Extremal Problems for a Class of Functionals Defined on Convex Sets

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1. Let X = (X,2) be a measurable space, and let T be a class of positive measures M* defined on 2- . we consider a set H of non-negative functions belonging to \P{K) on X for all A e 77 (1 <, p < oo) , and we denote by C (H) the convex hull of H. If <r is an arbitrary positive measure on X, we define the functional $\Lambda(r)$ ($r_6C(H)$,$L^1(0'))$ by

$$(1) \qquad A(r) = \sup \frac{[\int_X r^p d\mu]^{\frac{1}{p}}}{\int_X r d\sigma} .$$

The following result is a useful tool in the treatment of numerous extremal problems involving eigenvalues of differential and integral equations.

Theorem I. rf j\.{r) JLS the functional defined by (1) , then

$$(2) \qquad \sup_{r \in C(H)} A(r) = \sup_{s \in H} A(s)$$

The proof of (2) is very simple. Since $H \subset C(H)$, (2) will follow from the inequality

$$(3) \qquad \sup_{r \in C(H)} A(r) \le \sup_{s \in H} A(s),$$

and it is sufficient to establish (3) for finite sums of the form

$$(4) \qquad r = \alpha_1 s_1 + \cdots + \alpha_n s_n, \ \alpha_k > 0, \ \sum_{k=1}^{n} \alpha_k = 1, \ s_k \in H.$$

By Minkowski's inequality, we have

$$\Big[\int_X r^p d\mu \Big]^{\frac{1}{p}} \le \sum_{k=1}^{n} \alpha_k \Big[\int_X s_k^p d\mu \Big]^{\frac{1}{p}}$$

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and thus, by (1),

$$\text{(5)} \qquad \left[\int_X r^p d\mu\right]^{\frac{1}{p}} \le \sum_{k=1}^{n} \alpha_k \Lambda(s_k) \int_X s_k d\sigma .$$

Since this holds for all $\mu \in \mathcal{M}$, it follows from (1) and (5) that

$$\Lambda(r) \int_X r\, d\sigma \le \sum_{k=1}^{n} \alpha_k \Lambda(s_k) \int_X s_k d\sigma$$

$$\le \sup_{s \in H} \Lambda(s) \sum_{k=1}^{n} \alpha_k \int_X s_k d\sigma$$

$$= \sup_{s \in H} \Lambda(s) \int_X r\, d\sigma .$$

Thus,

$$\Lambda(r) \le \sup_{s \in H} \Lambda(s)$$

if r is of the form (4). Since these functions are dense in $C(H)$, this implies (3) and completes the proof of Theorem I.

2. As an example of a functional which can be brought into the form (1), we consider the lowest eigenvalue $\lambda = \lambda(R)$ of the differential system

$$\text{(6)} \qquad y^{(2n)} + (-1)^n \lambda R(x) y = 0, \quad U(y) = 0, \quad (\lambda = \lambda(R))$$

where $R > 0$, $R \in L^1$ on an interval $[a,b]$, $U(y) = 0$ is a set of self-adjoint boundary conditions, and n is a positive integer. By classical results,

$$\frac{1}{\lambda(R)} = \sup \int_a^b R\, d\mu ,$$

where $d\mu = u^2(x)\, dx$ and $u(x)$ ranges over the class of functions with the following properties: (a) u satisfies the conditions $U(u) = 0$; (b) $u^{(n)}$ is of class L^2 on $[a,b]$ and is normalized by the condition

$$\int_a^b [u^{(n)}]^2 dx = 1.$$

In this case, we thus have

(7) $$[\lambda(R^p)]^{-1} = \lambda^p(R) \int_a^b R^p \, d\sigma,$$

and Theorem I shows that

(8) $$\inf_{R^p \in C(H)} \lambda^p(R) \int_a^b R^p d\sigma = \inf_{T^p \in H} \lambda^p(T) \int_a^b T^p d\sigma .$$

If the value of the right-hand side of (8) can be found, (8) thus provides the exact lower bound for the expression (7), where R ranges over $C(H)$ or over a subset of $C(H)$ which contains H.

3. The use of Theorem I as a source of estimates for functionals $\psi(f)$ is most likely to be successful in the case of convex sets $C(H)$ which are spanned by sets H of functions of very simple type. There are many such sets which are of interest in the applications. Two well-known examples are:

(a) the class of bounded non-increasing non-negative functions on an interval $[a,b]$; in this case H may be identified with the set of functions $A\chi_t$ $(t \in (a,b])$, where A is a suitable positive constant and χ_t is the characteristic function of the interval $[a,t]$;

(b) the class of non-negative concave functions on an interval $[a,b]$; this class is spanned by the functions $g(x,t)$ $(t \in [a,b])$, where $g(x,t) = A(x - a)(b - t)$ for $x \in [a,t]$ and $g(x,t) = A(t - a)(b - x)$ for $x \in [t,b]$.

Another example of this type--which does not seem to be found in the literature—is described in the following statement.

Theorem II. *Let* $(X, \mathcal{S}, \mu)$ *be a finite positive measure space, and let* $K = K(m, M, \mu)$ *be the class of measurable functions*

F on X for which

(9) $$-\infty < m \le F \le M < \infty$$

and

(10) $$\int_X F\,d\mu = [\gamma M + (1-\gamma)m]\,\mu(X)$$

$(0 \le \gamma \le 1)$, where m and M are, respectively, the essential infimum and the essential supremum of F on X.

If H denotes the subset of K consisting of the functions $g = m + (M - m)\chi(X_0)$, where $X_0 \subset X$, $\mu(X_0) = \gamma\mu(X)$, and $\chi(X_0)$ is the characteristic function of X_0; then $K \subset C(H)$.

If we set $F = m + (M - m) f$, (9) and (10) take the form $0 \le f \le 1$ and

(10') $$\int_X f\,d\mu = \gamma\,\mu(X),$$

respectively. It is thus sufficient to prove Theorem II for the case $m = 0$, $M = 1$.

Another simplification which can be made is the assumption that f be a step-function which takes only the values $0, \epsilon, 2\epsilon, \ldots, N\epsilon$, where $\epsilon N = 1$ and N is an arbitrary positive integer. Indeed, f may be approximated by functions f^* defined by setting $f^* = \epsilon k$ on the subset of X on which $\epsilon(k - \theta) < f \le \epsilon(k + 1 - \theta)$, where θ is a number in $(0,1)$, and $k = 0,1,\ldots,N$. Evidently, $\inf f^* = 0$, $\sup f^* = 1$, and

$$-\epsilon\theta\,\mu(X) < \int_X f\,d\mu - \int_X f^*\,d\mu \le \epsilon(1 - \theta).$$

Since $\int_X f^*\,d\mu$ is a continuous function of θ, this shows that θ may be so chosen that $\int_X f^*\,d\mu = \int_X f\,d\mu$ and thus, by (10'),

$$\int_X f^*\,d\mu = \gamma\,\mu(X).$$

If S, denotes the subset of X on which f >_ ek (k = 1,2,...,N - 1), we have

(11) $^{S}k+l£ S_{k}$, k = 1,...,N - 2

and

(12) eNytt(S_{N1}) ≤ $\int_X$ f d A ≤ eNμ(S_1).

Since, by (10'),

(13) $\int_X$ fd/* = 7€Nyk(X) = ^>k(X)sup *t*,

(12) implies that

(14) / i (S_{N-1}) ≤ $\eta\mu$(X) ≤ μ(S_1).

We denote by S_1^* a subset of S_1 for which

(15) yMs$_1^*$) = *zMx)*

and which, in addition, is such that

(16) $S_1^* \supseteq S_{N-1}$.

The right-hand inequality (14) shows that there are subsets S_1^* of S_1 for which (15) holds and it follows from (11) and the left-hand inequality (14) that S_1^* may be so chosen as to satisfy (16).

We now consider the function

(17) f_1 = f - €/(S|).

Since S| C Sp we have fT >. °- Because of (.16), we have

$$\sup f_1 = \sup f - e = (N - \cdot 1)€$$

and, by (13) and (15),

(18) $\int_X$ f l d/* = ?£(N " 1)/A t(X) = *J/*W**J?* f!

A comparison of (13) and (18) shows that the procedure leading from (13) to (18) can be repeated. There will thus exist a subset S3; of X such that the function

$$f_2 = f_x - € /(S_2^*)$$

is non-negative and satisfies

$$\int_X f_2 d\mu = \wedge e(N - 2)\mu(X) = \zeta /t(X) \sup f_2.$$

By applying this process N times, we arrive at a function f^ which vanishes identically, and we thus obtain a decomposition

$$(19) \qquad f = \sum_{k=1}^{N} € X \wedge(S_K^*)$$

We set

$$g\wedge = Ne7(S£) = / (S_k^*),$$

and we observe that, by (15) (and the corresponding formulas for S_k^*, $k = 2, \ldots, N$)

$$\int_X g_k d\mu = \eta\mu(X),$$

i.e., $9v^{€H*}$ Since, with $o\wedge_k = e = N^{-1}$, (19) may be written in the form

$$f = \sum_{k=1}^{N} \alpha_k g_k, \quad \sum_{k=1}^{N} \alpha_k = 1,$$

this shows that f€C(H), and Theorem II is proved.

4. As an illustration of the type of explicit inequality obtainable by means of Theorem I, we consider the eigenvalue problem (6) with the boundary conditions

$$(20) \quad u(a) = u'(a) = \cdots = u^{(n-1)}(a) = u^{(n)}(b) = u^{(n+1)}(b) = \cdots = u^{(2n-1)}(b) = 0.$$

If the coefficient R(x) belongs to the class listed under (a) in Section 3, we have the following result.

Theorem III. Let $\lambda = \lambda(R) = \lambda(R;a,b)$ be the lowest eigenvalue of the differential equation

$$(21) \qquad y^{(2n)} - (-1)^n \lambda R(x) y = 0$$

with the boundary conditions (20), where $R > 0$, $R \in L^1$ on [a,b] and n is a positive integer. If R(x) is non-increasing in [a,b], then

$$(22)\qquad \lambda^{\frac{1}{p}}(R)\int_a^b [(x-a)^{2n}R(x)]^{\frac{1}{p}}\frac{dx}{x-a} \geq \frac{p}{2n}\lambda^{\frac{1}{p}}(1;0,1)$$

for any $p \geq 1$. There will be equality in (22) whenever R(x) coincides with a characteristic function $\chi[a,t]$, where $t \in (a,b]$.

If we set

$$d\varphi = (x-a)^{\frac{2n}{p}-1}\,dx,$$

it follows from (8) that (22) will be established if we can show that

$$\inf_{t \in (a,b]} \lambda^{\frac{1}{p}}(\chi_t)\int_a^b [(x-a)^{2n}\chi_t]^{\frac{1}{p}}\frac{dx}{x-a} = \frac{p}{2n}\lambda^{\frac{1}{p}}(1;0,1),$$

where $\chi_t = \chi[a,t]$. Since

$$\int_a^b [(x-a)^{2n}\chi_t]^{\frac{1}{p}}\frac{dx}{x-a} = \int_a^t (x-a)^{\frac{2n}{p}-1}dx = \frac{p}{2n}(t-a)^{\frac{2n}{p}},$$

this will follow from the identity

$$(23)\qquad \lambda^{\frac{1}{p}}(\chi_t)(t-a)^{\frac{2n}{p}} = \lambda^{\frac{1}{p}}(1;0,1).$$

To establish (23) we note that, by an elementary argument,

$$\lambda(\chi_t;a,b) = \lambda(1;a,t);$$

moreover, since

$$\lambda(1;a,t) = \inf \frac{\int_a^t [u^{(n)}]^2 dx}{\int_a^t u^2 dx},$$

where u is subject to the boundary conditions (20) (with b = t), it is evident that

$$\lambda(1;a,t) = (t-a)^{-2n}\lambda(1;0,1).$$

This completes the proof of Theorem III.

For n = 1, we have $\lambda(1;0,1) = \pi^2$, and Theorem III yields the inequality

$$\lambda^{\frac{1}{p}}(R)\int_a^b \ldots {}^{2} \ldots {}^{\frac{1}{p}} dx \ldots p \ldots {}^{\frac{2}{p}} \ldots$$

for the lowest eigenvalue of the problem

$$y'' + \lambda R(x)y = 0, \quad y(a) = y'(b) = 0.$$

For $p = 2$, this reduces to the known inequality [2]

$$\lambda^{\frac{1}{2}}(R) \int_a^b R^{\frac{1}{2}}(x)dx \geq \ldots$$

5, If the coefficient $R(x)$ in (21) satisfies the condition $0 \leq m \leq R(x) \leq M < \infty$, an application of Theorem II leads to the following result.

~~Theorem~~ IV. ~~Let~~ $\lambda = \lambda(R)$ ~~be, the lowest eigenvalue of the differential equation~~ (21) ~~with the boundary conditions~~ (20), ~~where~~ $0 \leq m \leq R(x) \leq M < \infty$ ~~and~~ n is ~~a positive integer~~. ~~If the~~ number γ ~~is defined by~~

$$(24) \qquad \int_a^b R^{\frac{1}{p}}(x)dx = (b - a)\left[M^{\frac{1}{p}}\gamma + m^{\frac{1}{p}}(1 - \gamma)\right], \quad (0 \leq \gamma \leq 1),$$

then

$$\lambda(R) \geq \lambda(R_0),$$

where $R_0 = m$ for $a \leq x < a\gamma + b(1 - \gamma)$ and $R_0 = M$ for $a\gamma + b(1 - \gamma) \leq x \leq b$.

By (8) and Theorem II,

$$\lambda^{\frac{1}{p}}(R) \int_a^b R^{\frac{1}{p}}dx \geq \inf \lambda^{\frac{1}{p}}(T) \int_a^b T^{\frac{1}{p}}dx,$$

if T ranges over the class of functions $T = m + (M - m)\chi(X_0)$, and X_0 is a subset of $[a,b]$ of Lebesgue measure $\gamma(b - a)$, where γ is defined in (24). Since

$$\int_a^b R^{\frac{1}{p}}dx = \int_a^b T^{\frac{1}{p}}dx,$$

we thus have

(25) $$A(R) \geq \inf A(T) .$$

If y_R is the solution of (21)-(20) associated with the lowest eigenvalue, it is well known that y_R^2 is non-decreasing in [a,b] if R is non-negative. Since, for a non-decreasing y^2, the value of

$$\int_a^b [m + (M - m) f(X_Q)]y^2dx$$

is largest if X_o is the interval [a y + b(l -*) ,b] , it follows that

$$\frac{1}{\lambda(T)} = \int_a^b Ty_T^2dx \leq \int_a^b R_o y_T^2dx \leq \wedge \wedge y .$$

In view of (25), this proves Theorem IV.

For n = 1, p = 1, Theorem IV reduces to a result of Krein [1].

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