) $\prod^{\infty} (1 + \delta_i) = 1 + \delta_o$

(5) $\eta_i \to 0$

By hypothesis we can select for each i a finite sequence $\underset{\sim}{x}^{i} = (x_1^i, x_2^i, \ldots, x_{N_i}^i)$ which satisfies

(6_1) $\quad 0 < x_j^{\,i} < \eta_i \epsilon$

(6_2) $\quad N_O(\underset{\sim}{x}^{i}) = \epsilon$

and (6_3) $\quad N_O(\epsilon, \underset{\sim}{x}^{i}) \leq (1 + \delta_i)\epsilon$.

Now let $\underset{\sim}{x}$ denote the countable sequence

$$\underset{\sim}{x} = (\underset{\sim}{x}^{1}, \underset{\sim}{x}^{2}, \underset{\sim}{x}^{3}, \ldots)$$

(5) and (6_1) imply $\underset{\sim}{x}$ is a null sequence. We proceed to show that $\underset{\sim}{x} \in B_{N_O}$. It follows from ($6_2$) and the definition of N_O that

(7) $N_O(\underset{\sim}{x}^{1};\ldots;\underset{\sim}{x}^{q}) = N_O(N_O(\epsilon;\underset{\sim}{x}^{2});\underset{\sim}{x}^{3};\ldots;\underset{\sim}{x}^{q})$.

Using (6_3) we deduce from (7)

(8)
$$\begin{aligned} N_O(\underset{\sim}{x}^{1};\ldots;\underset{\sim}{x}^{q}) &\leq N_O((1 + \delta_2)\epsilon;\underset{\sim}{x}^{3};\ldots;\underset{\sim}{x}^{q}) \\ &\leq (1+\delta_2)N_O(\epsilon;(1+\delta_2)^{-1}\underset{\sim}{x}^{3};\ldots;(1+\delta_2)^{-1}\underset{\sim}{x}^{q}) \\ &\leq (1+\delta_2)N_O(\epsilon;\underset{\sim}{x}^{3};\ldots;\underset{\sim}{x}^{q}) \end{aligned}$$

by the monotonicity of N_O. By repeating the computations in (7) and (8) we deduce that

(9) $N_O(\underset{\sim}{x}^{1};\underset{\sim}{x}^{2};\ldots;\underset{\sim}{x}^{q}) \leq (1+\delta_2)(1+\delta_3)N_O(\epsilon;\underset{\sim}{x}^{4};\ldots;\underset{\sim}{x}^{q})$

By induction we conclude

$$(10) \qquad N_0(\underset{\sim}{x}^1 ; \underset{\sim}{x}^2 ; \ldots ; \underset{\sim}{x}^q) \le \epsilon \prod_2^q (1+\delta_i) < \prod_2^\infty ((1+\delta_i))]\epsilon$$
$$= (1+\delta_0)\, \epsilon .$$

(10) implies that $\underset{\sim}{x} \in B_{N_0}$.

In order to see that B_{N_0} has no base by the monotomicity of N_0 observe that

$$(11) \qquad N_0(\underset{\sim}{x}^K ; \underset{\sim}{x}^{K+1} ; \ldots) \ge N_0(\underset{\sim}{x}^K ; 0,0,\ldots) = \epsilon$$

so that $N_0(\underset{\sim}{x}^K ; \underset{\sim}{x}^{K+1} ; \ldots) \not\to 0$ as $K \to \infty$. Thus $\{e^i\}_{i\ge 1}$ is not a base of B_{N_0}. Hence by Remark 3 B_{N_0} does not admit a base.

Conversely suppose the sequence $\{e^i\}_{i\ge 1}$ is not a base for the space B_{N_0}. Then by earlier results we can find a null sequence $\underset{\sim}{x} = (x_1, x_2, \ldots) \in B_{N_0}$ such that for some $\epsilon > 0$

(12) $N_0(x) = A < \infty$

(13) $A \ge N_0(x_m, x_{m+1}, \ldots) > 2\epsilon > 0$ for $m=1,2,\ldots$

By discarding an initial segment of x if necessary we can suppose without loss of generality that $0 < x_i < \eta\epsilon$ for a preassigned $\eta > 0$. Now define a sequence of integers $\{J_i\}_{i\ge 1}$ such that

$$\epsilon < N_0(x_{q_i+1}, \ldots, x_{q_i+N_{i+1}}) \le 2\epsilon$$

where $q_i = \sum_{t=1}^{i-1} J_t$

By (13) this is certainly possible for sufficiently small η.

Let us decompose the sequence $\underset{\sim}{x}$ as

$$x = (x_1, \ldots, x_{J_1};\ x_{J_1+1}, \ldots, x_{J_1+J_2}; \ldots)$$
$$= (\underset{\sim}{x}^1;\ \underset{\sim}{x}^2; \ldots)$$

where $2\epsilon \geq N_o(\underset{\sim}{x}^i) > \epsilon$.

Now multiply the finite sequences $\underset{\sim}{x}^i$ by $\theta_i = \dfrac{\epsilon}{N_o(\underset{\sim}{x}^i)}$

(clearly $\frac{1}{2} \leq \theta_i < 1$). Thus if $y^i = \theta_i \underset{\sim}{x}^i$, $i \geq 1$, than

(14) $y_o{}^i < \eta\epsilon$ and $N_o(\underset{\sim}{y}^i) = \epsilon$.

By the monotonicity of N_o, $\underset{\sim}{y} = (\underset{\sim}{y}^i;\ \underset{\sim}{y}^2; \ldots) \in B_{N_o}$ since $x \in B_{N_o}$.

We complete the proof by showing that for each δ there is a finite sequence $x_1, \ldots, x_N$ such that $0 < x_i < \eta\epsilon$, $N_o(x_1, \ldots, x_N) = \epsilon$ and $N_o(\epsilon, x_1, \ldots, x_N) \leq (1+\delta)\epsilon$. If this is false there exists a $\delta^1 > 0$ for which $0 < x_i < \eta\epsilon$ and $N_o(x_1, x_2, \ldots, x_N) = \epsilon$ imply $N_o(\epsilon, x_1, x_2, \ldots, x_N) \geq (1+\delta^1)\epsilon$. However in this case we claim that for every sequence

$$\underset{\sim}{x} = (\underset{\sim}{x}^1;\ \underset{\sim}{x}^2; \ldots,\ \underset{\sim}{x}^{2^K}) = (x_1, \ldots x_{N_K})$$

which satisfies $0 < x_j^i < \eta\epsilon$, $N_o(\underset{\sim}{x}^i) = \epsilon$ that

(16) $N_o(\underset{\sim}{x}) \geq (1 + \delta^1)^K\epsilon$, $K \geq 1$. This claim is justified as follows by induction. For $K = 1$ (16) is clearly satisfied. Suppose (16) is true for some $K > 1$. Then

(17) $$N_o(\underset{\sim}{x}^1;\ \underset{\sim}{x}^2; \ldots; \underset{\sim}{x}^{2^K};\ \underset{\sim}{x}^{2^K+1}; \ldots,\ \underset{\sim}{x}^{2^{K+1}})$$
$$= N_o(N_o(\underset{\sim}{x}^1;\ \underset{\sim}{x}^2; \ldots; \underset{\sim}{x}^{2^K});\ \underset{\sim}{x}^{2^K+1}; \ldots; \underset{\sim}{x}^{2^{K+1}})$$
$$\geq N_o((1+\delta^1)^K\epsilon;\ \underset{\sim}{x}^{2^K+1}; \ldots;\ x^{2^{K+1}})$$

and in addition

(18) $\quad N_o(x^{2^K+1};\ldots;x^{2^{K+1}}) \geq (1+\delta^1)^K\epsilon$

By (18) we see that for some c, $0 < c \leq 1$,

$$N_o(c\underset{\sim}{x}^{2^K+1};\ldots;\ c\underset{\sim}{x}^{2^{K+1}}) = (1+\delta^1)^K\epsilon$$

while by using the monotonicity and (15) we obtain

(19) $\quad N_o(\underset{\sim}{x}^1;\ \underset{\sim}{x}^2;\ldots;\ \underset{\sim}{x}^{2^K};\ \underset{\sim}{x}^{2^K+1};\ldots;\underset{\sim}{x}^2)$

$$\geq N_o((1+\delta^1)^K\epsilon;\ C\ x^{2^K+1};\ldots;\ C\ x^{2^{K+1}})$$

$$= (1+\delta^1)^K N_o(\epsilon;\ (1+\delta^1)^{-K}Cx^{2^K+1};\ldots;(1+\delta^1)^{-K}Cx^{2^{K+1}})$$

$$\geq (1+\delta^1)^{K+1}\epsilon.$$

This completes the induction argument.

Now (16) clearly implies that $\underset{\sim}{y} = (\underset{\sim}{y}^1;\ \underset{\sim}{y}^2;\ldots) \in B_{N_O}$ which is the desired contradiction.

Let us recall that α denotes the increasing function defined by $\alpha(s) = N_1(1,s)-1$, $s \geq 0$.

Theorem 4. B_{N_O} fails to have a base provided the function α satisfies the following condition for each $\lambda > 1$,

(*) $\quad (\ln 1/\eta)^{-1}\ (\min\limits_{t\leq\eta} \frac{\alpha(\lambda t)}{\alpha(t)}) \to \infty \quad$ as $\quad \eta \to 0$.

Proof. By the lemma it suffices to show that for some fixed $\epsilon > 0$ there exists for each pair $\delta,\ \eta > 0$ a sequence $x_1,\ldots,x_N$ such that (1) $0 < x_i \leq \eta\epsilon$ (2) $N_o(x_1,\ldots,x_N) = \epsilon$ and (3) $N_o(\epsilon,\ x_1,\ldots,x_N) \leq (1+\delta)\epsilon$. Moreover we may without loss of generality require η to be sufficiently small so that the following conditions are satisfied

(4_1) $\alpha(\eta) \leq 1/2, \quad \eta \leq 1/2,$

(4_2) $(\ln 1/\eta)^{-1} \left(\min_{t\leq\eta} \frac{\alpha((1+\delta/2)t)}{\alpha(t)}\right) \geq \frac{4}{\ln(1+\delta/4)},$

and (4_3) $[N_1(1,\eta)]^2 (1+\delta/2)(1+\delta/4) \leq 1+\delta.$

We now construct the desired sequence by taking

(5) $x_1 = \eta\epsilon, \quad x_2 = x_3 = \ldots = x_{N-1} = \eta^2\epsilon$ where N is determined by

(6) $N_o(x_1,x_2,\ldots,x_{N-1}) < \epsilon \leq N_o(x_1,x_2,\ldots,x_{N-1}, \eta^2\epsilon)$

and then selecting $x_N \epsilon\ (0,\eta^2\epsilon)$ so that (2) holds. That there exists an N satisfying (6) follows from the fact that B_{N_o} contains only null sequences.

If we denote $N_o(x_1,\ldots,x_j) = c_j$, $1 \leq j \leq N$ then the scalars c_j are recursively determined by

(7) $c_1 = x_1, \quad c_{j+1} = c_j N_1(1, \frac{x_{j+1}}{c_j}) = c_j[1+\alpha(\frac{x_{j+1}}{c_j})]$

Hence by (2)

(8) $x_1 \prod_{j=1}^{N-1} (1 + \alpha(\frac{x_{j+1}}{c_j})) = c_N = \epsilon$

Moreover

(9) $c_j \leq \epsilon, \quad \frac{x_{j+1}}{c_j} \leq \frac{x_{j+1}}{x_1} \leq \eta, \quad j = 1,\ldots, N-1.$

On the other hand if we denote $N_o(\epsilon,x_1,\ldots,x_j) = d_j$ for $1 \leq j \leq N$ then the scalars d_j are recursively determined by (10) $d_1 = N_o(\epsilon, x_1) = \epsilon N_1(1,\eta)$

$$d_{j+1} = d_j N_1(1, \frac{x_{j+1}}{c_j}) = d_j [1 + \alpha(\frac{x_{j+1}}{d_j})]$$

Thus we have

$$(11) \quad \epsilon N_1(1,\eta) \prod_{j=1}^{N-1} [1 + \alpha(\frac{x_{j+1}}{d_j})] = d_N = N_o(\epsilon, x_1, \ldots, x_N)$$

and (12) $d_j \geq \epsilon$. Select the index m so that

$$(13) \quad x_1 \prod_{j=1}^{m-1} [1 + \alpha(\frac{x_{j+1}}{c_j})] \leq \frac{\epsilon}{1+\delta/2} < x_1 \prod_{j=1}^{m} [1 + \alpha(\frac{x_{j+1}}{c_j})]$$

Then by (8) we deduce that

$$(14) \quad \prod_{j=m+1}^{N-1} [1 + \alpha(\frac{x_{j+1}}{c_j})] \leq 1 + \delta/2 .$$

We proceed to appraise $N_o(\epsilon, x_1, \ldots, x_N)$. By monotonicity of $\alpha(s)$ for $s \geq 0$ we conclude since $d_j > c_j$ that

$$(15) \quad \prod_{j=M+1}^{N-1} (1 + \alpha(\frac{x_{j+1}}{d_j})) \leq 1 + \delta/2.$$

On the other hand since (14) ensures that $c_j \leq \frac{\epsilon}{1+\delta/2}$, $1 \leq j \leq m - 1$. We have

$$(16) \quad \prod_{j=1}^{m-1} [1 + \alpha(\frac{x_{j+1}}{d_j})] = \prod_{j=1}^{m-1} [1 + \alpha(\frac{c_j}{d_j} \frac{x_{j+1}}{c_j})]$$

$$\leq \prod_{j=1}^{m-1} [1 + \alpha(\frac{1}{1+\delta/2} \frac{x_{j+1}}{c_j})]$$

$$\leq \exp [\sum_{j=1}^{m-1} \alpha(\frac{1}{1+\delta/2} \frac{x_{j+1}}{c_j})]$$

Now by (13) we have

$$(17) \quad \sum_{j=1}^{m-1} \ln [1 + \alpha(\frac{x_{j+1}}{e_j})] \leq \ln \frac{\epsilon}{(1+\delta/2)x_1} = \ln \frac{1}{(1+\delta/2)\eta}$$

By (9) and (4_1) $\alpha(\frac{x_{j+1}}{c_j}) \leq \alpha(\eta) \leq 1/2$

whereby $\ln [1+\alpha(\frac{x_{j+1}}{c_j})] = \sum_{K=1}^{\infty} \frac{(-1)^{K+1}}{K} [\alpha(\frac{x_{j+1}}{c_j})]^K$

$$> 1/2 \, \alpha(\frac{x_{j+1}}{c_j}) .$$

Thus (17) implies

$$(18)\quad \sum_{j=1}^{m-1} \alpha\left(\frac{x_{j+1}}{c_j}\right) \leq 2 \ln \frac{1}{(1+\delta/2)\eta}$$

From (4_2) and (4_1) it follows that

$$(19)\quad \sum_{j=1}^{m-1} \alpha\left(\frac{1}{1+\delta/2}\, \frac{x_{j+1}}{c_j}\right) \leq \frac{\ln(1+\delta/4)}{4 \ln 1/\eta} \cdot 2 \ln \frac{1}{(1+\delta/2)\eta} \leq \ln\,(1+\delta/4).$$

Substituting (19) into (16) we obtain

$$(20)\quad \sum_{j=1}^{m-1} [1+\alpha(\frac{x_{j+1}}{d_j})] \leq 1+\delta/4.$$

Finally inserting (15) and (20) into (12) we obtain

$$N_o(\epsilon, x_1, x_2, \ldots, x_n)$$
$$\leq \epsilon\, N_1(1,\eta)\,(1+\delta/4)\,[1+\alpha(\frac{x_{m+1}}{d_m})]\,(1+\delta/2)$$
$$\leq \epsilon\, [N_1(1,\eta)]^2\,(1+\delta/4)\,(1+\delta/2)$$

Using (4_3) it is verified that

$$N_o(\epsilon, x_1, \ldots, x_n) \leq \epsilon\,(1+\delta).$$

The proof is complete.

Adopting techniques similar to the proof of the preceeding theorem we obtain the following necessary condition for the nonexistence of a base.

Theorem 5. In order that B_{N_O} may not have a base it is necessary that for each $\lambda > 1$

$$\max_{t\leq\eta} \left(\frac{\alpha(\lambda t)}{\alpha(t)}\right) \rightarrow \infty \quad \text{as} \quad \eta\rightarrow 0$$

Next we proceed to construct an example of a nonseparable Banach space B_{N_O} whose elements are null sequences.

Example. Let (x_1, x_2) represent the coordinates of a point in the plane with reference to a pair of orthogonal axes. Consider the arc determined by $x_1 + x_1 e^{-x_1/x_2} = 1$, $3/4 \leq x_1 \leq 1$ and the line segment joining (0,1) and the point on the above arc corresponding to $x_1 = 3/4$. The above curve together with the line segments joining the origin to (0,1) and to (1,0) forms the boundary of a fan. This fan together with its reflections through the origin and across the axes is a convex set and determines a Minkowskian norm N_1 on R^2. It is verified that $N_1(1,s) = 1 + e^{-1/s}$ and $\alpha(s) = e^{-1/s}$ for small $s \geq 0$. Since $N_1(1,s) > 1$ for $s \geq 0$ every member of B_{N_O} is a null sequence. Futher since

$$(\ln \frac{1}{\eta})^{-1} \left(\min_{t \leq \eta} \frac{\alpha(\lambda t)}{\alpha(t)} \right) \longrightarrow \infty \quad \text{as} \quad \eta \to 0$$

it follows by theorem 4 that B_{N_O} does not admit a base. From the remark 3 we conclude that B_{N_O} is not even separable.

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